



Hilbert Schemes of Degree Four Curves

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Abstract. In this paper we determine the irreducible components of the Hilbert schemes $H_{4,g}$ of locally Cohen-Macaulay space curves of degree four and arbitrary arithmetic genus g : there are roughly $\sim (g^2/24)$ of them, most of which are families of multiplicity structures on lines. We give deformations which show that these Hilbert schemes are connected. For $g \leq -3$ we exhibit a component that is disjoint from the component of extremal curves and use this to give a counterexample to a conjecture of Ait-Amrane and Perrin.

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1. Introduction

Liaison theory has played a prominent role in classifying algebraic curves in \mathbb{P}^3 since the pioneering work of Max Noether. It has only recently become clear that locally Cohen-Macaulay curves are the natural objects of study, even if one is only interested in smooth connected curves. This is because each biliaison class is generated (via a rather explicit procedure known to the expert as the Lazarsfeld–Rao property [4, 16]) by an essentially unique minimal curve, which need only be locally Cohen–Macaulay. For example, in the classification of smooth irreducible curves of degree 8 and genus 5 [7], one family of curves is in the biliaison class of double lines of genus -2 (non-reduced curves) and another is in the biliaison class of the disjoint unions of a line and a twisted cubic (nonconnected curves). This explains the interest in Hilbert schemes $H_{d,g}$ parametrizing locally Cohen–Macaulay curves in \mathbb{P}^3 of degree d and arithmetic genus g .

In general, we know that the Hilbert scheme $H_{d,g}$ is non-empty precisely when $g = \frac{1}{2}(d-1)(d-2)$ (plane curves) or when $d > 1$ and $g \leq \frac{1}{2}(d-2)(d-3)$ [10, 3.3 and 3.4] and that $H_{d,g}$ is reducible for $d \geq 3$ and $g \leq \frac{1}{2}(d-3)(d-4)$ with the exceptions $(d, g) = (3, 0)$ and $(3, -1)$ [18]. More recently it has been shown that $H_{d,g}$ is connected for $g > \binom{d-3}{2} - 2$ [1, 23, 26] and that $H_{3,g}$ is connected for all g [20].

The connectedness results above were obtained by deforming curves to *extremal curves* as introduced by Martin-Deschamps and Perrin [17]: these are the curves C

which have the largest Rao function $h^1\mathcal{I}_C(n)$ with respect to d and g . Geometrically speaking, the extremal curves are precisely the curves of degree d which contain a planar subcurve of degree $d-1$ (unless C is planar or $(d, g) \in \{(3, 0), (4, 1)\}$) and they form an irreducible component $E \subset H_{d,g}$ [18]. The existence of a component of curves with the largest Rao function led Hartshorne to ask the following questions:

QUESTION 1. Is $H_{d,g}$ connected for all d and g ?

QUESTION 2. Does each irreducible component $B \subset H_{d,g}$ meet the extremal component E ?

Hartshorne showed that extremal curves can be connected to various families of curves, for example smooth rational and elliptic curves, smooth curves of degree $d \geq g+3$, arithmetically Cohen–Macaulay curves and many others [11]. Perrin showed that any curve with a Koszul Rao module can be deformed to an extremal curve [25]. Related to this is a conjecture of Ait-Amrane and Perrin stating that if X is a family of curves whose cohomology does not exceed that of a family X_0 and the Rao module of the general curve in X is a flat deformation of a subquotient of the Rao module of curves in X_0 , then $\bar{X} \cap X_0 \neq \emptyset$: they have shown [2] that semi-continuity alone is insufficient.

Now consider curves of degree $d=4$. The Hilbert scheme $H_{4,3}$ parametrizes plane curves and is smooth irreducible of dimension 17; $H_{4,1}$ is smooth irreducible of dimension 16 by [6] and [10, 3.3 and 3.5], its general member being a complete intersection of two quadrics. The Hilbert scheme $H_{4,0}$ has two irreducible components, whose general members are respectively rational quartic curves and disjoint unions of a plane cubic and a line. Hartshorne first noticed that these two families can be connected; now there are several published proofs: [11, 13, 20]. One of our motivations is to extend the systematic study of these Hilbert schemes and complete the picture when $d=4$.

Our first theorem describes the irreducible components of $H_{4,g}$ and gives their dimensions. Our method is to first identify components whose general curve C is special in the sense that C is contained in a quadric surface ($h^0(\mathcal{I}_C(2)) \geq 1$) or has large speciality ($h^1(\mathcal{O}_C(-1)) \geq 2$). There are very few such components: if the general curve C of a family lies on a quadric, then either C is an *extremal* curve, a *subextremal* curve (these are the curves with largest Rao function among the nonextremal curves [21]) or a double conic. If the general curve C of an irreducible component of $H_{4,g}$ has large speciality but does *not* lie on a quadric, we show (Proposition 4.3) that C is either a *thick 4-line* or the union of a conic and a double line meeting at a double point. A thick 4-line [3] is a curve of degree 4 supported on a line L and containing the first infinitesimal neighborhood of L in \mathbb{P}^3 .

Having disposed of these very special components, it is relatively easy to list those remaining. Their general member is either (a) a quasiprimitive (i.e. nonthick [3])

4-line, (b) the disjoint union of a line and a general curve of an irreducible component of $H_{3,g}$, or (c) the disjoint union of two double lines. Asymptotically $H_{4,g}$ has $\sim(g^2/24)$ irreducible components, most of which arise from curves supported on lines. For example, from Table III we find that $H_{4,-100}$ has 530 components (377 of these are families of 4-lines) while $H_{4,-1,000}$ has 42755 components (of which 41252 arise from 4-lines).

Our second theorem states that $H_{4,g}$ is connected when it is not empty. The main novelty for $g \leq -3$ is the presence of a component G_4 consisting entirely of thick 4-lines. We prove the connectedness theorem by showing that each irreducible component can be connected either to the extremal component E or to G_4 , and that the component of subextremal curves meets both E and G_4 . Specifically, the quasi-primitive 4-lines and the curves with large speciality deform to thick 4-lines (Sections 2 and 4). We show that a disjoint union of double lines specializes to a quasi-primitive 4-line on a double quadric surface in Section 3. Section 5 is devoted to showing that families of unions of triple lines and reduced lines can be connected to the extremal component.

The component G_4 of thick 4-lines turns out to be rather interesting. Since these curves are scheme-theoretically (although not cohomologically) the most special, they cannot specialize to extremal curves, answering Question 2 in the negative. Since their Rao modules are flat deformations of subquotients of the Rao modules of extremal curves (Example 6.7), we obtain a counterexample to the conjecture of Ait-Amrane and Perrin above. Question 1 remains open.

Throughout the paper we work over an algebraically closed field k of arbitrary characteristic. A *curve* for us is a locally Cohen–Macaulay scheme over k of pure dimension 1. We will freely use the sentence ‘the family of curves of degree d and genus g with property P is irreducible of dimension m ’ to mean that there is a (unique) irreducible m -dimensional constructible subset S of the Hilbert scheme $H_{d,g}$ whose closed points parametrize the curves of degree d and genus g with property P . Note that, since S is constructible, the closure of S in the Hilbert scheme is also an m -dimensional irreducible subset.

2. Multiplicity Four Structures on Lines

In this section we study locally Cohen–Macaulay curves in \mathbb{P}^3 which are supported on a line; we will simply call these d -lines, where d is the degree of the curve.

We begin with the general theory of Banica and Forster [3, § 3]. Let C be a locally Cohen–Macaulay curve on a smooth threefold X with smooth support Y . Letting $Y^{(i)}$ be the subscheme of X defined by \mathcal{I}_Y^i and C_i be the subscheme of X obtained by removing the embedded points from $C \cap Y^{(i)}$, we obtain the *Cohen–Macaulay filtration* for C :

$$Y = C_1 \subset C_2 \subset \cdots \subset C_k = C \tag{1}$$

for some $k \geq 1$. The quotients $L_j = \mathcal{I}_{C_j}/\mathcal{I}_{C_{j+1}}$ are vector bundles on Y and the multiplicity is given by $\mu(C) = 1 + \sum \text{rank } L_j$. The natural inclusions $\mathcal{I}_{C_i}\mathcal{I}_{C_j} \subset \mathcal{I}_{C_{i+j}}$ induce generically surjective multiplication maps $L_i \otimes L_j \rightarrow L_{i+j}$ and in particular we obtain generic surjections $L_1^j \rightarrow L_j$.

As in [3, § 4], the curve C is said to be *thick* if $Y^{(2)} \subset C$; this is equivalent to the condition that C has embedding dimension three at each point. In terms of filtration (1) above, this says that $C_2 = Y^{(2)}$, in which case $L_1 = \mathcal{I}_Y/\mathcal{I}_Y^2$ is the conormal bundle of Y on X . If further $\mu(C) = 4$, then $\text{rank } L_2 = 1$ and there is an exact sequence

$$0 \rightarrow \frac{\mathcal{I}_C}{\mathcal{I}_Y^3} \rightarrow \frac{\mathcal{I}_Y^2}{\mathcal{I}_Y^3} \rightarrow L_2 \rightarrow 0. \tag{2}$$

For a line $Y \subset \mathbb{P}^3$, we obtain the following.

PROPOSITION 2.1. *For $g \leq 1$, the thick 4-lines in $H_{4,g}$ form an irreducible closed subset of dimension $9 - 3g$.*

Proof. The condition that C contain $Y^{(2)}$ is clearly closed. If C is a thick 4-line with support $Y \subset \mathbb{P}^3$, then $\mu(C) = 4$, $\mathcal{I}_Y^2/\mathcal{I}_Y^3 \cong \mathcal{O}_Y(-2)^{\oplus 3}$ and $L_2 \cong \mathcal{O}_Y(-g - 1)$ (because $\text{Pic } Y \cong \mathbb{Z}$). It follows from sequence (2) above that giving such a curve C is the same as giving a surjective morphism $\mathcal{O}_Y(-2)^{\oplus 3} \rightarrow \mathcal{O}_Y(-g - 1)$ modulo an automorphism of $\mathcal{O}_Y(-g - 1)$. Thus the set of thick 4-lines of genus g is parametrized by an open subset of a \mathbb{P}^{5-3g} -bundle over the Hilbert scheme of lines in \mathbb{P}^3 . \square

On the other hand, a curve C is *quasiprimitive* if it is not thick, which is equivalent to having generic embedding dimension two. In this case $\text{rank } L_1 = 1$ and the generic surjections $L_1^j \rightarrow L_j$ yield effective divisors D_j such that $L_j \cong L_1^j(D_j)$; the multiplication maps show that $D_i + D_j \leq D_{i+j}$. If C is a quasiprimitive 4-line in \mathbb{P}^3 , then the Cohen-Macaulay filtration takes the form $Y \subset D \subset W \subset C$ where Y is a line, D a double line and W a triple line. Setting $a = \text{deg } L_1, b = \text{deg } D_2$ and $c = \text{deg } D_3$, the triple (a, b, c) is the *type* of C . Since the surjection $\mathcal{I}_Y \rightarrow \mathcal{O}_Y(a)$ factors through the conormal bundle $\mathcal{I}_Y/\mathcal{I}_Y^2 \cong \mathcal{O}_Y(-1)^2$, we necessarily have $a \geq -1$, while by construction $b, c \geq 0$. That C has arithmetic genus $p_a(C) = -6a - b - c - 3$ follows from the isomorphisms $\mathcal{I}_{Y,D} \cong \mathcal{O}_Y(a)$, $\mathcal{I}_{D,W} \cong \mathcal{O}_Y(2a + b)$ and $\mathcal{I}_{W,C} \cong \mathcal{O}_Y(3a + c)$.

