12

The Weinberg–Salam electroweak theory for leptons

We shall now couple the lepton fields to all the gauge boson fields: the electromagnetic field, the W^+ and W^- fields, and the Z field. We know that at low energies the theory must reproduce the phenomenology of Chapter 9. This consideration and the principles of $U(1) \times SU(2)$ local gauge symmetry determine the couplings uniquely.

We have seen how the Higgs mechanism gives mass to the $\mathrm{W}^{\pm}$ and Z bosons. To give mass to the charged leptons: the electron, the muon, the tau, they too must be coupled to the Higgs field. We shall finally arrive at the Weinberg–Salam unified theory of the electroweak interaction.

12.1 Lepton doublets and the Weinberg–Salam theory

We shall first construct a Lagrangian density for lepton fields that is invariant under $U(1)$ and $SU(2)$ transformations. The left-handed electron spinor e_{L} and the electron neutrino spinor ν_{eL} are put together in an $SU(2)$ doublet, like the Higgs fields in equation (11.1),

$$\mathbf{L} = \begin{pmatrix} \nu_{\mathrm{eL}} \\ e_{\mathrm{L}} \end{pmatrix} = \begin{pmatrix} \mathrm{L}_{\mathrm{A}} \\ \mathrm{L}_{\mathrm{B}} \end{pmatrix}. \tag{12.1}$$

We are now again specialising our notation; two-component left-handed and right-handed spinors were denoted by ψ_{L} and ψ_{R}, respectively, in Chapter 6. Under an $SU(2)$ transformation, this doublet transforms in exactly the same way as the Higgs doublet:

$$\mathbf{L} \to \mathbf{L}' = \mathbf{U}\mathbf{L}. \tag{12.2}$$

Since $SU(2)$ transformations mix the two spinor fields making up the doublet, to maintain Lorentz invariance only fields with the same Lorentz transformation properties can be combined together into a doublet.

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From the phenomenology of Chapter 9 the right-handed lepton fields do not couple to the W boson field so that e_R and ν_{eR} are invariant under $SU(2)$ transformations:

$$e_R \to e'_R = e_R. \quad \nu_{eR} \to \nu'_{eR} = \nu_{eR}. \tag{12.3}$$

To be consistent with the transformation rule (12.2), all $SU(2)$ gauge derivatives must be of the same form, $\partial_\mu + \mathrm{i}(g_2/2)\mathbf{W}_\mu$, where $g_2 \sin\theta_w = e$, as in (11.8) and (11.38). This is a consequence of the non-Abelian nature of the group $SU(2)$. However, there is no similar constraint on the coupling constant to the $U(1)$ gauge field B_μ. (See Problem 12.1.) We may take

$$D_\mu \mathbf{L} = [\partial_\mu + i(g_2/2)\mathbf{W}_\mu + \mathrm{i}(g'/2)B_\mu)]\mathbf{L}, \tag{12.4}$$

where g' remains at our disposal. We must choose g' so that the neutrino is neutral and the electron has charge $-e$. The terms in $D_\mu \mathbf{L}$ which couple to the electromagnetic field A_μ are linear combinations of W^3_μ and B_μ. Using (11.7) and (11.29) the terms in A_μ are

$$\begin{pmatrix} \partial_\mu + \{\mathrm{i}(\mathrm{g}_2/2)\sin\theta_\mathrm{w} + \mathrm{i}(\mathrm{g}'/2)\cos\theta_\mathrm{w}\}\mathrm{A}_\mu, & 0 \\ 0, & \partial_\mu + \{-\mathrm{i}(g_2/2)\sin\theta_\mathrm{w} + \mathrm{i}(\mathrm{g}'/2)\cos\theta_\mathrm{w}\}A_\mu \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}.$$

The gauge derivatives $\partial_\mu \nu_{eL}$ and $(\partial_\mu - \mathrm{i}eA_\mu)e_L$ which leave the neutrino electrically neutral but impart electric charge $-e$ to e_L, are obtained with the choice

$$g' \cos\theta_\mathrm{w} = -g_2 \sin\theta_\mathrm{w} = -e.$$

The complete gauge derivative of the left-handed fields is then

$$D_\mu \mathbf{L} = \begin{pmatrix} \partial_\mu + \mathrm{i}(e/\sin 2\theta_\mathrm{w})Z_\mu, & \mathrm{i}\{e/(\sqrt{2}\sin\theta_\mathrm{w})\}W^+_\mu \\ \mathrm{i}\{e/(\sqrt{2}\sin\theta_\mathrm{w})\}W^-_\mu, & \partial_\mu - \mathrm{i}eA_\mu - \mathrm{i}e\cot(2\theta_\mathrm{w})Z_\mu \end{pmatrix} \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \tag{12.5}$$

where we have used (11.7), (11.17) and (11.29).

