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QUASIMODULAR FORMS AND VECTOR BUNDLES

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Abstract

Modular forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ can be identified with holomorphic sections of line bundles over the modular curve U corresponding to Γ, and quasimodular forms generalize modular forms. We construct vector bundles over U whose sections can be identified with quasimodular forms for Γ.

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1. Introduction

Modular forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ are closely linked to the geometry of the quotient $\Gamma\backslash\mathcal{H}$ of the Poincaré upper half plane $\mathcal{H}$ by the linear fractional action of Γ. One such link is given by the interpretation of modular forms as holomorphic sections of line bundles over $\Gamma\backslash\mathcal{H}$. The goal of this paper is to extend such interpretation to the case of quasimodular forms.

Quasimodular forms generalize classical modular forms and were introduced by Kaneko and Zagier in [3]. Since then, they have been studied actively not only in number theory but also in other branches of pure and applied mathematics (see, for example, [2, 4, 5]). One of the useful properties of quasimodular forms is that, unlike modular forms, derivatives of quasimodular forms are also quasimodular forms. If f is a quasimodular form for Γ of weight w and depth at most $m \geq 0$, then there are holomorphic functions $f_0, f_1, \ldots, f_m$ on $\mathcal{H}$ satisfying

$$\frac{1}{(cz+d)^w} f\left(\frac{az+b}{cz+d}\right) = f_0(z) + f_1(z)\left(\frac{c}{cz+d}\right) + \cdots + f_m(z)\left(\frac{c}{cz+d}\right)^m$$

for all $z \in \mathcal{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then it can be shown that the associated polynomial

$$F(z, X) = \sum_{r=0}^{m} f_r(z) X^r,$$

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known as a quasimodular polynomial, is invariant under a certain right action of Γ. In fact, the above correspondence determines an isomorphism between the space of quasimodular forms and that of quasimodular polynomials.

In this paper we use the above-mentioned right action of Γ on the space of quasimodular polynomials to construct vector bundles over $\Gamma\backslash\mathcal{H}$ whose sections can be identified with quasimodular polynomials and therefore with quasimodular forms.

2. Quasimodular forms

In this section we describe quasimodular forms for a discrete subgroup of $SL(2, \mathbb{R})$. We also discuss some basic properties of quasimodular polynomials, which can be identified with quasimodular forms.

Let $\mathcal{H}$ be the Poincaré upper half plane on which $SL(2, \mathbb{R})$ acts as usual by linear fractional transformation. Thus, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$, we may write

$$\gamma z = \frac{az+b}{cz+d} \in \mathcal{H}.$$

For the same γ and z we set

$$\mathfrak{J}(\gamma, z) = cz + d, \quad \mathfrak{K}(\gamma, z) = \frac{c}{cz+d}, \tag{2.1}$$

so that we obtain the maps $\mathfrak{J}, \mathfrak{K} : SL(2, \mathbb{R}) \times \mathcal{H} \to \mathbb{C}$. The map $\mathfrak{J}$ is a well-known automorphy factor satisfying the cocycle condition

$$\mathfrak{J}(\gamma\gamma', z) = \mathfrak{J}(\gamma, \gamma' z)\mathfrak{J}(\gamma', z) \tag{2.2}$$

for all $z \in \mathcal{H}$ and $\gamma\gamma' \in SL(2, \mathbb{R})$. The other function $\mathfrak{K}$, on the other hand, satisfies

$$\mathfrak{K}(\gamma\gamma', z) = \mathfrak{K}(\gamma', z) + \mathfrak{J}(\gamma', z)^{-2}\mathfrak{K}(\gamma, \gamma' z). \tag{2.3}$$