The following lemma helps us construct 4-lines from quasiprimitive 3-lines.

LEMMA 2.2. *Let $W \subset \mathbb{P}^3$ be a quasiprimitive 3-line of type (a, b, c) with support Y and underlying double line D . Set $\mathcal{J} = \mathcal{I}_W\mathcal{I}_Y + \mathcal{I}_D^2$. Then in the short exact sequence*

$$0 \rightarrow \frac{\mathcal{I}_Y\mathcal{I}_D}{\mathcal{J}} \rightarrow \frac{\mathcal{I}_W}{\mathcal{J}} \rightarrow \frac{\mathcal{I}_W}{\mathcal{I}_Y\mathcal{I}_D} \rightarrow 0$$

we have isomorphisms

$$\frac{\mathcal{I}_Y\mathcal{I}_D}{\mathcal{J}} \cong \mathcal{O}_Y(3a + b) \quad \text{and} \quad \frac{\mathcal{I}_W}{\mathcal{I}_Y\mathcal{I}_D} \cong \mathcal{O}_Y(-a - b - 2).$$

In particular, when $a \geq 0$, the sequence is split exact.

Proof. On the open set where W has embedding dimension two, the quotient $\mathcal{I}_Y \mathcal{I}_D / \mathcal{J}$ is an invertible \mathcal{O}_Y -module. It follows that the multiplication map $\mathcal{I}_{Y,D} \otimes_{\mathcal{O}_Y} \mathcal{I}_{D,W} \xrightarrow{\mu} \mathcal{I}_Y \mathcal{I}_D / \mathcal{J}$ is an isomorphism (it is surjective a priori; if $\ker \mu \neq 0$, then the image has finite support) and $\mathcal{I}_Y \mathcal{I}_D / \mathcal{J} \cong \mathcal{O}_Y(3a + b)$. Further, $\mathcal{I}_D / \mathcal{I}_Y \mathcal{I}_D \cong \mathcal{O}_Y(2a) \oplus \mathcal{O}_Y(-a - 2)$ by [20, 2.3] and the exact sequence

$$0 \rightarrow \mathcal{I}_W / \mathcal{I}_Y \mathcal{I}_D \rightarrow \mathcal{I}_D / \mathcal{I}_Y \mathcal{I}_D \rightarrow \mathcal{O}_Y(2a + b) \rightarrow 0$$

shows that $\mathcal{I}_W / \mathcal{I}_Y \mathcal{I}_D \cong \mathcal{O}_Y(-a - b - 2)$. When $a \geq 0$, the sequence must split because

$$\text{Ext}_Y^1(\mathcal{O}_Y(-a - b - 2), \mathcal{O}_Y(3a + b)) \cong H^1(Y, \mathcal{O}_Y(4a + 2b + 2)) = 0. \quad \square$$

If a quasiprimitive 4-line C has type (a, b, c) with $a = -1$, then C necessarily lies in a double plane and, hence, is a flat limit of double conics [14, 8.1 and 8.2]. Since we are mainly interested in families that form irreducible components of the Hilbert scheme, we will assume that $a \geq 0$ in the sequel. The following proposition is based on [3, § 3.8].

PROPOSITION 2.3. *Let (a, b, c) be a triple of integers satisfying $a \geq 0$ and $0 \leq b \leq c$. Then the quasiprimitive 4-lines of type (a, b, c) in $H_{4,g}$ form an irreducible constructible subset of dimension $9a + 2b + 2c + 13$.*

Proof. The set of double lines of type a is parametrized by an open subscheme V_2 of a \mathbb{P}^{2a+3} -bundle over the Grassmannian of lines in \mathbb{P}^3 [20, 1.6]. Indeed, to give a double structure D of type a on a line Y is equivalent to give a surjective morphism $\mathcal{I}_Y / \mathcal{I}_Y^2 \cong \mathcal{O}_Y(-1) \xrightarrow{[g]} \mathcal{O}_Y(a)$ modulo an automorphism of $\mathcal{O}_Y(a)$.

Similarly, the quasiprimitive triple lines of type (a, b) which contain a double line $D \in V_2$ are determined by surjective morphisms $\mathcal{I}_D / \mathcal{I}_Y \mathcal{I}_D \cong \mathcal{O}_Y(2a) \oplus \mathcal{O}_Y(-a - 2) \xrightarrow{[p,q]} \mathcal{O}_Y(2a + b)$ modulo automorphisms of $\mathcal{O}_Y(2a + b)$, the isomorphism τ being given in [20, 2.3]. It follows that the set of triple lines of type (a, b) is parametrized by an open subscheme V_3 of a $\mathbb{P}^{3a+2b+3}$ -bundle over V_2 , hence is irreducible of dimension $5a + 2b + 10$.

In the same way, the 4-lines of type (a, b, c) which contain a fixed quasiprimitive 3-line W of type (a, b) with support Y are determined by surjections $\mathcal{I}_W / \mathcal{I}_Y \mathcal{I}_W \xrightarrow{\phi} \mathcal{O}_Y(3a + c)$. Since $\mathcal{I}_D^2 \subset \mathcal{I}_Y \mathcal{I}_W$ on an open set, the image of \mathcal{I}_D^2 in $\mathcal{I}_W / \mathcal{I}_Y \mathcal{I}_W$ is torsion so that ϕ factors through $\mathcal{I}_W / \mathcal{J}$, where $\mathcal{J} = \mathcal{I}_Y \mathcal{I}_W + \mathcal{I}_D^2$, allowing us to use Lemma 2.2. Thus the surjections $\phi: \mathcal{I}_W / \mathcal{J} \cong \mathcal{O}_Y(3a + b) \oplus \mathcal{O}_Y(-a - b - 2) \xrightarrow{[r,s]} \mathcal{O}_Y(3a + c)$ modulo automorphisms of $\mathcal{O}_Y(3a + c)$ are parametrized by an open subset V_4 of a $\mathbb{P}^{4a+2c+3}$ -bundle over V_3 , that is, by an irreducible variety of dimension $9a + 2b + 2c + 13$.

We now show that each surjection $\bar{\phi}$ gives a quasiprimitive 4-line. Since W is quasiprimitive, there are local coordinates s and t on an open set with $I_Y = (s, t)$, $I_D = (s, t^2)$, and $I_W = (s, t^3)$ [3, 2.1]. In particular, $(s, t)^3 = I_Y^3 \not\subset \mathcal{J} = (s^2, st, t^3)$ so that the image of \mathcal{I}_Y^3 is nonzero in $\mathcal{I}_Y \mathcal{I}_D / \mathcal{J} \cong \mathcal{O}_Y(3a + b)$ and for any surjection ϕ ,

the induced map $\mathcal{O}_Y(3a + b) \rightarrow \mathcal{O}_Y(3a + c)$ is injective, whence the image of \mathcal{I}_Y^3 is nonzero in $\mathcal{O}_Y(3a + c)$ and so $C \not\subset Y^{(3)}$: from this it follows that C is quasiprimitive via the Cohen-Macaulay filtration. \square

The following result is an important ingredient in our connectedness theorem 6.4.

PROPOSITION 2.4. *Let (a, b, c) be a triple with $a \geq 0$ and $c \geq b \geq 0$. Then there exists a flat family of curves $C \subset \mathbb{P}^3_{k[t]}$ such that*

- (1) *the fibre C_t is a quasiprimitive 4-line of type (a, b, c) for $t \neq 0$*
- (2) *the fibre C_0 is a thick 4-line.*

Proof. The outline of the proof is as follows. Fixing a double line Z of type (a) supported on the line L , we have seen above that a quasiprimitive triple line of type (a, b) containing Z is determined by a surjective morphism $\mathcal{I}_Z \xrightarrow{\psi} \mathcal{O}_L(2a + b)$. We construct a family of such surjections ψ_t for $t \neq 0$ whose limit at $t = 0$ is a morphism $\psi_0: \mathcal{I}_Z \rightarrow \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_D$ where D is an effective divisor on L . Then $\ker \psi_t$ defines a family W_t whose general fibre is a quasiprimitive 3-line of type (a, b) , while the special fibre W_0 is the first infinitesimal neighborhood of L plus some embedded points along D . We construct the desired family C_t by picking a morphism $\mathcal{I}_{W_t} \rightarrow \mathcal{O}_L(3a + c)$ in such a way that the ‘extra line’ in C_0 covers the embedded points of W_0 , so that C_0 is locally Cohen–Macaulay.

Fix coordinates so that $\mathbb{P}^3 = \text{Proj}(k[x, y, z, w])$ and let L_0 be the line with equations $x = y = 0$. Let Z_0 be the double structure on L_0 defined by the homogeneous ideal $(x^2, xy, y^2, xg - yf)$ where $f = z^{a+1}$ and $g = w^{a+1}$, thus Z_0 is a double line of genus $-a - 1$. Now consider the trivial families $L = L_0 \times \mathbb{A}^1$ and $Z = Z_0 \times \mathbb{A}^1$ over $\mathbb{A}^1 = \text{Spec } k[t]$ and let \mathcal{I}_Z denote the ideal sheaf of Z in $\mathbb{P}^3 \times \mathbb{A}^1$. Then we have an isomorphism $\mathcal{I}_Z \otimes \mathcal{O}_L \cong \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_L(2a)$ in which the natural epimorphism $\pi: \mathcal{I}_Z \rightarrow \mathcal{I}_Z \otimes \mathcal{O}_L \cong \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_L(2a)$ sends $xg - yf$ to $(1, 0)$ and x^2, xy and y^2 to $(0, f^2), (0, fg), (0, g^2)$ [20, 2.3].

Consider the injective morphism

$$\chi: \mathcal{O}_L(-a - b - 2) \xrightarrow{\begin{bmatrix} tw^b \\ z^{3a+b+2} \end{bmatrix}} \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_L(2a)$$

and set $\mathcal{G} = \text{coker}(\chi)$. Note that \mathcal{G} is flat over \mathbb{A}^1 because χ remains injective on the fibres over \mathbb{A}^1 . Now let $\psi: \mathcal{I}_Z \rightarrow \mathcal{G}$ be the composite surjection

$$\mathcal{I}_Z \xrightarrow{\pi} \mathcal{O}_L(-a - 2) \oplus \mathcal{O}_L(2a) \rightarrow \mathcal{G}$$

and define W by $\mathcal{I}_W = \ker(\psi)$.