The gauge derivative of e_R must be of the form

$$D_\mu e_R = [\partial_\mu + \mathrm{i}(g''/2)B_\mu]e_R. \tag{12.6a}$$

Since the electron has charge $-e$ we take $g'' = -2e/\cos\theta_\mathrm{w} = -2g_1$, (see (11.38)) so that, using (11.29) again,

$$D_\mu e_R = [(\partial_\mu - \mathrm{i}eA_\mu) + \mathrm{i}e\tan\theta_w Z_\mu]e_R. \tag{12.6b}$$

With $g'' = -2g_1$ and $g' = -g_1$, it can easily be checked that, under a local $U(1) \times SU(2)$ transformation

$$\mathbf{L} \to \mathbf{L}' = \mathrm{e}^{\mathrm{i}\theta(x)}\mathbf{U}(x)\mathbf{L},$$
$$e_R \to e'_R = \mathrm{e}^{2\mathrm{i}\theta(x)}e_R,$$

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the gauge derivatives satisfy

$$D_\mu{}'\mathrm{L}' = (\partial_\mu + \mathrm{i}(g_2/2)\mathbf{W}_\mu{}' + \mathrm{i}(g'/2)B_\mu{}')\mathbf{L}' = \mathrm{e}^{\mathrm{i}\theta}\mathbf{U}D_\mu\mathbf{L}$$
$$D_\mu{}'e_\mathrm{R}{}' = (\partial_\mu + \mathrm{i}(g''/2)B_\mu{}')e_R{}' = \mathrm{e}^{2\mathrm{i}\theta}\mathrm{D}_\mu e_\mathrm{R},$$

where the fields B_μ and $\mathbf{W}_\mu$ transform as in (11.4b) and (11.6).

We can now construct a gauge invariant and Lorentz invariant expression for the dynamical part of the Lagrangian density for the electron and the electron neutrino:

$$\mathcal{L}^e_{\mathrm{dyn}} = \mathbf{L}^\dagger\tilde{\sigma}^\mu \mathrm{i}D_\mu\mathbf{L} + e^\dagger_\mathrm{R}\sigma^\mu\mathrm{i}D_\mu e_\mathrm{R} + \nu^+_{\mathrm{eR}}\sigma^\mu\mathrm{i}\partial_\mu\nu_{e\mathrm{R}}. \tag{12.7}$$

The gauge invariance follows from our construction of the gauge derivatives, and the Lorentz invariance from the spinor properties set out in Section 5.4. (Remember that the $\tilde{\sigma}_\mu$ matrices act on the spinor indices, whereas the $SU(2)$ transformation acts independently on the components of the doublet of spinor fields.) Note that besides the interaction with the electromagnetic field we have fully determined, from the factor $D_\mu\mathbf{L}$, all the interactions with the heavy vector bosons.

Finally, we must give mass to the charged leptons. A gauge and Lorentz invariant contribution to the Lagrangian density that will impart mass to the electron but leave the neutrino massless is (neutrino mass will be introduced in Chapter 19)

$$\begin{aligned}\mathcal{L}^e_{\mathrm{mass}} &= -c_\mathrm{e}[(\mathbf{L}^\dagger\Phi)e_\mathrm{R} + e^\dagger_\mathrm{R}(\Phi^\dagger\mathbf{L})] \\ &= -c_\mathrm{e}[(\nu^\dagger_\mathrm{L}\Phi_\mathrm{A} + e^\dagger_\mathrm{L}\Phi_\mathrm{B})e_\mathrm{R} + e^\dagger_\mathrm{R}(\Phi^\dagger_\mathrm{A}\nu_\mathrm{L} + \Phi^\dagger_\mathrm{B}e_\mathrm{L})],\end{aligned} \tag{12.8}$$

where Φ is the Higgs doublet field and c_e is a dimensionless coupling constant. After symmetry breaking (see (11.23)), $\mathcal{L}^e_{\mathrm{mass}}$ becomes

$$\mathcal{L}^e_{\mathrm{mass}} = -c_\mathrm{e}\phi_0(e^\dagger_\mathrm{L}e_R + e^\dagger_\mathrm{R}e_\mathrm{L}) - \frac{c_\mathrm{e}h}{\sqrt{2}}\left(e^\dagger_\mathrm{L}e_\mathrm{R} + e^\dagger_\mathrm{R}e_\mathrm{L}\right). \tag{12.9}$$