Let $\mathcal{F}$ be the ring of holomorphic functions on $\mathcal{H}$, and denote by $\mathcal{F}_m[X]$ with $m \geq 0$ the complex algebra of polynomials in X over $\mathcal{F}$ of degree at most m. Given elements $f \in \mathcal{F}$, $F(z, X) \in \mathcal{F}_m[X]$, $\lambda \in \mathbb{Z}$, and $\gamma \in SL(2, \mathbb{R})$, we set

$$(f \mid_\lambda \gamma)(z) = \mathfrak{J}(\gamma, z)^{-\lambda} f(z) \tag{2.4}$$

$$(F \parallel_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} F(\gamma z, \mathfrak{J}(\gamma, z)^2(X - \mathfrak{K}(\gamma, z))) \tag{2.5}$$

for all $z \in \mathcal{H}$. If γ' is another element of $SL(2, \mathbb{R})$, using (2.2) and (2.3), it can be shown that

$$f \mid_\lambda (\gamma\gamma') = (f \mid_\lambda \gamma) \mid_\lambda \gamma',$$

$$((F \parallel_\lambda \gamma) \parallel_\lambda \gamma')(z, X) = (F \parallel_\lambda (\gamma\gamma'))(z, X).$$

Thus the operations $\mid_\lambda$ and $\parallel_\lambda$ determine right actions of $SL(2, \mathbb{R})$ on $\mathcal{F}$ and $\mathcal{F}_m[X]$, respectively.

We now consider a discrete subgroup Γ of $SL(2, \mathbb{R})$ and modify the usual definition of modular and quasimodular forms for Γ by suppressing the cusp conditions.

DEFINITION 2.1.

(i) Given an integer μ, an element $f \in \mathcal{F}$ is a *modular form for* Γ *of weight* μ if it satisfies

$$f \mid_\mu \gamma = f \tag{2.6}$$

for all $\gamma \in \Gamma$, where $|_\mu$ is the operation in (2.4). We denote by $M_\mu(\Gamma)$ the space of modular forms for Γ of weight μ.

(ii) Given integers ξ and m with $m \geq 0$, an element $f \in \mathcal{F}$ is a *quasimodular form for* Γ *of weight* ξ *and depth at most* m if there are functions $f_0, \ldots, f_m \in \mathcal{F}$ such that

$$(f \mid_\xi \gamma)(z) = \sum_{r=0}^{m} f_r(z) \mathfrak{K}(\gamma, z)^r \tag{2.7}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\mathfrak{K}(\gamma, z)$ is as in (2.3). We denote by $QM_\xi^m(\Gamma)$ the space of quasimodular forms for Γ of weight ξ and depth at most m.

If $f \in QM_\xi^m(\Gamma)$ is a quasimodular form satisfying (2.7), by using the identity element for γ, we obtain

$$f(z) = f_0(z)$$

for all $z \in \mathcal{H}$. On the other hand, for fixed $z \in \mathcal{H}$, by considering the right-hand side of (2.7) as a polynomial in $\mathfrak{K}(\gamma, z)$ and using the fact that the same equation is valid for all elements Γ of Γ, we see that the given quasimodular form $f \in QM_\xi^m(\Gamma)$ determines the coefficients $f_0, \ldots, f_m$ uniquely. We also see easily that

$$QM_\xi^0(\Gamma) = M_\xi(\Gamma)$$

for each $\xi \in \mathbb{Z}$.

Given a quasimodular form $f \in QM_\xi^m(\Gamma)$ satisfying (2.7), we define the corresponding polynomial $(\mathcal{Q}_\xi^m f)(z, X) \in \mathcal{F}_m[X]$ by

$$(\mathcal{Q}_\xi^m f)(z, X) = \sum_{r=0}^{m} f_r(z) X^r \tag{2.8}$$

for $z \in \mathcal{H}$, so that we obtain the complex linear map

$$\mathcal{Q}_\xi^m : QM_\xi^m(\Gamma) \to \mathcal{F}_m[X]$$

for each pair of nonnegative integers ξ and m.