It is clear that W is a flat family of closed subschemes over \mathbb{A}^1 . For $t \neq 0$, $\mathcal{G}_t = \mathcal{G} \otimes k(t)$ is isomorphic to $\mathcal{O}_{L_0}(2a + b)$ so that W_t is a quasiprimitive triple line of type (a, b) . On the other hand, \mathcal{G}_0 is isomorphic to $\mathcal{O}_{L_0}(-a - 2) \oplus \mathcal{O}_D$, where D is the divisor in L_0 defined by the global section $s = z^{3a+b+2}$ of $\mathcal{O}_{L_0}(3a + b + 2)$. By construction W_0 contains $L_0^{(2)}$ and $\mathcal{I}_{L_0^{(2)}, W_0} \cong \mathcal{O}_D$.

We now claim that, over the coordinate ring $R = k[t][x, y, z, w]$ of $\mathbb{P}^3_{k[t]}$, the saturated ideal I_W of $W \subset \mathbb{P}^3_{k[t]}$ has a free graded resolution

$$0 \rightarrow F_3 \xrightarrow{M_3} F_2 \xrightarrow{M_2} F_1 \xrightarrow{M_1} F_0 = R,$$

where

$$\begin{aligned} F_1 &= R(-3)^{\oplus 4} \oplus R(-a-3)^{\oplus 2} \oplus R(-a-b-2), \\ F_2 &= R(-4)^{\oplus 3} \oplus R(-a-4)^{\oplus 4} \oplus R(-a-b-3)^{\oplus 2}, \\ F_3 &= R(-a-5)^{\oplus 2} \oplus R(-a-b-2), \end{aligned}$$

and the maps are defined by the matrix transposes

$$M_1^t = \begin{bmatrix} x^3 \\ x^2y \\ xy^2 \\ y^3 \\ x(xg-yf) \\ y(xg-yf) \\ x^2z^{a+b} + tw^b(xg-yf) \end{bmatrix}, \quad M_3^t = \begin{bmatrix} g & 0 & z^{a+b} \\ -f & g & 0 \\ 0 & -f & 0 \\ y & 0 & 0 \\ -x & y & 0 \\ 0 & -x & 0 \\ 0 & y & -tw^b \\ 0 & 0 & -y \\ 0 & 0 & x \end{bmatrix},$$

$$M_2^t = \begin{bmatrix} y & -x & 0 & 0 & 0 & 0 & 0 \\ 0 & y & -x & 0 & 0 & 0 & 0 \\ 0 & 0 & y & -x & 0 & 0 & 0 \\ -g & f & 0 & 0 & x & 0 & 0 \\ 0 & -g & f & 0 & y & 0 & 0 \\ 0 & 0 & -g & f & 0 & y & 0 \\ 0 & 0 & 0 & 0 & -y & x & 0 \\ z^{a+b} & 0 & 0 & 0 & tw^b & 0 & -x \\ 0 & z^{a+b} & 0 & 0 & 0 & tw^b & -y \end{bmatrix}, \quad P^t = \begin{bmatrix} -tf^3z^{c-b}w^b \\ -tf^2gz^{c-b}w^b \\ -tfg^2z^{c-b}w^b \\ -tg^3z^{c-b}w^b \\ z^{4a+c+3} \\ z^{3a+c+2}w^{a+1} \\ w^{4a+b+c+2} \end{bmatrix}.$$

(In the second line we have included a matrix P which will be used a few lines below).

We use the Buchsbaum-Eisenbud criterion [5] to see that the complex above is acyclic. For this it is enough to observe that the ideal $I_3(M_3)$ generated by the 3×3 minors of M_3 contains the regular sequence (x^3, y^3, z^{3a+b+2}) and that $I_6(M_2)$ contains the regular sequence (x^6, y^6) . Therefore the complex is a resolution of the ideal J defined by M_1 , hence J is the saturated homogeneous ideal of a closed subscheme of $\mathbb{P}^3_{k[t]}$. It is clear that $J \subseteq I_W$, and comparing the Hilbert polynomials we see $J = I_W$. This proves the claim.

We now construct a family C of 4-lines as in the statement of the Proposition. Recalling that $f = w^{a+1}$ and $g = z^{a+1}$, one checks that the map $F_1 \xrightarrow{P} R(3a+c)$ satisfies $PM_2 = 0 \pmod{I_L}$, where $I_L = (x, y)$ is the ideal of L (here P is the matrix above). We thus obtain a map $I_W \xrightarrow{P} T(3a+c)$ where $T = R/I_L \cong A[z, w]$. Sheafifying, we obtain a morphism $\phi: \mathcal{I}_W \rightarrow \mathcal{O}_L(3a+c)$ and $\mathcal{I}_C = \ker \phi$ defines C . Since ϕ_t is

surjective for all $t \in \mathbb{A}^1$, we see from the proof of Proposition 2.3 above that C_t is a quasiprimitive 4-line of type (a, b, c) for $t \neq 0$.

Finally, we check that C_0 is a thick 4-line. To this end, observe that

$$\mathcal{I}_{L_0}^3 \subset \mathcal{I}_{W_0} \subset \mathcal{I}_{L_0}^2$$

and that ϕ_0 vanishes on $\mathcal{I}_{L_0}^3$, hence factors through $\mathcal{I}_{W_0}/\mathcal{I}_{L_0}^3$. On the other hand, looking at the presentation of $I_W \otimes T$, we see that $\mathcal{I}_{W_0}/\mathcal{I}_{L_0}^3$ is isomorphic to

$$\mathcal{O}_{L_0}(-a-3)^{\oplus 2} \oplus \mathcal{O}_{L_0}(-a-b-2).$$

Thus we have a commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{I}_{C_0}/\mathcal{I}_{L_0}^3 & \rightarrow & \mathcal{I}_{W_0}/\mathcal{I}_{L_0}^3 & \xrightarrow{\bar{\phi}_0} & \mathcal{O}_{L_0}(3a+c) & \rightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow z^{3a+b+2} & & \\ 0 & \rightarrow & \mathcal{I}_{\bar{C}}/\mathcal{I}_{L_0}^3 & \rightarrow & \mathcal{I}_{L_0}^2/\mathcal{I}_{L_0}^3 & \xrightarrow{\beta} & \mathcal{O}_{L_0}(6a+b+c+2) & \rightarrow & 0 \end{array}$$

where, identifying $\mathcal{I}_{W_0}/\mathcal{I}_{L_0}^3$ with $\mathcal{O}_{L_0}(-a-3)^{\oplus 2} \oplus \mathcal{O}_{L_0}(-a-b-2)$ and $\mathcal{I}_{L_0}^2/\mathcal{I}_{L_0}^3$ with $\mathcal{O}_{L_0}(-2)^{\oplus 3}$, the morphisms are

$$\alpha = \begin{bmatrix} w^{a+1} & 0 & z^{a+b} \\ -z^{a+1} & w^{a+1} & 0 \\ 0 & -z^{a+1} & 0 \end{bmatrix}, \quad \beta^t = \begin{bmatrix} z^{2a+2} w^{4a+b+c+2} \\ z^{a+1} w^{5a+b+c+3} - z^{6a+b+c+4} \\ w^{6a+b+c+4} - 2z^{5a+b+c+3} w^{a+1} \end{bmatrix},$$

$$\bar{\phi}_0 = [z^{4a+c+3} \quad z^{3a+c+2} w^{a+1} \quad w^{4a+b+c+2}].$$

Since α is injective and

$$\deg(\mathcal{I}_{L_0}^2/\mathcal{I}_{L_0}^3) - \deg(\mathcal{I}_{W_0}/\mathcal{I}_{L_0}^3) = 3a + b + 2,$$

we see that $C_0 = \bar{C}$ is a thick 4-line, and this concludes the proof. □

Remark 2.5. We don't know what kind of specializations might occur between quasiprimitive four-lines in general, but we formulate the following necessary condition: if a family of quasiprimitive four-lines of type (a, b, c) specializes to a four-line of type (a', b', c') , then $a' \leq a$. To see this, consider the deformation of the underlying double line Z : the general such Z has genus $-1 - a$, hence the limit Z' consists of a double line of genus $-1 - a'$ and possibly some embedded points. Since the arithmetic genus is constant, we conclude that $a' \leq a$. Similarly, if a family of triple lines of type (a, b) specializes to another of type (a', b') , then $a' \leq a$. Examples of both types of specialization are seen in [20, 3.6 and 3.10].

3. A Deformation on the Double Quadric

In this section we apply results from our general study of curves on double surfaces [24] to show that a disjoint union of double lines specializes to certain quasiprimitive 4-lines on a double quadric.

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface and let $X \subset \mathbb{P}^3$ be the effective divisor $2Q$. For a curve $C \subset X$, let P be the curve part of the scheme-theoretic intersection $C \cap Q$. We may write

$$\mathcal{I}_{C \cap Q, Q} = \mathcal{I}_{Z, Q}(-P)$$

where Z is zero-dimensional. The inclusion $P \subset C \cap Q$ generates the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathcal{I}_R(-Q) & \xrightarrow{f} & \mathcal{I}_C & \rightarrow & \mathcal{I}_{Z, Q}(-P) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathcal{O}_{\mathbb{P}^3}(-Q) & \xrightarrow{f} & \mathcal{I}_P & \rightarrow & \mathcal{O}_Q(-P) \rightarrow 0 \\
 & & \downarrow & & \downarrow \phi & & \downarrow \\
 0 & \rightarrow & \mathcal{O}_R(-Q) & \xrightarrow{\sigma} & \mathcal{L} & \rightarrow & \mathcal{O}_Z(-P) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{3}$$

in which f is the equation for $Q \subset \mathbb{P}^3$. This diagram defines the residual curve R to C in Q and we obtain a triple $T(C) = \{Z, R, P\}$ in which $R \subseteq P$ are effective divisors on Q . Using depth arguments and results on generalized divisors [9] as in [14], Z is a Gorenstein divisor on R , $\mathcal{L} \cong \mathcal{O}_R(Z - Q)$ is a rank one reflexive \mathcal{O}_R -module, and σ gives the section of $\mathcal{L}(Q)$ defining Z as a generalized divisor on R [24, § 2]. The arithmetic genus of C is given by the formula

$$g(C) = g(P) + g(R) + \deg_R \mathcal{O}_R(Q) - \deg(Z) - 1. \tag{4}$$

PROPOSITION 3.1. *Let $C \subset \mathbb{P}^3$ be a general quasiprimitive 4-line of type $(0, b, c)$ with Cohen-Macaulay filtration $L \subset D \subset W \subset C$. Then there exists a smooth quadric surface $Q \subset \mathbb{P}^3$ such that*

- (1) $D \subset Q$ and $C \subset 2Q$.
- (2) The triple associated to $C \subset 2Q$ has form $T(C) = \{Z, D, D\}$, where Z consists of $c - b$ simple points and $b + 2$ double points, none of which is contained in L .
- (3) $H^1(\mathcal{O}_D(Z + D - Q)) = 0$.