Comparing this with the Dirac Lagrangian density (5.12), we identify $c_\mathrm{e}\phi_0$ with the electron mass m_e. Introducing mass by following the principles of symmetry has left us no option but to introduce an interaction between the electron field and the Higgs field $h(x)$. Hence the coupling constant to the Higgs field is

$$\frac{c_\mathrm{e}}{\sqrt{2}} = \frac{m_\mathrm{e}}{\sqrt{2}\phi_0} = 2.01 \times 10^{-6} \tag{12.10}$$

(using (11.39)). It is just as well that c_e is small: we do not want this term to upset the calculations of QED!

The total Lagrangian density $\mathcal{L}^e$ for the electron and its neutrino is given by (12.7) and (12.8):

$$\mathcal{L}^e = \mathcal{L}^e_{\mathrm{dyn}} + \mathcal{L}^e_{\mathrm{mass}}. \tag{12.11}$$

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From $\mathcal{L}^e$ we can pick out the terms

$$\begin{aligned}\mathcal{L}^{e}_{\text{Dirac}} = \nu^{\dagger}_{\text{eL}}\tilde{\sigma}^{\mu}\text{i}(\partial_{\mu}\nu_{\text{eL}})+e^{\dagger}_{\text{L}}\tilde{\sigma}^{\mu}\text{i}(\partial_{\mu} - \text{i}eA_{\mu})e_{\text{L}} + \nu^{\dagger}_{\text{eR}}\sigma^{\mu}\text{i}\partial_{\mu}\nu_{\text{eR}} \\ + e^{\dagger}_{\text{R}}\sigma^{\mu}\text{i}(\partial_{\mu} - \text{i}eA_{\mu})e_{\text{R}} - m_{\text{e}}\left(e^{\dagger}_{\text{L}}e_{\text{R}} + e^{\dagger}_{\text{R}}e_{\text{L}}\right),\end{aligned} \tag{12.12}$$

which correspond to the expressions we found in Chapter 6 and Chapter 7 for a Dirac massless neutrino, and a Dirac electron of mass m_{e} and charge $-e$ in an electromagnetic field.

The Lagrangian densities $\mathcal{L}^{\mu}$ and $\mathcal{L}^{\tau}$ for the muon and tau leptons and their neutrinos differ from (12.11) only in their mass parameters and, hence, their couplings to the Higgs field:

$$\frac{c_{\mu}}{\sqrt{2}} = \frac{m_{\mu}}{\sqrt{2}\phi_0} = 4.15 \times 10^{-4}, \qquad \frac{c_{\tau}}{\sqrt{2}} = \frac{m_{\tau}}{\sqrt{2}\phi_0} = 6.98 \times 10^{-3}. \tag{12.13}$$

The coupling constant g_2 of the $SU(2)$ gauge theory, or, equivalently, the Weinberg angle θ_{w} (see (11.38)), which determines the coupling to the $W^{\pm}$ and Z fields, must be the same for all leptons, a feature of the theory that is forced on us by the $SU(2)$ group, and that is known as *lepton universality.*

The complete Lagrangian density $\mathcal{L}^{\text{ws}}$ of the Weinberg–Salam theory (Weinberg, 1967; Salam, 1968) is the sum of the lepton contributions, and the boson contributions given by (11.31) and (11.32):

$$\mathcal{L}^{\text{ws}} = \mathcal{L}^{\text{e}} + \mathcal{L}^{\mu} + \mathcal{L}^{\tau} + \mathcal{L}^{\text{bosons}}, \tag{12.14}$$

The form of $\mathcal{L}^{\text{ws}}$ has been determined by considerations of symmetry: invariance under Lorentz transformations, and under $U(1)$ and $SU(2)$ transformations. Massive bosons and leptons appear through the Higgs mechanism of local symmetry breaking. It has been proved by t'Hooft (1976), who introduced radically new methods of analysis, that the theory is renormalisable. We shall see in Chapter 13 that there is a great body of data that supports it.