DEFINITION 2.2. A *quasimodular polynomial for* Γ *of weight* ξ *and degree at most* m is an element of $\mathcal{F}_m[X]$ that is Γ-invariant with respect to the right Γ-action in (2.5). We denote by

$$QP_\xi^m(\Gamma) = \{F(z, X) \in \mathcal{F}_m[X] \mid F \parallel_\xi \gamma = F \text{ for all } \gamma \in \Gamma\}$$

the space of all quasimodular polynomials for Γ weight ξ and degree at most m.

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LEMMA 2.3.

(i) *If $f \in \mathcal{F}$ is a quasimodular form belonging to $QM_\xi^m(\Gamma)$, then*

$$(\mathcal{Q}_\xi^m f)(z, X) \in QP_\xi^m(\Gamma).$$

(ii) *Let $F(z, X)$ be a quasimodular polynomial of the form*

$$F(z, X) = \sum_{r=0}^{m} f_r(z) X^r$$

belonging to $QP_\xi^m(\Gamma)$. Then f_0 is a quasimodular form belonging to $QM_\xi^m(\Gamma)$ such that the condition (2.7) is satisfied for $f = f_0$. Furthermore, for each $r \in \{0, 1, \ldots, m\}$ the coefficient f_r satisfies

$$(f_r|_{\xi-2r}\gamma)(z) = \sum_{\ell=r}^{m} \binom{\ell}{r} f_\ell(z)\mathfrak{K}(\gamma, z)^{\ell-r} = \sum_{\ell=0}^{m-r} \binom{\ell+r}{r} f_{\ell+r}(z)\mathfrak{K}(\gamma, z)^{\ell} \tag{2.9}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

PROOF. These results can be proved by using the definition of the operation $\|_\xi$ in (2.5) and the relations in (2.2) and (2.3) (see, for example, [1]). □

By Lemma 2.3 the map $\mathcal{Q}_\xi^m$ given by (2.8) induces an isomorphism

$$\mathcal{Q}_\xi^m : QP_\xi^m(\Gamma) \to QM_\xi^m(\Gamma).$$

Furthermore, if $\mathcal{Q}_\xi^m f$ with $f \in QP_\xi^m(\Gamma)$ is as in (2.8), then

$$f_0 = f \in QM_\xi^m(\Gamma);$$

hence, the inverse of the isomorphism $\mathcal{Q}_\xi^m$ is the map

$$\mathcal{P}_0 : QP_\xi^m(\Gamma) \to QM_\xi^m(\Gamma)$$

sending a quasimodular polynomial $F(z, X) \in QP_\xi^m(\Gamma)$ to its constant term

$$(\mathcal{P}_0 F)(z) = F(z, 0)$$

for all $z \in \mathcal{H}$.

3. Vector bundles

Let Γ be a discrete subgroup of $SL(2, \mathbb{R})$ as in Section 2. In this section we construct vector bundles over the quotient space $\Gamma\backslash\mathcal{H}$ whose sections may be identified with quasimodular polynomials and therefore quasimodular forms for Γ.

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Given integers λ, k and r with $0 \leq k \leq r \leq m$, we consider a map

$$\Xi_r^{\lambda,k} : SL(2, \mathbb{R}) \times \mathcal{H} \to \mathbb{C}$$

defined by

$$\Xi_r^{\lambda,k}(\gamma, z) = \binom{k}{r} \mathfrak{J}(\gamma, z)^{\lambda-2r} \mathfrak{K}(\gamma, z)^{k-r} \tag{3.1}$$

for $\gamma \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$.

LEMMA 3.1. *The map $\Xi_r^{\lambda,k}$ given by (3.1) satisfies*

$$\Xi_r^{\lambda,k}(\gamma_1\gamma, z) = \sum_{\ell=r}^{k} \Xi_r^{\lambda,\ell}(\gamma_1, \gamma z) \Xi_\ell^{\lambda,k}(\gamma, z) \tag{3.2}$$

for all $\gamma_1, \gamma \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$.

