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INTERPOLATING SEQUENCE ON CERTAIN BANACH SPACES OF ANALYTIC FUNCTIONS

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Let G be a finitely connected domain and let X be a reflexive Banach space of functions analytic on G which admits the multiplication operator M_z as a polynomially bounded operator. We give some conditions that a sequence in G has an interpolating subsequence for X.

INTRODUCTION

Let X be a separable reflexive Banach space whose elements are analytic functions on a complex domain Ω. It is convenient and helpful to introduce the notation $\langle x, x^* \rangle$ to stand for $x^*(x)$, for $x \in X$ and $x^* \in X^*$. Assume $1 \in X$ and the operator M_z of multiplication by z maps X into itself and for each λ in Ω, the functional $e(\lambda) : X \to C$, the evaluation at λ given by $e(\lambda)(f) = \langle f, e(\lambda) \rangle = f(\lambda)$, is bounded.

For the algebra $B(X)$ of all bounded operators on a Banach space X, the weak operator topology is the one in which a net A_α converges to A if $A_\alpha x \to Ax$ weakly, $x \in X$.

A complex valued function φ on Ω for which $\varphi f \in X$ for every $f \in X$ is called a multiplier of X and the collection of all these multipliers is denoted by $M(X)$. Because M_z is a bounded operator on X, the adjoint $M_z^* : X^* \to X^*$ satisfies $M_z^* e(\lambda) = \lambda e(\lambda)$. In general each multiplier φ of X determines a multiplication operator M_φ defined by $M_\varphi f = \varphi f, f \in X$. Also $M_\varphi^* e(\lambda) = \varphi(\lambda) e(\lambda)$ ([8]). It is well-known that each multiplier is a bounded analytic function. Indeed $|\varphi(\lambda)| \leqslant \|M_\varphi\|$ for each λ in Ω. Also $M_\varphi 1 = \varphi \in X$. But $X \subset H(\Omega)$, thus φ is a bounded analytic function. We say that $M(X)$ is rotation invariant if whenever $h \in M(X)$, then $h_\theta \in M(X)$ where $h_\theta(z) = h(e^{-i\theta} z)$. Also we call that M_z is polynomially bounded in the sense that there is a constant $C > 0$ such that $\|M_p\| \leqslant C \|p\|_\infty$ for every polynomial p, where $\|p\|_\infty$ is the supremum norm of p on Ω. By $H(\overline{\Omega})$ we mean the set of all functions that are analytic in some fixed open set G containing $\overline{\Omega}$, with $f_k \to f$ uniformly on compact subsets of G.

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Main Results

First we give the Rosenthal–Dor Theorem which we need for the proof of our main theorem.

Rosenthal–Dor Theorem. *Suppose X is a Banach space and $\{e_n\}$ is a bounded sequence in X. Then there exists a subsequence $\{e_{n_k}\}_k$ such that either*

(i) *the map $\{a_k\}_{k=1}^{\infty} \to \sum\limits_{k=1}^{\infty} a_k e_{n_k}$ is an isomorphism of ℓ^1 into X, or*

(ii) *$\lim \varphi(e_{n_k})$ exists for every $\varphi \in X^*$.*

Proof: See [4] and [7]. ∎

The pseudo-hyperbolic distance $\rho(z,w)$ between points z,w in the unit disc U is defined by $\rho(z,w) = \left|(w-z)/(1-\overline{w}z)\right|$. Given any two pairs of points in U of equal pseudo-hyperbolic distance apart, there is an analytic automorphism of U mapping the first pair onto the second pair of points ([6]).

From now on we assume that X is a separable Banach space and the operator M_z is bounded on X.