12.2 Lepton coupling to the $W^{\pm}$

The coupling of the electron and the electron neutrino to the W^{+} and W^{-} gauge fields is given by the appropriate terms in (12.5) and (12.7), which are

$$\begin{aligned}\mathcal{L}_{\text{ew}} &= -\left(g_2/\sqrt{2}\right)\nu^{\dagger}_{\text{eL}}\tilde{\sigma}^{\mu}e_{\text{L}}W^{+}_{\mu} - \left(g_2/\sqrt{2}\right)e^{\dagger}_{\text{L}}\tilde{\sigma}^{\mu}\nu_{\text{eL}}W^{-}_{\mu} \\ &= -\left(g_2/\sqrt{2}\right)[j^{\mu\dagger}_{\text{e}}W^{+}_{\mu} + j^{\mu}_{\text{e}}W^{-}_{\mu}].\end{aligned} \tag{12.15}$$

The right-handed fields do not contribute to this interaction. As in Chapter 9 the currents are defined as

$$j^{\mu}_{\text{e}} = e^{\dagger}_{\text{L}}\tilde{\sigma}^{\mu}\nu_{\text{eL}}, \quad j^{\mu\dagger}_{\text{e}} = \nu^{\dagger}_{\text{eL}}\tilde{\sigma}^{\mu}e_{\text{L}}. \tag{12.16}$$

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There are similar muon and tau currents, giving a total lepton current

$$j^{\mu} = \left(e_{\rm L}^{\dagger}\tilde{\sigma}^{\mu}\nu_{\rm eL} + \mu_{\rm L}^{\dagger}\tilde{\sigma}^{\mu}\nu_{\mu{\rm L}} + \tau_{\rm L}^{\dagger}\tilde{\sigma}^{\mu}\nu_{\tau{\rm L}}\right), \tag{12.17}$$

and total interaction Lagrangian density

$$\mathcal{L}_{lW} = -(g_2/\sqrt{2})\left[j^{\mu\dagger}W_{\mu}^{+} + j^{\mu}W_{\mu}^{-}\right]. \tag{12.18}$$

The effective $\mathcal{L}_{\rm lepton}$ used in the discussion of muon decay in Section 9.4 can be obtained as the low energy limit of the Weinberg–Salam theory. Since the mass $M_{\rm w}$ is so large, at low energies the term $M_{\rm w}^2 W_{\mu}^{-}W^{+\mu}$ in (11.31) dominates in the W contribution to the Lagrangian density, and

$$\mathcal{L}_{\rm w} \approx M_{\rm w}^2 W_{\mu}^{-}W^{+\mu} - \left(g_2/\sqrt{2}\right)[j^{\mu\dagger}W_{\mu}^{\dagger} + j^{\mu}W_{\mu}^{-}]. \tag{12.19}$$

Physical field configurations correspond to stationary values of the action. Varying W_{μ}^{+} and W_{μ}^{-} independently gives the field equations

$$M_{\rm w}^2 W_{\mu}^{-} = \left(g_2/\sqrt{2}\right)j_{\mu}^{\dagger}, \quad M_{\rm w}^2 W_{\mu}^{+} = \left(g_2/\sqrt{2}\right)j_{\mu}, \tag{12.20}$$

and using these in (12.19) gives

$$\mathcal{L}_{w} \approx -\frac{1}{2}g_2^2 M_{\rm w}^{-2} j_{\mu}^{\dagger}j^{\mu}. \tag{12.21}$$

$\mathcal{L}_{\rm w}$ is equivalent to the effective $\mathcal{L}_{\rm lepton}$ of (9.8) if we make the identification

$$G_{\rm F} = \frac{g_2^2}{4\sqrt{2}M_{\rm w}^2} = \frac{e^2}{4\sqrt{2}M_{\rm w}^2\sin^2\theta_{\rm w}}. \tag{12.22}$$

Taking $M_{\rm w} = 80.33$ Gev, $M_{\rm z} = 91.187$ GeV, $\sin^2\theta_{\rm w} = 1 - M_{\rm w}^2/M_z^2$, gives $G_{\rm F} = 1.12 \times 10^{-5}\,{\rm GeV}^{-2}$, which is in good agreement with the accepted experimental value, $1.166 \times 10^{-5}\,{\rm GeV}^{-2}$. Historically, the knowledge of $G_{\rm F}$, together with an estimate of $\theta_{\rm w}$ (see Section 13.1) was used to predict the masses of the $W^{\pm}$ and Z bosons, and the CERN proton–antiproton collider was then built to find them.