Proof. That D is contained in a smooth quadric Q is [20, 1.5], hence $\mathcal{I}_{2Q} \subset \mathcal{I}_D^2 \subset \mathcal{I}_C$ and $C \subset 2Q$. The definition of type of C gives rise to the top row in the diagram:

$$\begin{array}{ccccccc}
 0 & \rightarrow & \mathcal{I}_W & \rightarrow & \mathcal{I}_D & \xrightarrow{\psi} & \mathcal{O}_L(b) \rightarrow 0 \\
 & & & & \pi \downarrow & & \uparrow [p \ q] \\
 & & & & \mathcal{I}_D \otimes \mathcal{O}_L & \xrightarrow{\sim} & \mathcal{O}_L \oplus \mathcal{O}_L(-2)
 \end{array} \tag{5}$$

in which ψ factors through π and the isomorphism along the bottom row takes the equation f for Q to the generator of the summand $\mathcal{O}_L(-2)$ [20, 2.3]; thus

$\psi(f) = q \in H^0 \mathcal{O}_L(b+2)$ and in particular $q \neq 0$ since $W \not\subset Q$. Now $W \subset 2Q$ and its associated triple has the form $\{Z_W, L, D\}$ for a zero-dimensional subscheme $Z_W \subset L$. Interpreting diagram (3) for W , the middle column is simply the top row of diagram (5), $\mathcal{L} \cong \mathcal{O}_L(b)$, and the map σ is multiplication by q ; in particular, $\deg Z_W = b+2$.

The triple for C has form $\{Z, D, D\}$ ($Z \subset D$ a zero-dimensional subscheme) and the inclusion $W \subset C$ induces a morphism from diagram (3) for C to diagram (3) for W in which the middle rows coincide. Thus the maps along the bottom row are surjective and when we assemble these along with their kernels we obtain the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathcal{I}_{L,D}(-Q) & \xrightarrow{\alpha} & \mathcal{I}_{W,C} \cong \mathcal{O}_L(c) & \rightarrow & \mathcal{I}_{Z_W,Z}(-D) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathcal{O}_D(-Q) & \rightarrow & \mathcal{I}_{D,C} & \rightarrow & \mathcal{O}_Z(-D) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \mathcal{O}_L(-Q) & \rightarrow & \mathcal{I}_{D,W} \cong \mathcal{O}_L(b) & \rightarrow & \mathcal{O}_{Z_W}(-D) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{6}$$

We immediately deduce the vanishing in statement 3: indeed, the long exact cohomology sequence associated to the middle row shows that $H^1(\mathcal{I}_{D,C}) = 0$, but in view of the discussion following diagram (3) and the fact that $D^2 = 0$ on Q we have

$$\mathcal{I}_{D,C} \cong \mathcal{O}_D(Z - Q) \cong \mathcal{O}_D(Z + D - Q)$$

so that $H^1(\mathcal{O}_D(Z + D - Q)) = 0$.

Now we analyze the map α , which arises from the natural inclusion $\mathcal{I}_L(-2) \xrightarrow{f} \mathcal{I}_L \mathcal{I}_D \subset \mathcal{I}_W$ by taking quotients. Thus α factors as

$$\begin{array}{ccccccc}
 \mathcal{I}_L/\mathcal{I}_D(-2) & \xrightarrow{\beta} & \mathcal{I}_L \mathcal{I}_D/\mathcal{J} & \rightarrow & \mathcal{I}_W/\mathcal{J} & \rightarrow & \mathcal{I}_W/\mathcal{I}_C \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{O}_L(-2) & \xrightarrow{q} & \mathcal{O}_L(b) & \rightarrow & \mathcal{O}_L(b) \oplus \mathcal{O}_L(-b-2) & \xrightarrow{[r,s]} & \mathcal{O}_L(c)
 \end{array} \tag{7}$$

where $\mathcal{J} = \mathcal{I}_Y \mathcal{I}_W + \mathcal{I}_D^2$ and the vertical maps are isomorphisms via Lemma 2.2. We conclude that α is multiplication by the global section $qr \in H^0(L, \mathcal{O}_L(c+2))$ so that $\mathcal{I}_{Z_W,Z} \cong \mathcal{O}_L/(qr)$. For a general 4-line C of type $(0, b, c)$, the form qr has simple zeros.

Now twist diagram (6) by $\mathcal{O}_D(Q) = \mathcal{O}_D(-2)$ and apply $\mathfrak{H}\mathcal{O}_{\mathcal{M}_{\mathcal{O}_D}}(-, \mathcal{O}_D) = (-)^\vee$. As D is a local complete intersection (hence locally Gorenstein), \mathcal{O}_L is a reflexive \mathcal{O}_D -module because it has local depth one [9, 1.6]. In particular $\mathfrak{E}\mathcal{T}_{\mathcal{O}_D}^1(\mathcal{O}_L, \mathcal{O}_D) = 0$ and the columns remain exact. The leftmost column becomes

$$0 \rightarrow (\mathcal{O}_L)^\vee \rightarrow (\mathcal{O}_D)^\vee \rightarrow (\mathcal{O}_L)^\vee \rightarrow 0$$

(recall that $\mathcal{I}_{L,D} \cong \mathcal{O}_L$). It's easy to check on stalks that $(\mathcal{O}_L)^\vee$ is a locally free \mathcal{O}_L -module; the sequence above shows that the image of $1 \in H^0(\mathcal{O}_D)^\vee$ is a global

generator, hence $(\mathcal{O}_L)^\vee \cong \mathcal{O}_L$. Since the dual of the reflexive sheaf $\mathcal{I}_{D,C}(Q) \cong \mathcal{O}_D(Z)$ is $\mathcal{I}_{Z,D}$, we obtain

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \leftarrow & \mathcal{O}_L/(qr) & \leftarrow & \mathcal{O}_L & \xleftarrow{qr} & \mathcal{O}_L(-c-2) & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \leftarrow & \mathcal{O}_Z & \leftarrow & \mathcal{O}_D & \leftarrow & \mathcal{I}_{Z,D} & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \rightarrow & \mathcal{O}_L/(q) & \rightarrow & \mathcal{O}_L & \xleftarrow{q} & \mathcal{O}_L(-b-2) & \leftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
 \end{array} \tag{8}$$

This diagram shows that Z consists of $b + 2$ double points supported at the isolated zeros of q and $c - b$ reduced points. □

We end this section showing that a general quasiprimitive 4-line as in Proposition 3.1 is a specialization on $X = 2Q$ of a disjoint union of double lines having arithmetic genera $-1 - b$ and $-1 - c$, a fact we will later use in Theorems 6.2 and 6.4. The idea is to lift families of triples to families of curves on X . In [24] we construct the Hilbert scheme $H_{z,r,p}$ of curves $C \subset X$ having triples $\{Z, R, P\}$ such that the Hilbert polynomial of Z (resp. R and P) is z (resp. r and p). If $D_{z,r,p}$ is the Hilbert flag scheme for the corresponding triples, there is a natural map $\pi: H_{z,r,p} \rightarrow D_{z,r,p}$ assigning a curve to its triple. The structure of this map is partially described as follows:

THEOREM 3.2 ([24], Theorem 3.1). *Let $V \subset D_{z,r,p}$ be the open subset of triples $\{Z, R, P\}$ for which $H^1 \mathcal{O}_R(Z + P - F) = 0$. Then the induced map $\pi^{-1}(V) \rightarrow V$ is surjective and has the structure of an open immersion followed by an affine bundle projection.*

We apply this machinery to prove:

PROPOSITION 3.3. *Let C be a general quasiprimitive 4-line of type $(0, b, c)$ lying on the double quadric surface $2Q$. Then there is an irreducible flat family of curves on $2Q$ with special fibre C and general member a disjoint union of two double lines of respective arithmetic genera $-1 - b$ and $-1 - c$.*

Proof. As in Proposition 3.1, let $\{Z, D, D\}$ be the triple associated to C . Let p_1, p_2, \dots, p_{c-b} be the reduced points of Z , and choose planes H_j transverse to L , one containing each double point z_j of Z . Let $\{L_t: t \in \mathbb{P}^1\}$ be the family of rulings on Q with $L_0 = L$ and define

$$Z_t = \{p_1, \dots, p_{c-b}\} \cup \bigcup_{j=1}^{b+2} [H_j \cap (L_t \cup L_0)].$$

Setting $R_t = L_t \cup L_0$, we obtain a family of triples $\{Z_t, R_t, R_t\}$ specializing to $\{Z, D, D\}$. Moreover, for $t \neq 0$ we have the vanishing

$$H^1 \mathcal{O}_{R_t}(Z_t + D_t - Q) \cong H^1(\mathcal{O}_{L_0}(c + 2 - 2)) \oplus H^1(\mathcal{O}_{L_t}(b + 2 - 2)) = 0.$$

Thus we obtain a map $\phi: \mathbb{P}^1 \rightarrow D_{z,r,p}$ with image contained in the open set V of triples for which $H^1 \mathcal{O}_R(Z + P - Q) = 0$. Over V the natural map $\pi: H_{z,r,p} \rightarrow D_{z,r,p}$ is surjective and has the structure of an open immersion followed by an affine bundle projection by Theorem 3.2, hence the restriction $U = \pi^{-1}(\phi(\mathbb{P}^1)) \xrightarrow{\pi} \mathbb{P}^1$ has the same structure and is irreducible. We obtain the desired family W by pulling back the universal family over $H_{z,r,p}$ to $U: C \in U$ by construction, and W_u is a disjoint union of double lines of respective genera $-1 - b$ and $-1 - c$ whenever $\pi(u) \neq 0$ by Equation (4) (cf. [24, Example 2.8 (3)]). □

4. Curves of Degree Four with Large Speciality

In this section we study curves which have large speciality. We express the speciality of a curve C by its *spectrum* [28], which can be defined as the nonnegative function

$$h_C(n) = \Delta^2 h^0(\mathcal{O}_C(n)) = h^0(\mathcal{O}_C(n)) - 2h^0(\mathcal{O}_C(n - 1)) + h^0(\mathcal{O}_C(n - 2)).$$

We represent this by the t-tuple of integers with exponents $\{n^{h_C(n)}\}$. The extremal curves, which by definition achieve the upper bounds on the Rao function $h^1(\mathcal{I}_C(n))$ given in [17], also have the largest speciality and comprise an irreducible component $E \subset H_{4,g}$ of dimension $15 - 2g$ [18, Theorems 2.5 and 3.7]. For $d = 4$ and $g \leq 0$ they may be characterized as (a) nonplanar curves containing a plane cubic curve or (b) curves with spectrum $\{g\} \cup \{0, 1, 2\}$ [21, 2.2].

Similarly, there are sharp upper bounds on the Rao function for nonextremal curves: the curves achieving these bounds are called subextremal and have spectrum $\{g + 1, 0, 1^2\}$, although they are not characterized by this fact [21, 2.11, 2.14 and 2.15]. We characterize curves of degree four with subextremal speciality as follows:

LEMMA 4.1. *Let C be a curve of degree 4 and genus $g \leq -2$. Then C has spectrum $\{g + 1, 0, 1^2\}$ if and only if C contains a subcurve T of degree 3 and genus 0.*

Proof. If C contains such a curve $T \in H_{3,0}$, then the principal spectrum spectrum $\{0, 1^2\}$ of T is contained in that of C [28, § 3] and the remaining element $g + 1$ is determined by the genus of C .