PROOF. If $\gamma_1, \gamma \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$, from (3.1) we obtain

$$\Xi_r^{\lambda,k}(\gamma_1\gamma, z) = \binom{k}{r} \mathfrak{J}(\gamma_1\gamma, z)^{\lambda-2r} \mathfrak{K}(\gamma_1\gamma, z)^{k-r}.$$

However, using (2.2) and (2.3),

$$\begin{aligned}
\mathfrak{J}(\gamma_1\gamma, z)^{\lambda-2r} &= \mathfrak{J}(\gamma_1, \gamma z)^{\lambda-2r} \mathfrak{J}(\gamma, z)^{\lambda-2r}, \\
\mathfrak{K}(\gamma_1\gamma, z)^{k-r} &= (\mathfrak{K}(\gamma, z) + \mathfrak{J}(\gamma, z)^{-2} \mathfrak{K}(\gamma_1, \gamma z))^{k-r} \\
&= \sum_{j=0}^{k-r} \binom{k-r}{j} \mathfrak{J}(\gamma, z)^{-2j} \mathfrak{K}(\gamma_1, \gamma z)^{j} \mathfrak{K}(\gamma, z)^{k-r-j} \\
&= \sum_{\ell=r}^{k} \binom{k-r}{\ell-r} \mathfrak{J}(\gamma, z)^{-2\ell+2r} \mathfrak{K}(\gamma_1, \gamma z)^{\ell-r} \mathfrak{K}(\gamma, z)^{k-\ell}.
\end{aligned}$$

Hence, we see that

$$\begin{aligned}
\Xi_r^{\lambda,k}(\gamma_1\gamma, z) = \sum_{\ell=r}^{k} \binom{k}{r} \binom{k-r}{\ell-r} & \mathfrak{J}(\gamma_1, \gamma z)^{\lambda-2r} \mathfrak{J}(\gamma, z)^{\lambda-2\ell} \\
& \times \mathfrak{K}(\gamma_1, \gamma z)^{\ell-r} \mathfrak{K}(\gamma, z)^{k-\ell}.
\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}
\Xi_r^{\lambda,\ell}(\gamma_1, \gamma z) \Xi_\ell^{\lambda,k}(\gamma, z) &= \binom{\ell}{r} \mathfrak{J}(\gamma_1, \gamma z)^{\lambda-2r} \mathfrak{K}(\gamma_1, \gamma z)^{\ell-r} \\
&\qquad \times \binom{k}{\ell} \mathfrak{J}(\gamma, z)^{\lambda-2\ell} \mathfrak{K}(\gamma, z)^{k-\ell} \\
&= \binom{k}{\ell} \binom{\ell}{r} \mathfrak{J}(\gamma_1, \gamma z)^{\lambda-2r} \mathfrak{J}(\gamma, z)^{\lambda-2\ell} \\
&\qquad \times \mathfrak{K}(\gamma_1, \gamma z)^{\ell-r} \mathfrak{K}(\gamma, z)^{k-\ell}.
\end{aligned} \tag{3.4}$$

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From (3.3), (3.4) and the relation

$$\binom{k}{r}\binom{k-r}{\ell-r} = \frac{k!}{r!(\ell-r)!(k-\ell)!} = \binom{k}{\ell}\binom{\ell}{r},$$

formula (3.2) follows. $\square$

We fix a nonnegative integer m, and denote by $\mathbb{C}_m[X]$ the ring of polynomials in X over $\mathbb{C}$ of degree at most m. Given a polynomial of the form

$$F(X) = \sum_{r=0}^{m} c_r X^r \in \mathbb{C}_m[X] \tag{3.5}$$

with $c_0, \ldots, c_m \in \mathbb{C}$ and an integer λ, we now set

$$\gamma \odot_\lambda^m (z, F(X)) = \left(\gamma z, \sum_{r=0}^{m}\sum_{k=r}^{m} c_k \Xi_r^{\lambda,k}(\gamma, z) X^r\right) \tag{3.6}$$

for all $\gamma \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$.