12.3 Lepton coupling to the *Z*

The coupling of the leptons to the Z field can be extracted from the terms involving Z_{μ} in (12.7):

$$\begin{aligned}\mathcal{L}_{\rm eZ} = &-\nu_{\rm eL}^{\dagger}\tilde{\sigma}^{\mu}\nu_{\rm eL}\left(\frac{e}{\sin(2\theta_{\rm w})}\right)Z_{\mu} + e_{\rm L}^{\dagger}\tilde{\sigma}^{\mu}e_{\rm L}\left(\frac{e\,\cos(2\theta_{\rm w})}{\sin(2\theta_{\rm w})}\right)Z_{\mu}\\ &-e_{\rm R}^{\dagger}\sigma^{\mu}e_{\rm R}(e\,\tan\theta_{\rm w})Z_{\mu} \quad \text{(using (12.5) and (12.6b))}\\ = &\ \frac{-e}{\sin(2\theta_{\rm w})}(j_{\rm neutral})_{\mu}Z^{\mu},\end{aligned}$$

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where

$$(j_{\text{neutral}})^{\mu} = \nu_{\text{eL}}^{\dagger}\tilde{\sigma}^{\mu}\nu_{\text{eL}} - \cos(2\theta_{\text{w}})e_{\text{L}}^{\dagger}\tilde{\sigma}^{\mu}e_{\text{L}} + 2\sin^2\theta_{\text{w}}e_{\text{R}}^{\dagger}\sigma^{\mu}e_{\text{R}}. \tag{12.23}$$

There are similar expressions for $\mathcal{L}_{\mu z}$ and $\mathcal{L}_{\tau z}$. Note that the right-handed charged lepton fields also couple to the Z field but not the right-handed neutrino.

The low energy limit of $\mathcal{L}_{z}$ may be obtained in the same way as we obtained the low energy limit $\mathcal{L}_{w}$ in Section 12.2, with the same identification of coupling constants, and is identical with the effective Lagrangian density (9.15) if, comparing (12.23) with (9.17),

$$c_{\text{A}} = -\frac{1}{2}, \quad c_{\text{V}} = -\frac{1}{2} + 2\sin^2\theta_{\text{w}}. \tag{12.24}$$

The low energy muon neutrino–electron elastic scattering cross-sections calculated from the effective Lagrangian density are

$$\sigma(\nu_{\mu} + \text{e}^{-} \to \nu_{\mu} + \text{e}^{-}) = \frac{G_{\text{F}}^2 s}{\pi}\left[\frac{4}{3}\sin^4\theta_{\text{w}} - \sin^2\theta_{\text{w}} + \frac{1}{4}\right], \tag{12.25}$$

$$\sigma(\bar{\nu}_{\mu} + \text{e}^{-} \to \bar{\nu}_{\mu} + \text{e}^{-}) = \frac{G_{\text{F}}^2 s}{\pi}\left[\frac{4}{3}\sin^4\theta_{\text{w}} - \frac{1}{3}\sin^2\theta_{\text{w}} + \frac{1}{12}\right], \tag{12.26}$$

where s is the square of the centre of mass energy and $E_{\nu} \gg m_{\text{e}}$ (see Perkins, 1987, p. 327).

These low energy ($\ll M_{\text{z}}, M_{\text{w}}$) cross-sections have been measured at CERN (CHARM II Collaboration, 1994), and their ratio yields an estimate for $\sin^2\theta_{\text{w}} = 0.2324 \pm 0.0083$.

The Fermi constant G_{F} is also known experimentally from low energy phenomena, and e is of course well known. Hence within the framework of the Weinberg–Salam theory the masses of the Z and $\text{W}^{\pm}$ gauge bosons can be estimated from low energy data alone, using (12.22) and (11.37). (Earlier estimates of $\sin^2\theta_{\text{w}}$ came from neutrino–nuclear scattering.)

12.4 Conservation of lepton number and conservation of charge

The Weinberg–Salam Lagrangian density $\mathcal{L}^{\text{WS}}$ has also further independent global $U(1)$ symmetries. It is invariant under the $U(1)$ transformation $\mathbf{L}_{\text{e}} \to \text{e}^{\text{i}\alpha}\mathbf{L}_{\text{e}}$, $e_{\text{R}} \to \text{e}^{\text{i}\alpha}e_{\text{R}}$, where α is a constant phase (see (12.7) and (12.9)). Using the device (by now familiar) of varying α so that $\alpha \to \alpha + \delta\alpha(x)$, where $\delta\alpha$ is space and time dependent, the first-order variation in the action comes from the dynamical part of