Lemma 1. *Let $w_1, \dots, w_n \in U$, and $\varepsilon > 0$. Then there exists a function φ analytic on $\overline{U}$ such that $\varphi(w_i) = 1$ for $i = 1, 2, \dots, n$ and $\varphi(1) = -1$ and $\|\varphi\|_\infty \leqslant 1 + \varepsilon$.*

Proof: Consider the Blaschke product $B(z) = \prod\limits_{j=1}^{n} e^{i\theta_j}(z - w_j)/(1 - \overline{w_j}z)$. Clearly $\|B\|_\infty = 1$ and $B(w_j) = 0$ for $j = 1, \dots, n$. Now by the same method used in the proof of Lemma 9 in [1], consider the pseudo-hyperbolic distance $\rho(w,z)$ between points $w, z \in U$. Choose $\delta > 0$ such that $1/(1+\delta) = \rho\big(1/(1+\varepsilon), -1/(1+\varepsilon)\big)$. Thus $\rho\big(0, 1/(1+\delta)\big) = \rho\big(1/(1+\varepsilon), -1/(1+\varepsilon)\big)$. So there exists $b \in H(\overline{U})$ such that $\|b\|_\infty = 1$, $b(0) = 1/(1+\varepsilon)$ and $b\big(1/(1+\delta)\big) = -1/(1+\varepsilon)$.

Define $\varphi = (1+\varepsilon) b \circ \big(B/(1+\delta)\big)$. Clearly φ is analytic on $\overline{U}$, $\varphi(w_k) = (1+\varepsilon)b(0) = 1$ for $k = 1, \dots, n$ and $\varphi(1) = (1+\varepsilon) b\big(1/(1+\delta)\big) = -1$. Because b is analytic on $\overline{U}$ and $\|b\|_\infty = 1$, we conclude that φ is analytic on $\overline{U}$ and $\|\varphi\|_\infty \leqslant 1 + \varepsilon$. ∎

We now make the following.

Definition 2: An open connected subset Ω of the plane is called a Caratheodory region if its boundary equals the boundary of the unbounded component of $C \backslash \overline{\Omega}$.

It is easy to see that Ω is a Caratheodory region if and only if Ω is the interior of the polynomially convex hull of $\overline{\Omega}$. In this case the Farrell-Rubel–Shields Theorem holds [5, Theorem 5.1, p. 151]. Let f be a bounded analytic function on Ω. Then there is a sequence $\{p_n\}$ of polynomials such that $\|p_n\|_\Omega \leqslant C$ for a constant C and $p_n(z) \to f(z)$ for all $z \in \Omega$. In the following we suppose that Ω is the unit disc U.

Lemma 3. *Let M_z be polynomially bounded in the sense that for some $C > 0$, $\|M_p\| \leqslant C\,\|p\|_\infty$ for all polynomial p. Then $\|M_\varphi\| \leqslant C\,\|\varphi\|_\infty$ for all φ in $M(X)$.*

PROOF: Since $M(X) \subset H^\infty(U)$ and U is a Caratheodory region, there is a sequence $\{p_n\}$ of polynomials such that $\|p_n\|_\infty \leqslant \|\varphi\|_\infty$ and $p_n(z) \to \varphi(z)$ for every $z \in U$. For each $\lambda \in U$ we have

$$\langle M_{p_n} f, e(\lambda)\rangle = (p_n f)(\lambda) = p_n(\lambda)f(\lambda) \to \varphi(\lambda)f(\lambda) = \langle M_\varphi f, e(\lambda)\rangle.$$

Because $X^* = \operatorname{span}\{\mathrm{e}(\lambda) : \lambda \in \mathrm{U}\}$, we conclude that $\langle M_{p_n} f, g\rangle \to \langle M_\varphi f, g\rangle$ for all f in X and g in X^*. Now

$$|\langle M_{p_n} f, g\rangle| \leqslant \|M_{p_n}\| \; \|f\| \; \|g\| \leqslant C \|p_n\|_\infty \|f\| \; \|g\| \leqslant C \|\varphi\|_\infty \|f\| \; \|g\|.$$

Let $n \to \infty$, then $|\langle M_\varphi f, g\rangle| \leqslant C \|\varphi\|_\infty \|f\| \; \|g\|$ for all f in X and g in X^*. This completes the proof. ∎

DEFINITION 4: A sequence $\{w_n\}$ of points of Ω is said an interpolating sequence for X if there exists a positive weight sequence $\{k_n\}$ so that the sequence $\{f(w_n)k_n\}_{n=1}^\infty$ is in ℓ^∞ for all f in X and conversely every sequence in ℓ^∞ can be written in that form.

In the following $e(\lambda)$ is the functional of evaluation at λ.