On the other hand, suppose that C has spectrum $\{g + 1, 0, 1^2\}$. Then $h^0 \mathcal{O}_C(g + 1) = 1$ and choosing $0 \neq \alpha \in H^0 \mathcal{O}_C(g + 1)$ gives a map $\mathcal{O}_C \rightarrow \mathcal{O}_C(g + 1)$ with image \mathcal{O}_D for some closed subscheme $D \subset C$. The local depth of D is one because $\mathcal{O}_D \subset \mathcal{O}_C(g + 1)$, hence D is a locally Cohen–Macaulay curve. The inclusion above also shows that $h^0 \mathcal{O}_D(1) \leq 2$: for $g < -2$ this is because $h^0 \mathcal{O}_C(g + 2) = 2$. If $g = -2$, then $h^0 \mathcal{O}_C = 3$ and the inclusion $H^0(\mathcal{O}_D(1)) \subset H^0(\mathcal{O}_C)$ is strict, as otherwise we obtain

a surjection $\mathcal{O}_D(1) \rightarrow \mathcal{O}_C$, which is absurd. It follows that $D = L$ is a line. This yields an exact sequence

$$0 \rightarrow \mathcal{O}_L(-g - 1) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_T \rightarrow 0 \tag{9}$$

for a closed subscheme $T \subset C$ of degree 3 and genus 0.

If T is not purely one-dimensional, then the purely one-dimensional part $P \subset T$ is planar because $g(P) > 0$ and $\deg P = 3$ [10, 3.1], but this is not possible because C is not extremal. Thus T is locally Cohen–Macaulay, finishing the proof. \square

PROPOSITION 4.2. *Let C be a curve of degree 4 and genus $g \leq -1$ having spectrum $\{g + 1, 0, 1^2\}$. Then there is a line L such that the Rao module $M_C = H_*^1(\mathcal{I}_C)$ is a graded module over the coordinate ring $S = H_*^0(\mathcal{O}_L)$ with resolution*

$$0 \rightarrow S(-j) \oplus S(j - 5 + g) \xrightarrow{\sigma} S(-2)^{\oplus 3} \rightarrow S(-g - 1) \rightarrow M_C \rightarrow 0 \tag{10}$$

for some integer $2 \leq j \leq n(g) = \lfloor \frac{5-g}{2} \rfloor$. The cohomology of C is determined by the integer j and we denote the corresponding family of curves by H_j .

Proof. First we suppose that $g \leq -2$; then C contains a curve T of degree 3 and genus 0 by Lemma 4.1. In particular, T is arithmetically Cohen–Macaulay and the total ideal I_T is generated by three quadrics [10, 3.5]. Moreover, the proof of Lemma 4.1 produces a line L such that $\mathcal{I}_{T,C} \cong \mathcal{O}_L(-g - 1)$. Thus we obtain the composite surjection $\mathcal{O}(-2)^{\oplus 3} \rightarrow \mathcal{I}_T \rightarrow \mathcal{O}_L(-g - 1)$: factoring this map through $\mathcal{O}_L(-2)^{\oplus 3}$ and writing the kernel of the induced map as $\mathcal{O}_L(-j) \oplus \mathcal{O}_L(j - 5 + g)$ for some integer $j \in [2, \lfloor (5 - g)/2 \rfloor]$, we obtain resolution (10).

Now assume that $g = -1$. The spectrum shows that C is neither ACM (because $h_C(0) > 1$) nor extremal, hence [21, Theorem 2.11] gives the bounds $h^1\mathcal{I}_C \leq 1$, $h^1\mathcal{I}_C(1) \leq 2$ and $h^1\mathcal{I}_C(2) \leq 1$. The first two of these are equalities in view of the Euler characteristics (since $h^3\mathcal{I}_C = h^3\mathcal{I}_C(1) = 0$) and the vanishing $h^2\mathcal{I}_C(1) = 0$, which may be read from the spectrum. If $h^1\mathcal{I}_C(2) = 1$, then C is subextremal by definition and has (a degenerate form of) resolution (10) with $j = 2$.

If $h^1\mathcal{I}_C(2) = 0$, then \mathcal{I}_C is 3-regular (hence $h^1\mathcal{I}_C(n) = 0$ for $n \geq 2$) and $h^0\mathcal{I}_C(2) = h^2\mathcal{I}_C(2) = 0$. The condition $h^1\mathcal{I}_C = 1$ implies that $h^1\mathcal{I}_C(n) = 0$ for $n < 0$ [19, 2.8], hence the Rao module M_C has a generator m in degree 0. If m does not generate M_C as a module over the homogeneous coordinate ring $S = k[x, y, z, w]$ of \mathbb{P}^3 , then m is annihilated by 3 independent linear forms, which implies that C lies on a quadric by [27, 3.4.5], a contradiction. Thus m generates the Rao module and $M_C \cong S/(x, y, z^2, zw, w^2)$ after a change of coordinates, so that M_C has resolution (10) for $I_L = (x, y)$ with $j = 3$. \square

Finally, we describe the families H_j and how they fit together in the Hilbert scheme. We let $G_4 \subset H_{4,g}$ denote the closed family of thick 4-lines. By Proposition 2.1 G_4 is irreducible of dimension $9 - 3g$.

PROPOSITION 4.3. *For fixed $g \leq -1$, let H_j be the family of curves defined in Proposition 4.2. Then*

- (1) *The family H_2 is irreducible and meets G_4 . It consists of subextremal curves and has dimension $13 - 2g$ (resp. 16 if $g = -1$).*
- (2) *The family H_3 is irreducible and meets G_4 . It has dimension $13 - 2g$ (resp. 16 if $g = -1$). If $g = -1$ or -2 , then $G_4 \subset \overline{H_3}$.*
- (3) *Suppose $g \leq -3$. The families H_j are contained in G_4 for $3 < j \leq n(g)$ and $G_4 = \overline{H_{n(g)}}$.*

Proof. We consider the last statement first: suppose that $C \in H_j$ for $3 < j \leq n(g)$. Sequence (10) shows that $h^1 \mathcal{I}_C(3) = -g - 3$, hence $h^0 \mathcal{I}_C(3) = 4$. If L and T are as in the proof of Proposition 4.1 above, then $\dim(I_L I_T)_3 \leq 4$ because $I_L I_T \subset I_C$ and so $I_T = I_L^2$ by Lemma 4.5 below. It follows that C is a thick 4-line supported on L , hence $H_j \subset G_4$. As we saw in Proposition 2.1, the thick 4-lines supported on L are parametrized by the open subset

$$U \subset \text{Hom}_{\mathcal{O}}(\mathcal{I}_T, \mathcal{O}_L(-g-1)) \cong \text{Hom}_{\mathcal{O}_L}(\mathcal{O}_L(-2)^3, \mathcal{O}_L(-g-1))$$

corresponding to surjective maps. For $j = 2$ and 3, the specific surjections given by $(w^{1-g}, w^{3-g-j} z^{j-2}, z^{1-g})$ show that G_4 meets H_j . Since I_C is determined by its image in $I_T/I_L I_T \cong S_L(-2)^3$ (and hence by the image of the first map in sequence (10) above), we find by counting dimensions that H_j is irreducible of dimension $5 + 2j - 2g$ (except if $2j = 5 - g$, when the dimension is $4 + 2j - 2g$). For $j = n(g) = \lfloor \frac{5-g}{2} \rfloor$, the closure of $H_{n(g)}$ is irreducible of dimension $9 - 3g$, hence is equal to G_4 .

If $j = 2$, then H_j consists of subextremal curves: indeed, for $C \in H_2$, resolution (10) gives Rao function $\rho_C(n) = \binom{n-g}{1} - 2\binom{n-1}{1} + \binom{n+g-2}{1}$, which agrees with the subextremal bound in [21, 2.11]. In particular, these curves are obtained from extremal curves of degree 2 and genus $g' = g - 1$ by a height one biliaison on a quadric surface [21, 2.14]. Let then γ, ρ (resp. γ', ρ') be the gamma and Rao functions for the extremal curves of degree 2 and genus $g - 1$ (resp. subextremal curves of degree 4 and genus g). Letting $\mathcal{B}_{\gamma, \rho, 2, 1}$ denote the universal biliaison scheme of Martin-Deschamps and Perrin [16, VII § 4], we have smooth irreducible projections

$$\begin{array}{ccc} \mathcal{B}_{\gamma, \rho, 2, 1} & \xrightarrow{q_2} & H_{\gamma', \rho'} \\ q_1 \downarrow & & \\ E = H_{\gamma, \rho} & & \end{array}$$

to the spaces $H_{\gamma, \rho}$ (resp. $H_{\gamma', \rho'}$) of curves with constant cohomology. The family $E = H_{\gamma', \rho'}$ of extremal curves is irreducible of dimension $7 - 2g$ [18, 2.5] and using [16, VII, 4.8] we compute that the fibre dimension of q_1 is 8 (resp. 9 if $g = -1$) and the fibre dimension of q_2 is 2, hence the family $H_{\gamma, \rho}$ of subextremal curves is irreducible of dimension $13 - 2g$ (resp. 16 if $g = -1$).

For $j = 3$, we take an indirect approach. Consider the family of arithmetically Cohen-Macaulay curves D with resolution of the form

$$0 \rightarrow \mathcal{O}(2g - 1) \oplus \mathcal{O}(g - 2) \rightarrow \mathcal{O}(g - 1)^3 \rightarrow \mathcal{I}_D \rightarrow 0.$$

It is irreducible of dimension $2\binom{-g+3}{3} + \binom{-g+2}{2} + 2$ (resp. 12 if $g = -1$) by [6] and the general member is smooth and irreducible (the numerical character has no gaps).

Let D be a general such curve in this family. Then a general (disjoint) union $D \cup L$ with a line L lies on an integral surface of degree $-g + 2$; To see this, we consider the map

$$\tau: \mathbb{P}^3 - D - L \xrightarrow{\Delta} \mathbb{P}^3 - D \times \mathbb{P}^3 - L \rightarrow \mathbb{P}^2 \times \mathbb{P}^1$$

given by the linear systems $\mathbb{P}H^0\mathcal{I}_D(-g + 1)$ and $\mathbb{P}H^0\mathcal{I}_L(1)$. The fibres of τ are the intersections $\{S \cap H: S \in H^0\mathcal{I}_D(-g + 1), H \in H^0\mathcal{I}_L(1)\}$, which are generally of dimension one, and hence the image of τ has dimension two. Composing with the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$, we apply Jouanolou’s Bertini theorem [15, 6.10] to see that the general surface of degree $-g + 2$ containing $D \cup L$ is irreducible. Furthermore, the resolution for \mathcal{I}_D shows that $H^1\mathcal{O}_{D \cup L}((-g + 2) - 4 - (g + 1)) \neq 0$ and we find that [16, III, 2.7(b)] $D \cup L$ can be bilinked on a surface of degree $-g + 2$ with height $g + 1$ to a curve C , which lies in H_3 by direct calculation.