PROPOSITION 3.2. *Equation (3.6) determines a left action of $SL(2, \mathbb{R})$ on the Cartesian product $\mathcal{H} \times \mathbb{C}_m[X]$.*

PROOF. Given elements $\gamma, \gamma_1 \in SL(2, \mathbb{R})$, $z \in \mathcal{H}$ and a polynomial $F(X) \in \mathbb{C}_m[X]$ as in (3.5), using (3.6), we obtain

$$\begin{aligned}\gamma_1 \odot_\lambda^m (\gamma \odot_\lambda^m (z, F(X))) &= \left(\gamma_1\gamma z, \sum_{r=0}^{m}\sum_{\ell=r}^{m}\sum_{k=\ell}^{m} c_k \Xi_\ell^{\lambda,k}(\gamma, z)\Xi_r^{\lambda,\ell}(\gamma, z) X^r\right)\\ &= \left(\gamma_1\gamma z, \sum_{r=0}^{m}\sum_{k=r}^{m}\sum_{\ell=r}^{k} c_k \Xi_\ell^{\lambda,k}(\gamma, z)\Xi_r^{\lambda,\ell}(\gamma, z) X^r\right).\end{aligned} \tag{3.7}$$

On the other hand,

$$(\gamma_1\gamma) \odot_\lambda^m (z, F(X)) = \left(\gamma_1\gamma z, \sum_{r=0}^{m}\sum_{k=r}^{m} c_k \Xi_\ell^{\lambda,k}(\gamma_1\gamma, z) X^r\right). \tag{3.8}$$

Combining (3.7) and (3.8) with (3.2), we obtain

$$\gamma_1 \odot_\lambda^m (\gamma \odot_\lambda^m (z, F(X))) = (\gamma_1\gamma) \odot_\lambda^m (z, F(X));$$

hence, the proposition follows. $\square$

Let Γ be a discrete subgroup of $SL(2, \mathbb{R})$, and denote the quotient of the space $\mathcal{H} \times \mathbb{C}_m[X]$ by Γ with respect to the action shown in Proposition 3.2 by

$$\mathcal{V}_\lambda^m = \Gamma\backslash\mathcal{H} \times \mathbb{C}_m[X]. \tag{3.9}$$

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If we denote the modular curve associated with Γ by $U = \Gamma\backslash\mathcal{H}$, then the natural projection map $\mathcal{H} \times \mathbb{C}_m[X] \to \mathcal{H}$ induces a surjective map $\varpi : \mathcal{V}_\lambda^m \to U$ such that $\varpi^{-1}(x)$ is isomorphic to $\mathbb{C}_m[X]$ for each $x \in U$. Thus $\mathcal{V}_\lambda^m$ has the structure of a complex vector bundle over U whose fiber is the $(m+1)$-dimensional complex vector space $\mathbb{C}_m[X]$ of polynomials in X. We denote by $\mathbf{\Gamma}_0(U, \mathcal{V}_\lambda^m)$ the space of all holomorphic sections of $\mathcal{V}_\lambda^m$ over U.

THEOREM 3.3. *The space $\mathbf{\Gamma}_0(U, \mathcal{V}_\lambda^m)$ of holomorphic sections of $\mathcal{V}_\lambda^m$ over $U = \Gamma\backslash\mathcal{H}$ is canonically isomorphic to the space $QP_\lambda^m(\Gamma)$ of all quasimodular polynomials for Γ of weight λ and depth at most m.*