THEOREM 5. *Let U be the open unit disc for which each point is a bounded point evaluation for a reflexive Banach space X of functions analytic on U which contains the constant functions and admits M_z to be polynomially bounded. Also assume that $M(X)$ is rotation invariant and $H(\overline{U}) \subset M(X)$. If $\{w_n\}$ is a sequence in U such that $w_n \to \partial U$, then some subsequences of $\{w_n\}$ is interpolating for X.*

PROOF: Put $e_n = (e(w_n))/\|e(w_n)\|$ for all $n \in \mathbb{N}$. Then $\{e_n\}_n$ is a bounded sequence in X^*. Use the Rosenthal–Dor Theorem for the Banach space X^* and let $\{e_{n_k}\}$ be the subsequence of $\{e_n\}_n$ promised by the Rosenthal–Dor Theorem, and suppose that case (i) of the Theorem holds. Let T denotes the isomorphism from ℓ^1 into X^* given by case (i) of the Rosenthal–Dor Theorem. Because X is reflexive and T is one to one with closed range, the dual T^* maps X onto ℓ^∞. Now let $a = \{a_n\} \in \ell^\infty$. Since T^* is onto, there exists $f \in X$ such that $T^* f = a$. Recall that $T^* f = f \circ T$. So $f \circ T = a$. Apply both sides of the equation $f \circ T = a$ to the vector in ℓ^1 that is 0 except for a 1 in the kth coordinates, getting $f(e_{n_k}) = a_k$ for every k. Thus

$$a_k = \langle e_{n_k}, f\rangle = \langle f, e_{n_k}\rangle = \left\langle f, \frac{e(w_{n_k})}{\|e(w_{n_k})\|}\right\rangle = \frac{f(w_{n_k})}{\|e(w_{n_k})\|}$$

for all k. On the otherhand for all f in X,

$$\left|\frac{f(w_{n_k})}{\|e(w_{n_k})\|}\right| = \left|\left\langle f, \frac{e(w_{n_k})}{\|e(w_{n_k})\|}\right\rangle\right| \leqslant \|f\|.$$

Thus indeed $\{w_{n_k}\}_k$ is interpolating for X if we can prove that case (ii) of the Rosenthal–Dor Theorem never holds. For this let $\{\varepsilon_k\}_k$ be a sequence of positive numbers such that

$\prod_{k=1}^{\infty}(1+\varepsilon_k) < \infty$. Similar to the proof of [1, Proposition 4, p. 416], by using Lemma 1 we can choose inductively an increasing sequence $n_1 < n_2 < \cdots$ of positive integers and a sequence $\varphi_1, \varphi_2, \ldots$ of functions analytic on $\overline{U}$ such that

$$\begin{gathered}(\varphi_1 \cdots \varphi_{k-1})(w_{n_k}) \approx (-1)^{k-1},\\ \varphi_k(w_{n_1}) = \cdots = \varphi_k(w_{n_k}) = 1,\\ \varphi_k(1) = -1,\\ \|\varphi_k\|_\infty \leqslant 1 + \varepsilon_k.\end{gathered}$$

By Lemma 3, $\|M_\varphi\| \leqslant C\,\|\varphi\|_\infty$ for all $\varphi \in M(X)$, thus we get

$$\|M_{\varphi_1\varphi_2\ldots\varphi_k}\| \leqslant C\,\|\varphi_1\varphi_2\ldots\varphi_k\|_\infty \leqslant C\prod_{i=1}^{k}\|\varphi_i\|_\infty \leqslant C\prod_{i=1}^{k}(1+\varepsilon_i).$$