Let γ, ρ (resp. γ', ρ') be the gamma and Rao functions for curves in H_3 (resp. $D \cup L$). Letting $\mathcal{B}_{\gamma, \rho, -g+2, -g-1}$ be the universal biliaison scheme [16, VII Section 4], we obtain smooth irreducible projections

$$\begin{array}{ccc} \mathcal{B}_{\gamma, \rho, -g+2, -g-1} & \xrightarrow{q_2} & H_{\gamma', \rho'} \\ q_1 \downarrow & & \\ H_3 = H_{\gamma, \rho} & & \end{array}$$

From the last paragraph, the image of q_2 is dense in the irreducible component consisting of the closure of the family of disjoint unions $D \cup L$ considered above. Using the resolutions given, we compute the dimension of the fibres of q_1 and q_2 via [16, VII 4.8] and conclude that H_3 is irreducible of dimension $13 - 2g$ (resp. 16 if $g = -1$). □

Remark 4.4. One can check by a dimension count that the general members of the families H_2 and H_3 are described as follows.

1. For $g = -1$, the general member of H_2 is a disjoint union of conics. For $g \leq -2$, the general member of H_3 is the union of a double line Z of genus $g - 2$ and two disjoint lines L_1 and L_2 , each meeting Z in a scheme of length 2.
2. For $g = -1$, the general member of H_3 is a disjoint union of a line and a twisted cubic curve. For $g \leq -2$, the general member of H_3 is the union of a double line of genus $g - 1$ and a smooth conic meeting in a scheme of length 2.

The following lemma and its proof are well known:

LEMMA 4.5. *Let $L \subset \mathbb{P}^N$ be a linear subvariety of codimension two and let $V \subset H^0(\mathbb{P}^N, \mathcal{O}(d))$ be a subspace of dimension r generating the homogeneous ideal I . Then the image W under the multiplication map $V \otimes H^0(\mathbb{P}^N, \mathcal{I}_L(1)) \rightarrow H^0(\mathbb{P}^N, \mathcal{O}(d+1))$ satisfies $\dim(W) \geq r+1$ with equality if and only if $I = I_L^{-1}f$ for some form f of degree $d-r+1$.*

Proof. Let $S = \text{Sym } H^0(\mathbb{P}^N, \mathcal{I}_L(1)) \cong k[x, y]$ be the symmetric algebra and set $\mathbb{P}^1 = \text{Proj}(S)$. Let \mathcal{F} denote the image of the sheafification of the natural map $V \otimes_k S \rightarrow \bigoplus_n H^0(\mathbb{P}^N, \mathcal{O}(n))$ of free graded S -modules. This yields an exact sequence $0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{F} \rightarrow 0$ of locally free $\mathcal{O}_{\mathbb{P}^1}$ -modules. Since $h^0(\mathcal{E}) = 0$ by hypothesis and we have $\text{rank}(\mathcal{E}) \leq r-1$, so

$$\mathcal{E} \cong \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}^1}(-a_i)$$

with $s \leq r-1$ and $a_i > 0$, hence $h^0(\mathcal{E}(1)) \leq r-1$. Since $H^0(\mathcal{E}(1))$ is the kernel of the surjection $V \otimes S_1 \rightarrow W$, we see that $\dim(W) \geq r+1$ with equality if and only if $s = r-1$ and $a_i = 1$ for $1 \leq i \leq r-1$, which is equivalent to saying that $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}(r-1)$. □

5. Triple Lines Union a Line

In this section we are interested in unions $C = W \cup L$ of a quasi-primitive triple line W of type (a, b) and a reduced line L , where $W \cap L$ is nonempty. The arithmetic genus of C is given by

$$g(C) = -3a - 3 - b + \text{length}(W \cap L).$$

Fix g and $a \geq 0$ and let $Z \subset W$ denote the underlying double line of genus $-1 - a$. Then we have four families of such curves C in $H_{4,g}$ as shown in Table I.

The main results of this section are that families F_1 and F_3 are irreducible (Proposition 5.2) and that the other two families lie in their closures (Proposition 5.4).

Our arguments hinge on the following construction: fix a double line $Z \subset \mathbb{P}^3$ of type $a \geq 0$ with support Y and let $b \geq 0$ be an integer. Let $h \in H^0 \mathcal{I}_Z(a+b+2)$ be the equation of a surface S which does not contain the first infinitesimal neighborhood $Y^{(2)}$ of Y . Removing the embedded points from the scheme cut out by

Table I. Families of unions $W \cup L$, W a triple line of type (a, b)

Label	Length ($W \cap L$)	Length ($Z \cap L$)	Relation
F_1	3	2	$b = -3a - g$
F_2	2	2	$b = -3a - g - 1$
F_3	2	1	$b = -3a - g - 1$
F_4	1	1	$b = -3a - g - 2$

$I = (I_Y I_Z, h)$ yields a locally Cohen–Macaulay curve W supported on Y . Since $Y^{(2)} \not\subset S$, the surface S is smooth at a general point p of Y , so as in Section 2 we may find local equations $s = h_p$ and t for Y at p such that $\mathcal{I}_{D,p} = (s, t^2)$, and $\mathcal{I}_{W,p} = (s, t^3)$. It follows that W is quasiprimitive of degree three. Thus we obtain the map

$$\Phi: \{h \in H^0(\mathcal{I}_Z(a + b + 2)) : h \notin I_Y^2\} \rightarrow \{\text{Quasiprimitive 3-lines } W \supset Z\}.$$

LEMMA 5.1. *Let Φ be the map above. Then*

- (1) *The image of Φ is the set of quasiprimitive 3-lines $W \supset Z$ of type (a, b') with $b' \leq b$.*
- (2) *For h as above, set $I = (I_Y I_Z, h)$. Then the following are equivalent.*
 - (a) *$W = \Phi(h)$ has type (a, b) .*
 - (b) *The scheme defined by I is locally Cohen–Macaulay.*
 - (c) *h is irreducible modulo $I_Y I_Z$.*

If any of these conditions hold, then $I = I_W$ is saturated.

Proof. For $h \in \{(I_Z)_{a+b+2} : h \notin I_Y^2\}$, let W be the purely one-dimensional part of the scheme V defined by the ideal $I = (I_Y I_Z, h)$. Since W is quasi-primitive and contains Z , W has type (a, b') for some $b' \geq a$ and the total ideal may be written $I_W = (I_Y I_Z, h')$ with h' of degree $a + b' + 2$ by [20, 2.3]. The inclusions $(I_Y I_Z, h) \subset I_V \subset I_W$ now show that $b' \leq b$. On the other hand, if W is a quasi-primitive 3-line of type (a, b') with $b' \leq b$, then writing $I_W = (I_Y I_Z, h')$ as above and choosing a hypersurface F of degree $b - b'$ with equation f meeting Z properly, we see that $\Phi(fh') = W$.

For the equivalences in statement 2, let C be the scheme defined by $I = (I_Y I_Z, h)$ so that $W = \Phi(h)$ is obtained from C by removing possible embedded points.

(a) \Rightarrow (b) If W has type (a, b) , then by [20, 2.3] the total ideal for W takes the form $I_W = (I_Y I_Z, h')$ with $\deg h' = a + b + 2 = \deg h$. The inclusions $I \subset I_C \subset I_W$ show that all three ideals are equal, so $C = W$ is locally Cohen–Macaulay.

(b) \Rightarrow (c) Suppose that $h = h't$ modulo $I_Y I_Z$. If T is the surface with equation t , then we may assume T meets Y properly (since if both $t \in I_Y$ and $h' \in I_Y$, then $h \in I_Y^2$, contrary to hypothesis). In this case $(I_Y I_Z, t)$ defines a scheme of length $4 \deg T$ (because $I_Y I_Z$ defines a locally Cohen–Macaulay 4-line) and if C is locally Cohen–Macaulay, then (I, t) defines a scheme of length $3 \deg T$; since $(I_Y I_Z, t) = (I, t)$, we must conclude that C is not locally Cohen–Macaulay.

(c) \Rightarrow (a) If W has type (a, b') for $b' < b$, then $I_W = (I_Y I_Z, h')$ with $\deg h' = a + b' + 2$. The inclusion $I \subset I_W$ shows that there exists t of degree $b - b'$ such that $h = h't$ modulo $I_Y I_Z$. □

PROPOSITION 5.2. *With the notation of Table I, we have*

- (1) *The family $F_1 \subset H_{4,g}$ is irreducible of dimension $11 - 2g - a$ if $0 \leq a \leq -\frac{g}{3}$ and empty if $a > -(g/3)$.*

(2) *The family $F_3 \subset H_{4,g}$ is irreducible of dimension $10 - 2g - a$ if $0 \leq a < (-g - 1)/3$ and empty if $a \geq (-g - 1)/3$.*

Proof. We first prove statement 1, then indicate the changes to obtain statement 2. Let $H \subset H_{3,-a}$ be the family of unions $Z \cup_{2P} L$. By [20, 3.2(a)], H is irreducible of dimension $9 + 2a$. For $b = -3a - g$ we interpret $H^0 \mathcal{O}_{\mathbb{P}^3}(a + b + 2)$ as an affine scheme. Pulling back the universal family over H we obtain a diagram

$$\begin{array}{ccc} \mathcal{Z} \cup \mathcal{L} \subset \mathbb{P}^3 \times H \times H^0(\mathcal{O}_{\mathbb{P}^3}(a + b + 2)) & & \\ \searrow & & \downarrow \\ H \times H^0(\mathcal{O}_{\mathbb{P}^3}(a + b + 2)) & & \end{array}$$

Consider the closed subset

$$V = \{(Z \cup L, h) \in H \times H^0(\mathcal{O}_{\mathbb{P}^3}(a + b + 2)) : h \in I_Z \cap (I_Y^3, I_L)\} \tag{11}$$

with first projection $V \xrightarrow{p_1} H$. The fibres of p_1 are vector subspaces of dimension $\binom{a+b+5}{3} - 3a - 2b - 7$. Indeed, after a change of coordinates we may write $I_Z = (x^2, xy, y^2, xg - yf)$ ([20, 1.4(c)]) and $I_L = (x, z)$, when the fibre is identified with the kernel K of the composite map

$$H^0(\mathcal{I}_Z(a + b + 2)) \hookrightarrow (x, z, y^2)_{a+b+2} \rightarrow ((x, z, y^2)/(x, z, y^3))_{a+b+2} \tag{12}$$

(the inclusion has the correct target because Z meets L in the double point $2P$). Since \mathcal{I}_Z is $(a + 2)$ -regular,

$$h^0(\mathcal{I}_Z(a + b + 2)) = \chi \mathcal{I}_Z(a + b + 2) = \binom{a + b + 5}{3} - 3a - 2b - 6$$

and $p_1: V \rightarrow H$ is an affine bundle with fibres of dimension $\binom{a+b+5}{3} - 3a - 2b - 7$. In particular, V is irreducible.