PROOF. Let $\sigma : U \to \mathcal{V}_\lambda^m$ be a holomorphic section of $\mathcal{V}_\lambda^m$ over $U = \Gamma\backslash\mathcal{H}$, and denote by $q : \mathcal{H} \to U$ the natural projection map. Given $z \in \mathcal{H}$, then

$$\sigma(q(z)) = \left[\left(z, \sum_{r=0}^{m} c_{r,z} X^r\right)\right] \in \Gamma\backslash\mathcal{H} \times \mathbb{C}_m[X]$$

for some $c_{0,z}, \ldots, c_{m,z} \in \mathbb{C}$, where $[(\cdot)]$ denotes the Γ-orbit of the element $(\cdot)$ of $\mathcal{H} \times \mathbb{C}_m[X]$. We define the $\mathbb{C}$-valued functions $f_0^\sigma, \ldots, f_m^\sigma$ on $\mathcal{H}$ by

$$f_r^\sigma(z) = c_{r,z} \tag{3.10}$$

for all $z \in \mathcal{H}$ and $0 \le r \le m$. Given $\gamma \in \Gamma$, since for each $z \in \mathcal{H}$ the Γ-orbits of z and γz are the same, then

$$\sigma(q(z)) = \sigma(q(\gamma z)) = \left[\left(\gamma z, \sum_{r=0}^{m} c_{r,\gamma z} X^r\right)\right]. \tag{3.11}$$

On the other hand, since

$$[\gamma \odot_\lambda^m (z, f(X))] = [(z, f(X))]$$

for each $(z, f(X)) \in \mathcal{H} \times \mathbb{C}_m[X]$, using (3.6) leads to

$$\begin{aligned}\sigma(q(z)) &= \left[\gamma \odot_\lambda^m \left(z, \sum_{r=0}^{m} c_{r,z} X^r\right)\right] \\ &= \left[\left(\gamma z, \sum_{r=0}^{m} \sum_{k=r}^{m} c_{k,z}\, \Xi_r^{\lambda,k}(\gamma, z) X^r\right)\right].\end{aligned}$$

Comparing this with (3.11) and using (3.1) and (3.10), we see that

$$\begin{aligned}f_r^\sigma(\gamma z) = c_{r,\gamma z} &= \sum_{k=r}^{m} c_{k,z}\, \Xi_r^{\lambda,k}(\gamma, z) \\ &= \sum_{k=r}^{m} \binom{k}{r} \mathfrak{J}(\gamma, z)^{\lambda-2r} \mathfrak{K}(\gamma, z)^{k-r} f_k^\sigma(z);\end{aligned}$$

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hence, we obtain

$$(f_r^\sigma|_{\lambda-2r}\gamma)(z) = \sum_{k=r}^{m} \binom{k}{r} \mathfrak{K}(\gamma, z)^{k-r} f_k^\sigma(z)$$

for $0 \le r \le m$. In particular,

$$(f_0^\sigma|_{\lambda-2r}\gamma)(z) = \sum_{k=0}^{m} \mathfrak{K}(\gamma, z)^{k-r} f_k^\sigma(z),$$

and therefore it follows that the polynomial

$$F^\sigma(z, X) = \sum_{r=0}^{m} f_r^\sigma(z) X^r$$

is a quasimodular polynomial belonging to $QP_\lambda^m(\Gamma)$. On the other hand, we assume that $G(z, X) = \sum_{r=0}^m g_r(z) X^r$ is a quasimodular polynomial belonging to $QP_\lambda^m(\Gamma)$. We define the map $\sigma_G : U \to \mathcal{V}_\lambda^m$ by

$$\sigma_G(q(z)) = \left[\left(z, \sum_{r=0}^{m} g_r(z) X^r\right)\right]$$

for all $z \in \mathcal{H}$. Then for each $\gamma \in \Gamma$, using (2.9), (3.1) and (3.6),

$$\begin{aligned}
\sigma_G(q(\gamma z)) &= \left[\left(\gamma z, \sum_{k=0}^{m} g_k(\gamma z) X^{k+\delta}\right)\right] \\
&= \left[\left(\gamma z, \sum_{k=0}^{m} \sum_{k=r}^{m} \binom{k}{r} \mathfrak{J}(\gamma, z)^{\lambda-2r} \mathfrak{K}(\gamma, z)^{k-r} g_k(z)\right)\right] \\
&= \left[\left(\gamma z, \sum_{k=0}^{m} \sum_{k=r}^{m} \Xi_r^{\lambda,k}(\gamma, z) g_k(z)\right)\right] \\
&= \left[\gamma \odot_\lambda^m \left(z, \sum_{r=0}^{m} g_r(z) X^r\right)\right],
\end{aligned}$$

and therefore σ_Γ is well defined. Since clearly $\varpi \circ \sigma_G = 1_U$, it follows that σ_G is a holomorphic section of $\mathcal{V}_\lambda^m$ over U; hence, the proof of the theorem is complete. □