Hence the sequence $\{M_{\phi_1\phi_2\ldots\phi_k}\}_k$ is norm bounded. Since X is reflexive, the unit ball of X is weakly compact. Therefore the unit ball of $B(X)$ is compact in the weak operator topology. We may assume, by passing to a subsequence if necessary, that $M_{\varphi_1\varphi_2\ldots\varphi_k} \to A$ in the weak operator topology, for some operator A. Thus $M^*_{\varphi_1\varphi_2\ldots\varphi_k}e(\lambda) \to A^*e(\lambda)$ in the weak star topology. On the otherhand $M^*_{\varphi_1\varphi_2\ldots\varphi_k}e(\lambda) = (\varphi_1\varphi_2\ldots\varphi_k)(\lambda)e(\lambda)$, so there exists a function φ such that $A^*e(\lambda) = \varphi(\lambda)e(\lambda)$ and thus $A^* = M^*_\varphi$. Hence $A = M_\varphi$ on X which implies that $\varphi \in M(X)$ and if $\{w_n\}$ is a sequence in U such that $|w_n| \to 1$, then φ satisfies $\varphi(w_{n_k}) = (-1)^k$ and $\lim_k \varphi(w_{n_k})$ does not exist. Now for suitable choices of $\theta_k, e^{-i\theta_k}w_{n_k}^k$ is a positive real number for all k. Now consider the sequence $\{a_k\}_k$ of positive real numbers such that the function $\psi(z) = \sum_{k=0}^{\infty} a_k e^{-i\theta_k} z^k$ be in X. Then $\psi(w_{n_k})$ is a positive real number. Define $h = \varphi\psi$. Since $\varphi \in M(X)$, the function h is in X and we have:

$$h(e_{n_k}) = \langle h, e_{n_k}\rangle = \langle \varphi\psi, e_{n_k}\rangle = \frac{\varphi(w_{n_k})\psi(w_{n_k})}{\|e(w_{n_k})\|} = (-1)^k\frac{\psi(w_{n_k})}{\|e(w_{n_k})\|}.$$

But

$$0 \leqslant \frac{\psi(w_{n_k})}{\|e(w_{n_k})\|} = \frac{\langle \psi, e(w_{n_k})\rangle}{\|e(w_{n_k})\|} \leqslant \|\psi\|,$$

for all k. So $\lim_k h(e_{n_k})$ does not exist. This completes the proof. ∎

Corollary 6. *Let $w_n \to \partial U$. Then there exists a function h in $H^\infty(U)$ such that $\lim_n h(w_n)$ does not exist.*

Proof: Let ϕ, ψ, h are defined as in the proof of the above theorem. Note that we could choose ψ being in $M(X)$. But $M(X)$ is an algebra, thus $h = \phi\psi \in M(X)$. Since $\lim_k h(w_{h_k})$ does not exist, the proof is complete. ∎

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Corollary 7. *Let $X \subset H(\Omega)$ when Ω is one of the sets $\{z : |z| > r\}$ or $\{z : |z| < r\}$. If $\{w_n\}$ is a sequence in Ω such that $w_n \to \partial\Omega$, then under the assumptions of the theorem there exists $h \in H^\infty(\Omega)$ such that $\lim_n h(w_n)$ does not exist.*

Proof: Since Ω is the conformal image of the open unit disc U, by the above corollary it is clear. ∎

Consider the circular domain $G = U \backslash K_1 \cup \ldots \cup K_N$ where $K_i = \overline{D_i} = \{z : |z - z_i| \leqslant r_i\}$ are disjoint closed subdiscs of the open unit disc U. Put $G_i = (\mathbb{C} \cup \{\infty\}) \backslash K_i$ for $i = 1, 2, \ldots, N$. Then by the Cauchy integral formula it is proved that

$$H^\infty(G) = H^\infty(G_0) + H_0^\infty(G_1) + \cdots + H_0^\infty(G_N) \tag{1}$$

where $G_0 = U, H_0^\infty(G_i) = H^\infty(G_i) \cap H_0(G_i)$ and $H_0(G_i)$ is the space of all analytic functions on G_i that vanish at infinity ([**2, 3**]).

The above Theorem can be extended for the case of circular domain instead of the open unit disc.

Corollary 8. *Theorem (5) is also true for any circular domain G, if in addition we suppose that $M(X) = H^\infty(G)$.*

Proof: By the same way as the Theorem we can prove that if case (i) of the Rosenthal–Dor Theorem is satisfied, then there exists some subsequence of $\{w_n\}$ that is interpolating for X. So it is sufficient to prove that case (ii) of the Theorem does not hold. Let $w_n \to \partial(G_i)$ for some $i = 0, 1, \ldots, N$. By the above corollary there is $h \in H^\infty(G_i)$ such that $\lim_n h(w_n)$ does not exist. By the decomposition (1), $H^\infty(G_0)$ and $H_0^\infty(G_i)$ are subsets of $H^\infty(G)$. So $h \in H^\infty(G) \subset M(X) \subset X$ and this completes the proof. ∎

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