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$\mathcal{L}^{e}_{\rm dyn}$ (equation (12.7)), and is

$$\delta S = -\int \mathbf{L}^{\dagger}\tilde{\sigma}^{\mu}\mathbf{L}\partial_{\mu}(\delta\alpha)\,\mathrm{d}^4x - \int e^{\dagger}_{\rm R}\sigma^{\mu}e_{\rm R}\partial_{\mu}(\delta\alpha)\,\mathrm{d}^4x$$
$$= \int \left[\partial_{\mu}(\mathbf{L}^{\dagger}\tilde{\sigma}^{\mu}\mathbf{L}) + \partial_{\mu}\left(e^{\dagger}_{\rm R}\sigma^{\mu}e_{\rm R}\right)\right](\delta\alpha)\,\mathrm{d}^4x,$$

on integrating by parts. Setting $\delta S = 0$ for arbitrary $\delta\alpha$ yields

$$\partial_{\mu}\left(\nu^{\dagger}_{L}\tilde{\sigma}^{\mu}\nu_{\rm L} + e^{\dagger}_{\rm L}\tilde{\sigma}^{\mu}e_{\rm L}\right) + \partial_{\mu}\left(e^{+}_{\rm R}\sigma^{\mu}e_{\rm R}\right) = 0,$$

or

$$\partial_{\mu}\left(J^{\mu}_{\rm e}\right) = 0, \tag{12.27}$$

where

$$\begin{aligned} J^{0}_{\rm e} &= \nu^{\dagger}_{\rm L}\nu_{\rm L} + e^{\dagger}_{\rm L}e_{\rm L} + e^{\dagger}_{\rm R}e_{\rm R}, \\ J^{i}_{\rm e} &= \nu^{\dagger}_{\rm L}\tilde{\sigma}^{i}\nu_{\rm L} + e^{\dagger}_{\rm L}\tilde{\sigma}^{i}e_{\rm L} + e^{\dagger}_{\rm R}\sigma^{i}e_{\rm R}. \end{aligned} \tag{12.28}$$

Equation (12.28), which we may write as

$$\frac{\partial J^{0}_{e}}{\partial t} + \nabla\cdot\mathbf{J}_{\rm e} = 0, \tag{12.29}$$

expresses the conservation of electron lepton number. Similar $U(1)$ transformations applied to the muon and tau parts of $\mathcal{L}_{\rm ws}$ give the conservation of muon lepton number, and tau lepton number. We will see in Chapter 19 that the inclusion of Dirac neutrino mass into the Standard Model reduces these three conservation laws to one.

As in Chapters 4 and 5, the inhomogeneous Maxwell equations can be obtained by varying A_{μ}. There are contributions to the electric current from the charged $W^{\pm}$ fields, as well as from the charged leptons. Conservation of charge follows from Maxwell's equations, but can be obtained more directly from the $U(1)$ symmetry apparent in each term of the Weinberg–Salam Lagrangian density (12.14):

$$e_{\rm L} \to \mathrm{e}^{\mathrm{i}\alpha}e_{\rm L},\ e_{\rm R} \to \mathrm{e}^{\mathrm{i}\alpha}e_{\rm R};\ \mu_{\rm L} \to e^{\mathrm{i}\alpha}\mu_{\rm L},\ \mu_{\rm R} \to \mathrm{e}^{\mathrm{i}\alpha}\mu_{R};\ \tau_{\rm L} \to \mathrm{e}^{\mathrm{i}\alpha}\tau_{\rm L},\ \tau_{\rm R} \to \mathrm{e}^{\mathrm{i}\alpha}\tau_{\rm R};\ W^{+}_{\mu} \to \mathrm{e}^{-\mathrm{i}\alpha}W^{+}_{\mu},\ W^{-}_{\mu} \to \mathrm{e}^{\mathrm{i}\alpha}W^{-}_{\mu}. \tag{12.30}$$

12.5 *CP* symmetry

We saw in Chapter 5 (equation (5.27)) that under space inversion a left-handed spinor $\psi_{\rm L}$ transforms into a right-handed spinor $\psi_{\rm R}$, and *vice versa.* The Weinberg–Salam Lagrangian does not have space inversion symmetry, since only the left-hand components of the lepton wave functions are coupled to the $SU(2)$ gauge field $\mathbf{W}_{\mu}$.

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We also discussed in Chapter 7 the operation of charge conjugation,

$$\psi_L^C = -i\sigma^2\psi_R^*, \quad \psi_R^C = i\sigma^2\psi_L^*,$$

which relates solutions of the Dirac equation for particles to solutions for antiparticles. In the Weinberg–Salam theory there is no charge symmetry.