REMARK 3.4. If $m = 0$, then the bundle $\mathcal{V}_\lambda^0$ becomes a line bundle and we obtain the isomorphism

$$\mathbf{\Gamma}_0(U, \mathcal{V}_\lambda^m) \cong M_\lambda(\Gamma)$$

for each λ, which provides the usual identification between modular forms and holomorphic sections of a line bundle.

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4. Morphisms of vector bundles

Given $m \geq 0$, there are natural linear maps carrying quasimodular forms of depth at most m to those of depth at most $r \leq m$. In this section we construct morphisms of vector bundles over a modular curve corresponding to such linear maps.

Given a polynomial $F(z, X) \in \mathcal{F}_m[X]$ of the form

$$F(z, X) = \sum_{r=0}^{m} f_r(z) X^r \tag{4.1}$$

with $f_0, \ldots, f_m \in \mathcal{F}$, we set

$$(\Delta_p F)(z, X) = \sum_{r=0}^{m-p} \binom{r+p}{p} f_{r+p}(z) X^r \tag{4.2}$$

for each integer p with $0 \leq p \leq m$, so that we obtain the complex linear map

$$\Delta_p : \mathcal{F}_m[X] \to \mathcal{F}_{m-p}[X]. \tag{4.3}$$

LEMMA 4.1. *Given $\lambda \in \mathbb{Z}$, then*

$$\Delta_p(QP_p^m(\Gamma)) \subset QP_{\lambda-2p}^{m-p}(\Gamma)$$

for each $p \in \{0, 1, \ldots, m\}$.

PROOF. Let $F(z, X) \in QP_\lambda^m(\Gamma)$ be a quasimodular polynomial of the form given by (4.1). Then by Lemma 2.3(ii) the coefficients of $F(z, X)$ satisfies

$$(f_j|_{\lambda-2j}\gamma)(z) = \sum_{\ell=j}^{m} \binom{\ell}{j} f_\ell(z) \mathfrak{K}(\gamma, z)^{\ell-j}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. From this and Lemma 2.3(i) we see that f_p is a quasimodular form belonging to $QF_{\lambda-2p}^{m-p}(\Gamma)$, and therefore $(\Delta_p F)(z, X)$ is a quasimodular polynomial belonging to $QP_{\lambda-2p}^{m-p}(\Gamma)$. □

Given $p \in \{0, 1, \ldots, m\}$, we now define the map

$$\widetilde{\Delta}_p : \mathcal{H} \times \mathbb{C}_m[X] \to \mathcal{H} \times \mathbb{C}_{m-p}[X] \tag{4.4}$$

by

$$\widetilde{\Delta}_p(z, f(X)) = (z, \Delta_p f(X)) \tag{4.5}$$

for all $f(X) \in \mathbb{C}_m[X]$, where $\Delta_p : \mathbb{C}_m[X] \to \mathbb{C}_{m-p}[X]$ is the map obtained from (4.3) by restriction. We consider the vector bundles

$$\mathcal{V}_\lambda^m = \Gamma\backslash\mathcal{H} \times \mathbb{C}_m[X], \quad \mathcal{V}_{\lambda-2p}^{m-p} = \Gamma\backslash\mathcal{H} \times \mathbb{C}_{m-p}[X], \tag{4.6}$$

where the first bundle is as in (3.9) and the second quotient is with respect to the operation $\odot_{\lambda-2p}^{m-p}$ in (3.6) of Γ on $\mathcal{H} \times \mathbb{C}_{m-p}[X]$.