Consider the open subset $U = \{(Z \cup L, h) \in V : h \notin I_Y^3\}$. The correspondence of Lemma 5.1 shows that elements of U determine unions $W \cup L$ in which W is a quasiprimitive 3-line. If $\mathcal{S} \subset \mathbb{P}^3 \times U$ is the family of surfaces with equation h , we obtain flat families

$$\begin{array}{ccc} \mathcal{Y}, \mathcal{Z} \cup \mathcal{L}, \mathcal{S} \subset \mathbb{P}^3 \times U & & \\ \searrow & & \downarrow \\ & & U \end{array}$$

where \mathcal{Y} is the support of \mathcal{Z} . The ideal sheaf $\mathcal{I}_F = \mathcal{I}_Y \mathcal{I}_Z$ defines a flat subscheme $F \subset \mathbb{P}^3 \times U$ whose fibres are thick 4-lines of genus $-2a - 1$ (easy calculation on the fibres). The subscheme $\mathcal{W} \subset \mathbb{P}^3 \times U$ defined by the ideal sheaf $\mathcal{I}_W = \mathcal{I}_Y \mathcal{I}_Z + \mathcal{I}_S$ is also flat over U . To see this, observe that the relative ideal sheaf $\mathcal{I}_{W,F} = \mathcal{I}_W / \mathcal{I}_F$ is an \mathcal{O}_Y -module, whose fibre over the point $u = (Z \cup L, h)$ is generated by the equation h . This gives a right exact sequence

$$\mathcal{O}_{Y_u}(-a - b - 2) \xrightarrow{h} \mathcal{O}_{F_u} \rightarrow \mathcal{O}_{W_u} \rightarrow 0.$$

From the choice of V and U (and the construction of the map Ψ) we know that the fibre W_u is a scheme of degree 3. In considering the *degree* of the Hilbert

polynomials, we see that the map h is injective and that \mathcal{W}_u has Hilbert polynomial $p(z) = 3z + 3a + b + 3$ independent of u , so \mathcal{W} is flat [8, III, 9.9].

Finally, let $U' \subset U$ be the open set for which the fibres of \mathcal{W} are locally Cohen-Macaulay, taken with the induced reduced scheme structure. This is precisely the set for which the fibres of \mathcal{W} are quasiprimitive 3-lines of type (a, b) by Lemma 5.1. The curves used in the proof of Corollary 5.5 show that U' is non-empty; the double line Z with total ideal $I_Z = (x^2, xy, y^2, xz^{a+1} - yw^{a+1})$ has the line $L = \{x = w = 0\}$ as a double tangent and $h = z^b(xz^{a+1} - yw^{a+1}) - x^2w^{a+b} = 0$ satisfies the conditions above. The definition of V above makes it clear that the fibres of \mathcal{W} meet the lines L in triple points, so the family $\mathcal{W} \cup \mathcal{L}$ is also flat over U' . The universal property of the Hilbert scheme gives a map $U' \rightarrow H_{4,-3a-b}$ whose image is precisely the family F_1 of unions $W \cup_{3P} L$. In particular, F_1 is irreducible.

The structure of the map $U' \rightarrow H$ shows that U' has dimension $\binom{a+b+5}{3} - a - 2b + 2$. On the other hand, if W is a triple line arising in the construction above, then \mathcal{I}_W is $(a + b + 2)$ -regular [20, 2.4], hence $\dim H^0(\mathcal{I}_W(a + b + 2)) = \binom{a+b+5}{3} - 6a - 4b - 9$. Subtracting this redundancy shows that the family has dimension $5a + 2b + 11 = 11 - 2g - a$.

The proof of statement 2 goes through via the same outline. The main differences are as follows. The family $H \subset H_{3,-a-1}$ is now the family of unions $Z \cup_P L$, which is irreducible of dimension $10 + 2a$ by [20, 3.2(b)]. In definition (11) of V , I_Y^3 is replaced by I_Y^2 and in map (12) the exponents of y should be reduced by one. To see that U' is nonempty, we can use the same triple line W as in the proof above, but instead use the line L given by $\{x = z = 0\}$. The remaining modifications are clear. \square

Remark 5.3. Since the family of triple lines W is irreducible of dimension $10 + 5a + 2b$ [20, 2.6], we expect that the natural map $\{W \cup_{3P} L\} \xrightarrow{F} \{W\}$ which forgets the line L has generically one dimensional fibres. However, there are triple lines W for which the fibre $F^{-1}(W)$ has larger dimension. For example, the triple lines constructed in characteristic $p > 0$ by Hartshorne [10, 2.3] have a two-dimensional family of triple tangent lines.

PROPOSITION 5.4. *With the notation above, we have*

- (1) $F_2 \subset \bar{F}_1$ if $0 \leq a < \frac{-g-1}{3}$ and is otherwise empty.
- (2) $F_4 \subset \bar{F}_3$ if $0 \leq a < \frac{-g-1}{3}$ and is otherwise empty.

Proof. Let $W_0 \cup_{2P} L$ be a curve in the family F_2 , so that the underlying double line $Z \subset W_0$ satisfies $\text{length}(Z \cap L) = 2$. If W_0 has type $(a, -3a - g - 1)$ and support Y ; we may write $I_Y = (x, y)$, $I_Z = ((x, y)^2, xg - yf)$, $I_L = (x, z)$ and $I_{W_0} = (I_Y I_Z, h_0)$ in suitable coordinates [20, 2.3]. If $K \subset H^0(\mathcal{I}_Z(-2a - g + 2))$ is the vector subspace considered in the proof above, then $zh_0 \in K$. Fixing a member $(Z \cup L, h) \in U'$ as above, the deformation $(1 - t)zh_0 + th$ gives a map $\mathbb{A}^1 \xrightarrow{\psi} K$, which yields $\psi^{-1}(U') \rightarrow H_{4,g}$. This extends to a map $\bar{\psi}: T = \psi^{-1}(U') \cup \{0\} \rightarrow \text{Hilb}_4^g$ into the full

Hilbert scheme: by construction, it's clear that the limit curve $\bar{\psi}(0)$ contains $W_0 \cup L$. Since this curve has genus g , it is equal to $\bar{\psi}(0)$, completing the proof. The limit of the triple lines W_t is the triple line W_0 along with an embedded point, which is conveniently covered up by the line L . Statement 2 is similar. \square

COROLLARY 5.5. *The closure of family $F_{1,a}$ in $H_{4,g}$ contains extremal curves.*

Proof. Following [20, 3.6], the family of ideals I_t below give a deformation from a triple line W of type (a, b) to an extremal curve of the same arithmetic genus.

$$I_t = ((x, y)^3, (x, y)(xz^{a+1} - tyw^{a+1}), z^b t^2 (xz^{a+1} - tyw^{a+1}) - x^2 w^{a+b}).$$

We simply observe that the line $L = \{x = w = 0\}$ is triple tangent to the triple line W_t defined by I_t for all $t \neq 0$ and that this same line is a triple tangent to the limit extremal curve having ideal

$$I_0 = (x^2, xy, y^3, xz^{3a+b+3} - y^2 w^{3a+b+2}). \quad \square$$

Remark 5.6. The closure of the family F_3 above forms an irreducible component of the Hilbert scheme (Theorem 6.2) with one exception: We will show that the curves $C_0 = W_0 \cup_{2P} L \in F_3$ with W_0 of type $(0, b)$ are flat limits of curves $C_t = L \cup_{2P} Z_t \cup L_t$, where Z_t is a double line and L_t is a line disjoint from L (compare with [20, 3.3]).

Recall that the underlying double line $Z_0 \subset W_0$ lies on a smooth quadric with equation $q = 0$ [20, 1.5] and that the ideal of W_0 may be written $I_{W_0} = ((x, y)^3, xq, yq, h = pq - ax^2 - bxy - cy^2)$ [20, 2.6]. The double point $2P = W_0 \cap L$ is not contained in Z_0 since $C_0 \in F_3$.

Letting L_t be the ruling $\{x + wt = y + zt = 0\}$ of Q for $t \in \mathbb{A}^1$, the flat family $D_t = L_t \cup L_0$ has limit $D_0 = Z_0$ (linear equivalence on Q) and the flat family of subschemes $E_t = D_t \cup 2P$ has limit $Z_0 \cup 2P$. Since $\dim H^0 \mathcal{I}_{E_t}(-g + 1)$ is constant, $\pi_* (\mathcal{I}_E(-g + 1))$ is locally (and hence globally) free on \mathbb{A}^1 by Grauert's theorem [8, III, 12.9], where $E \subset \mathbb{P}^3 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the total family. Now E_1 is contained in a smooth surface of degree $-g + 1$ (in fact, $E_1 \subset D_1 \cup L$ and $D_1 \cup L$ is contained in a smooth surface of degree $-g + 1$ by [22, 2.6(a)]) with equation s_1 . Thus we can find a section s_t extending s_1 with $s_0 = h$ and we let S_t be the corresponding family of surfaces.

Now consider the family $C_t = L \cup (S_t \cap L_0^{(2)}) \cup L_t$. For $t \in U$, $U \subset \mathbb{A}^1$ a nonempty open subset, C_t is the disjoint union of L_t and $Z_t \cup_{2P} L$, where Z_t is a double line of genus g . Since $I_{Z_t} = ((x, y)^2, s_t)$ and $L_t \subset Q \cap S_t$, we obtain the containment

$$((x + wt, y + zt)(x, y)^2, (x, y)q, s_t) \subset I_{Z_t} \cap I_{L_t}$$

and conclude that $\lim_{t \rightarrow 0} Z_t \cup L_t \subset W_0$ in taking the limit as $t \rightarrow 0$. It follows that $\lim_{t \rightarrow 0} C_t \subset W_0 \cup L$ in the full Hilbert scheme. Since the Hilbert polynomials agree, $W_0 = \lim C_t$.

6. The Hilbert Schemes $H_{4,g}$

In this section we prove the main results of the paper. The first of these is Theorem 6.2 that describes the irreducible components of the Hilbert schemes $H_{4,g}$. The second is the fact 6.4 that $H_{4,g}$ is connected. The cases when $g \geq 0$ are well known and described in the introduction. We begin with the case $g = -1$, since it has a somewhat different statement due to the existence of more reduced curves.

PROPOSITION 6.1. *The Hilbert scheme $H_{4,-1}$ is connected and has irreducible components as shown in Table II.*

Proof. If a curve $C \in H_{4,-1}$ is not extremal, then its spectrum is necessarily $\{0^2, 1^2\}$, in which case C is in one of the families H_2 or H_3 of Proposition 4.2. These families have general members as described in Remark 4.4 and meet because both contain thick 4-lines. Finally, H_2 meets the family of extremal curves by [23] or [14]. □

THEOREM 6.2. *The Hilbert schemes $H_{4,g}$ have irreducible components as shown in Table III for $g \leq -2$.*

Proof. In the table G_j denotes the closure in the Hilbert scheme $H_{4,g}$ of the set of curves described in the corresponding row. The outline of the proof is as follows. First we show the families listed in the table are irreducible of the stated dimension. Then we show there is no inclusion relation among them. Finally, we prove every curve of degree 4 and genus $g \leq -2$ belongs to one of these families. The restrictions given in the table are necessary to ensure that a given family exists and is not contained in another family of the list.