The Weinberg–Salam Lagrangian does exhibit a symmetry under the combined *CP* (charge conjugation, parity) operation. This symmetry implies that the physics of particles described in a right-handed coordinate system is the same as the physics of antiparticles described in a left-handed coordinate system.

Under the combined *CP* operation, lepton fields transform according to

$$\psi_L^{CP} = -i\sigma^2\psi_L^*, \quad \psi_R^{CP} = i\sigma^2\psi_R^*. \tag{12.31}$$

The other fields in the electroweak theory transform as set out below:

Higgs field: $\begin{pmatrix}\Phi_A^{CP}\\ \Phi_B^{CP}\end{pmatrix} = \begin{pmatrix}\Phi_A^*\\ \Phi_B^*\end{pmatrix}$.

$U(1)$ gauge fields: $B_0^{CP} = -B_0$, $B_i^{CP} = B_i$.

$SU(2)$ gauge fields:

$$\begin{pmatrix} W_0^3 & W_o^1 - iW_0^2 \\ W_0^1 + iW_0^2 & -W_0^3 \end{pmatrix}^{CP} = -\begin{pmatrix} W_0^3 & W_0^1 + iW_0^2 \\ W_0^1 - iW_0^2 & -W_0^3 \end{pmatrix},$$
$$\begin{pmatrix} W_i^3 & W_i^1 - iW_i^2 \\ W_i^1 + iW_i^2 & -W_i^3 \end{pmatrix}^{CP} = \begin{pmatrix} W_i^3 & W_i^1 + iW_i^2 \\ W_i^1 - iW_i^2 & -W_i^3 \end{pmatrix}.$$

It follows that

$$\begin{aligned} W_0^{+CP} &= -W_0^-, & W_i^{+CP} &= W_i^-, \\ Z_0^{CP} &= -Z_0, & Z_i^{CP} &= Z_i, \\ A_0^{CP} &= -A_0, & A_i^{CP} &= A_i. \end{aligned} \tag{12.32}$$

Space derivatives of fields are replaced by their negatives.

To show that the Lagrangian density is invariant under these transformations requires some care. We demonstrate it here for just one term, but one which involves all the necessary steps in the complete argument, and we leave the remaining terms to the reader. Consider then the term from the expression (12.7)

$$e_R^\dagger\sigma^\mu i[\partial_\mu + i(g''/2)B_\mu]e_R = l, \text{ say}.$$

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Replacing the fields by their *CP* transforms, and ∂_i by $-\partial_i$, gives

$$l^{CP} = e_{\mathrm{R}}^{\mathrm{T}}(\sigma^\mu)^{\mathrm{T}}\mathrm{i}[\partial_\mu - \mathrm{i}(g''/2)B_\mu]e_{\mathrm{R}}^*,$$

where we have used the results

$$(\sigma^2)^2 = 1, \quad \sigma^2\sigma^i\sigma^2 = -(\sigma^i)^{\mathrm{T}}.$$

The operators ∂_μ now act on the conjugate fields. In fact l^{CP} is not identical to l, but differs from it only by a sum of total derivatives and, as explained in Section 3.1, a total derivative is of no consequence. If we add to l^{CP} the terms $-\mathrm{i}\partial_\mu[e_{\mathrm{R}}^{\mathrm{T}}(\sigma^\mu)^{\mathrm{T}}e_{\mathrm{R}}^*]$ we obtain

$$-\mathrm{i}\left(\partial_\mu e_{\mathrm{R}}^{\mathrm{T}}\right)(\sigma^\mu)^{\mathrm{T}}\, e_{\mathrm{R}}^* + \left(g''/2\right) B_\mu e_{\mathrm{R}}^{\mathrm{T}}\,(\sigma^\mu)^{\mathrm{T}}\, e_{\mathrm{R}}^*.$$

Transposing this expression introduces another minus sign, since e_{R} and $e_{\mathrm{R}}{}^\dagger$ are fermion fields and hence anticommute. We then recover l.

12.6 Mass terms in $\mathcal{L}$: an attempted generalisation

For later use, when the theory is extended to quarks, we finish this chapter by contemplating a possible generalisation of our Lagrangian density. The coupling of the three lepton families to the Higgs field was taken to be

$$\mathcal{L}_{\mathrm{mass}} = -\sum_{i=1}^{3} c_i\left[\left(\mathbf{L}_i^\dagger\Phi\right)r_i + r_i^\dagger\left(\Phi^\dagger\mathbf{L}_i\right)\right],$$

where the sum is over the three lepton families, and we have modified the notation of (12.8) in an obvious way. We might have taken a more general coupling,

$$\mathcal{L}_{\mathrm{mass}}^{\mathrm{gen}} = -\sum\left[G_{ij}\left(\mathbf{L}_i^\dagger\Phi\right)r_j + G_{ij}^* r_j^\dagger\left(\Phi^\dagger\mathbf{L}_i\right)\right].$$

This preserves the $U(1)\times SU(2)$ symmetry with G_{ij} any 3×3 complex matrix.