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THEOREM 4.2. *If $0 \le p \le m$, the map $\widetilde{\Delta}_p$ in (4.4) induces a morphism*

$$\mathcal{V}_\lambda^m \to \mathcal{V}_{\lambda-2p}^{m-p} \tag{4.7}$$

of vector bundles in (4.6) over $X = \Gamma\backslash\mathcal{H}$.

PROOF. Given $\lambda \in \mathbb{Z}$ and $p \in \{0, 1, \ldots, m\}$, it suffices to prove that

$$\widetilde{\Delta}_p(\gamma \odot_\lambda^m (z, f(X))) = \gamma \odot_{\lambda-2p}^{m-p} \widetilde{\Delta}_p(z, f(X))$$

for all $z \in \mathcal{H}$, $\gamma \in \Gamma$ and $f(X) \in \mathbb{C}_m[X]$. If $z \in \mathcal{H}$, $\gamma \in \Gamma$ and

$$f(X) = \sum_{r=0}^m c_r X^r \in \mathbb{C}_m[X],$$

using (3.6), (4.2) and (4.5), we obtain

$$\begin{aligned}
&\widetilde{\Delta}_p(\gamma \odot_\lambda^m (z, f(X))) \\
&\quad = \widetilde{\Delta}_p\bigg(\gamma z, \sum_{r=0}^m \sum_{k=r}^m \binom{k}{r} \mathfrak{J}(\gamma, z)^{\lambda-2r} \mathfrak{K}(\gamma, z)^{k-r} c_k X^r\bigg) \\
&\quad = \bigg(\gamma z, \sum_{r=0}^{m-p} \binom{r+p}{p} \sum_{k=r+p}^m \binom{k}{r+p} \mathfrak{J}(\gamma, z)^{\lambda-2r-2p} \mathfrak{K}(\gamma, z)^{k-r-p} c_k X^r\bigg) \\
&\quad = \bigg(\gamma z, \sum_{r=0}^{m-p} \sum_{k=r}^{m-p} \binom{r+p}{p} \binom{k+p}{r+p} \mathfrak{J}(\gamma, z)^{\lambda-2r-2p} \mathfrak{K}(\gamma, z)^{k-r} c_{k+p} X^r\bigg).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\gamma \odot_{\lambda-2p}^{m-p} \widetilde{\Delta}_p(z, f(X)) \\
&\quad = \gamma \odot_{\lambda-2p}^{m-p} \bigg(z, \sum_{r=0}^{m-p} \binom{r+p}{p} c_{r+p} X^r\bigg) \\
&\quad = \bigg(\gamma z, \sum_{r=0}^{m-p} \sum_{k=r}^{m-p} \binom{k+p}{p} \Xi_r^{\lambda-2p}(\gamma, z) X^r\bigg) \\
&\quad = \bigg(\gamma z, \sum_{r=0}^{m-p} \sum_{k=r}^{m-p} \binom{k+p}{p} \binom{k}{r} \mathfrak{J}(\gamma, z)^{\lambda-2r-2p} \mathfrak{K}(\gamma, z)^{k-r} c_{k+p} X^r\bigg) \\
&\quad = \bigg(\gamma z, \sum_{r=0}^{m-p} \sum_{k=r}^{m-p} \binom{r+p}{p} \binom{k+p}{r+p} \mathfrak{J}(\gamma, z)^{\lambda-2r-2p} \mathfrak{K}(\gamma, z)^{k-r} c_{k+p} X^r\bigg);
\end{aligned}$$

hence, the theorem follows. □

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REMARK 4.3. If $p = m$ in (4.7), then we obtain the morphism

$$\mathcal{V}_\lambda^m \to \mathcal{V}_{\lambda-2m}^0$$

from a vector bundle to a line bundle, where the holomorphic sections of the line bundle $\mathcal{V}_{\lambda-2m}^0$ can be identified with modular forms as in Remark 3.4.

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