We wish to show that this form has no essential difference from that already introduced. This is because an arbitrary complex matrix can always be put into real diagonal form with the help of two unitary matrices, $\mathbf{U}_{\mathrm{L}}$ and $\mathbf{U}_{\mathrm{R}}$ (Appendix A):

$$\mathbf{G} = \mathbf{U}_{\mathrm{L}}{}^\dagger\mathbf{C}\mathbf{U}_{\mathrm{R}},$$

with $C_{ij} = 0$ for $i \neq j$.

$\mathbf{U}_\mathrm{L}$ and $\mathbf{U}_\mathrm{R}$ are in general unique, except that both may be multiplied on the left by the same 'phase factor' matrix

$$\begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} \end{pmatrix}.$$

If we define $r_i{}' = U_{\mathrm{R}ij} r_j$, $L_i{}' = U_{\mathrm{L}ij} L_j$ we recover the original form for the coupling to the Higgs field. Since the dynamical terms in the Lagrangian density are of the same form after these unitary transformations (Problem 12.5), $\mathcal{L}^{\mathrm{gen}}_{\mathrm{mass}}$ is just a more complicated expression of the same physics. The three phase factors $\exp(i\alpha_k)$ correspond to the three $U(1)$ symmetries which lead to electron, muon, and tau number conservation.

Problems

12.1 Set the fields W_μ to be zero, and consider the dynamical Lagrangian density

$$\mathcal{L}_1 = \mathbf{L}^\dagger \tilde{\sigma}^\mu i\left(\partial_\mu + i(g'/2)\, B_\mu\right) \mathbf{L}.$$

With the gauge transformation (11.4b),

$$B_\mu \to B_\mu{}' = B_\mu + (2/g_1)\, \partial_\mu \theta,$$

show that $\mathcal{L}_1$ is invariant if $\mathbf{L}$ transforms as

$$\mathbf{L} \to \mathbf{L}' = \exp[-i(g'/g_1)\theta]\mathbf{L}.$$

Now set the fields B_μ to be zero, and consider

$$\mathcal{L}_2 = \mathbf{L}^\dagger \tilde{\sigma}^\mu i(\partial_\mu + i(g'/2)\mathbf{W}_\mu)\mathbf{L}.$$

With the gauge transformation (11.6),

$$\mathbf{W}_\mu \to \mathbf{W}_\mu{}' = \mathbf{U}\mathbf{W}_\mu\mathbf{U}^\dagger + (2i/g_2)(\partial_\mu \mathbf{U})\mathbf{U}^\dagger,$$

show that $\mathcal{L}_2$, can be made invariant only if

$$\mathbf{L} \to \mathbf{L}' = \mathbf{U}\mathbf{L} \quad \text{and} \quad g' = g_2.$$

12.2 Show that, to conform with the mathematical structure of Chapter 11, if two fields are to be put together in an $SU(2)$ doublet then they must differ by e in electric charge.

12.3 Inspection of (12.9) shows that the Higgs boson can decay into an e^+e^- pair. Show that, in the rest frame of the Higgs particle, the electron and positron must have equal and opposite momenta and the same helicity (i.e. both positive or both negative).

Show that the final density of momentum states for the decay is

$$\rho(E_f) = \frac{V}{(2\pi)^2} p_{\mathrm{e}} E_{\mathrm{e}},$$

where p_{e} and E_{e} are the momentum and energy of the electron.

Calculate the matrix elements for the transition, and hence show that to lowest order in perturbation theory,

$$\text{total decay rate} = \frac{c_e^2}{16\pi} m_{\mathrm{H}} \left(\frac{v_e}{c}\right)^3,$$

where v_{e} is the electron velocity.

12.4 Show that the ratio of the leptonic partial width of the Higgs particle to its mass is approximately

$$\frac{1}{16\pi}\left(\frac{m_\tau}{\phi_0}\right)^2 \approx 2 \times 10^{-6}.$$

12.5 Verify that the unitary transformations of Section 12.6 preserve the form of the dynamical terms in the Lagrangian density.

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