The family G_1 consists of extremal curves, and is an irreducible component of the Hilbert scheme of dimension $15 - 2g$ by [18, 4.3]. G_2, G_3 and G_4 are the closures of the families H_2, H_3 and $H_{n(g)}$ of Proposition 4.3, and are therefore irreducible of the stated dimension. Note that G_2 contains the subextremal curves and G_4 consists of thick 4-lines. Since any double conic lies on a double plane (if the double conic C has support Y , then $C \subset Y^{(2)}$, i.e. $I_Y^2 \subset I_C$), the closure G_5 of the family of double conics of genus g is irreducible of dimension $13 - 2g$ because the family of such curves in a *fixed* double plane $2H$ is irreducible of dimension $10 - 2g$ by [14, 2.1 and 4.3]. The

Table II. Irreducible components of the Hilbert scheme $H_{4,-1}$

Label	General curve	Dimension
H_1	external curve	17
H_2	disjoint union of two conics	16
H_3	disjoint union of line and twisted cubic	16

Table III. Irreducible components of the Hilbert scheme $H_{4,g}$ for $g \leq -2$

Label	General curve	Dimension	Restrictions
G_1	$D \cup Z$ D smooth conic $g(Z) = g - 3$, $\text{length}(D \cap Z) = 4$	$15 - 2g$	none
G_2	$L_1 \cup_{2P} Z \cup_{2Q} L_2$ $L_1 \cap L_2 = \emptyset$ $g(Z) = g - 2$	$13 - 2g$	none
G_3	$D \cup_{2P} Z$ D smooth conic $g(Z) = g - 1$	$13 - 2g$	none
G_4	general thick 4-line	$9 - 3g$	$g \leq -3$
G_5	double conic	$13 - 3g$	none
G_6	$Z \cup_{2P} L_1 \dot{\cup} L_2$ $g(Z) = g$	$11 - 2g$	$g \leq -3$
$G_{7,a}$	W quasiprimitive 3-line of type $(a, -3a - g)$	$11 - 2g - a$	$g \leq -3$ $0 < a \leq \frac{-g}{3}$
$G_{8,a}$	$W \cup_{2P} L$ W quasiprimitive 3-line of type $(a, -1 - 3a - g)$	$10 - 2g - a$	$g \leq -6$ $0 < a \leq \frac{-g-1}{3}$
$G_{9,a}$	$W \dot{\cup} L$ W quasiprimitive 3-line of type $(a, -3 - 3a - g)$	$8 - 2g - a$	$g \leq -6$ $0 < a \leq \frac{-g-3}{3}$
$G_{10,m}$	$D_1 \dot{\cup} D_2$ $\text{deg}(D_1) = 2, g(D_1) = -m$ $\text{deg}(D_2) = 2, g(D_2) = g + m + 1$	$d(-m) + d(g+m+1)$	$0 \leq m \leq \frac{-g-1}{2}$
$G_{11,a,b}$	Quasiprimitive 4-line of type $(a, b, -6a - b - g - 3)$	$7 - 2g - 3a$ $= 9a + 2b + 2c + 13$	$g \leq -9$ $0 < a \leq \frac{-g-3}{6}$ $0 \leq b \leq \frac{-6a-g-3}{2}$

Notation: The letter L denotes a line, the letter Z denotes a double line, and $\dot{\cup}$ indicates a disjoint union. $L \cup_{nP} C$ stands for the union of a line L and a curve C which intersect along the divisor nP on L . In the tenth entry we have $d(-m) = \dim H_{2,-m}$ equals $5 + 2m$ if $m > 1$, and 8 if $m = 0$ or $m = 1$.

families $G_{7,a}$ and $G_{8,a}$ are the closures of $F_{1,a}$ and $F_{3,a}$ from Proposition 5.2 and are therefore irreducible of the stated dimension. The irreducibility of $G_{11,a,b}$ is proven in Proposition 2.3.

Finally, families $G_6, G_{9,a}$ and $G_{10,m}$ are irreducible components of $H_{4,g}$ because the curves defining them are disjoint unions of curves that are general in their respective Hilbert schemes. The dimensions of these families can be computed out of [20, 1.6, 3.4, 3.5].

This shows all families in the statement are irreducible, and we now prove there are no inclusion relation among them. We have just seen that $G_1, G_6, G_{9,a}$ and

$G_{10,m}$ are irreducible components of $H_{4,g}$, so certainly none of them can be contained in any other family of the list.

The families G_2 , G_3 and G_5 could only be contained in G_1 or G_4 by reason of dimension. However, none of them is contained in G_1 by semicontinuity, and none of them is contained in G_4 because curves in G_4 are supported on a single line.

In the table, the only families of dimension larger than that of G_4 consist of curves C with $h^0\mathcal{I}_C(2) \neq 0$. These cannot specialize to a general thick 4-line T because $h^0\mathcal{I}_T(2) = 0$ for $g \leq -3$ by Proposition 4.3. Thus G_4 is not contained in any of the other families. Note however that G_3 contains all thick 4-lines when $g = -2$.

We now treat the case of $G_{7,a}$ and $G_{8,a}$. The general curves in $G_{7,a}$ and $G_{8,a}$ are not supported on a line, so they can't be contained in G_4 or $G_{11,a,b}$. $G_{7,a}$ and $G_{8,a}$ are not contained in G_1 , G_2 , G_3 or G_5 by semicontinuity – curves in the latter families have larger speciality by Proposition 4.3.

$G_{7,a}$ and $G_{8,a}$ are not contained in G_6 or $G_{10,m}$ because when two lines collapse the resulting double line has genus ≥ -1 , hence $a \leq 0$ contradicting the restriction imposed.

If $G_{7,a'} \subset G_{9,a}$, then $a' \leq a$ by Remark 2.5, which in turn implies that $\dim G_{7,a'} = 11 - 2g - a' > 8 - 2g - a = \dim G_{9,a}$, a contradiction. Similarly $G_{8,a'} \not\subset G_{9,a}$.

It remains to show that neither $G_{7,a}$ and $G_{8,a}$ contains the other. There can be no containment $G_{7,a'} \subset G_{8,a}$, because then $a' \leq a$ by Remark 2.5 and hence $\dim G_{8,a} = 10 - 2g - a < 11 - 2g - a' = \dim G_{7,a}$, a contradiction. Now suppose that $G_{8,a'} \subset G_{7,a}$. Remark 2.5 tells us again that $a' \leq a$, and since $\dim G_{8,a'} < \dim G_{7,a}$ we conclude that $a' = a$. In particular, the limit of the underlying family of double lines Z has no embedded points. This is not possible because the limit double line meets L in one point while the general member meets L in a double point.

Finally, $G_{11,a,b}$ cannot be contained in any of the families G_j with $j \leq 10$ by semicontinuity: indeed, since $a > 0$, every 4-line C in $G_{11,a,b}$ satisfies $h^1\mathcal{O}_C(-2) = 1$, while for any other curve $D \in H_{4,g}$ we have $h^1\mathcal{O}_D(-2) \geq 2$. On the other hand, there are no containments among the families $G_{11,a,b}$: if $G_{11,a,b} \subset G_{11,a',b'}$, then by Remark 2.5 we would have $a \leq a'$, while

$$7 - 2g - 3a' = \dim G_{11,a',b'} > \dim G_{11,a,b} = 7 - 2g - 3a$$

would yield $a' < a$, a contradiction.

To finish the proof, we still have to show our families cover the Hilbert scheme. Let $C \in H_{4,g}$ have support $B = C_{\text{red}}$.

Case 1. deg B = 4

Here $C = B$ is reduced, and all reduced curves of degree 4 satisfy $g \geq -1$ with the following two exceptions: either (a) C is the disjoint union of a conic (possibly degenerate) and two lines, when $g = -2$ and $C \in G_{10,0}$ or (b) C is the disjoint union of four lines, $g = -3$ and $C \in G_{10,1}$.

Case 2. deg B = 3

In this case $C = Z \cup D$, where Z is a double line with support L and D is a reduced curve of degree 2. In particular, $B = L \cup D$.

First suppose that D is planar and let $l = \text{length}(D \cap L)$. If $l = 0$, then $C \in G_{10,0}$. If $l = 1$, then $g(B) = 0$ and C belongs to one of the families G_2 , G_3 or G_4 by Proposition 4.3. If $l = 2$, then B is planar and hence C is extremal by [21, 2.2].

The other possibility is that $D = L_1 \cup L_2$ is a disjoint union of lines. If D does not meet Z , then $C \in G_{10,1}$. If Z meets L_1 but not L_2 , then $Z \cup L_1$ is a specialization of a double line meeting L_1 in a double point by [20, 3.2], so C lies in G_6 . If Z meets both L_1 and L_2 , then $g(B) = 0$ and C again belongs to one of the families G_2 , G_3 or G_4 by Proposition 4.3.

Case 3. deg B = 2

If B is a smooth conic, then C belongs to G_5 , so we suppose that $B = L \cup L'$ is a union of two lines. If C is a union of double lines, then either (a) the lines are disjoint and $C \in G_{10,m}$ for $m = -\max\{g(Z_1), g(Z_2)\}$ or (b) the lines meet and hence C is contained in a double plane; in this case $C \in G_5$ by [14, 8.1 and 8.2].

The remaining possibility is that $C = W \cup L$, where L is a line and W is a triple line which is quasi-primitive because $g(W) \leq -1$ (the only thick triple line has genus 0). If W has type (a, b) , let Z be the underlying double line and set $l = \text{length}(W \cap L)$.

If $l = 3$, then necessarily $\text{length}(Z \cap L) = 2$. If $a = -1$, then $Z \cup L$ is planar and C is extremal. If $a = 0$, then $Z \cup L \in H_{3,0}$ and C belongs to one of the families G_2 , G_3 or G_4 by Proposition 4.3. If $a > 0$, then C belongs to $G_{7,a}$.

Suppose $l = 2$. If $\text{length}(Z \cap L) = 2$ and $a = -1$ or 0 , we argue as in the case $l = 3$ above. If $\text{length}(Z \cap L) = 2$ and $a > 0$, then $C \in G_{7,a}$ by Proposition 5.4. Thus we may assume $Z \cap L = P$ a reduced point. If $a = -1$, then $Z \cup L \in H_{3,0}$ and $C \in G_2 \cup G_3$ by Proposition 4.3, while if $a = 0$, then $C \in G_6$ by Remark 5.6. If $a > 0$, then $C \in G_{8,a}$.

Suppose $l = 1$. Then either (a) $a > 0$ so that $C \in G_{8,a}$ by Proposition 5.4 (b) $a = 0$ and $C \in G_6$ by Proposition 5.4 and Remark 5.6 or (c) $a = -1$, $Z \cup L \in H_{3,0}$ and $C \in G_2 \cup G_3$ by Proposition 4.3.

If $l = 0$, then $C = W \cup L$ with W a quasi-primitive 3-line of type $(a, -3 - 3a - g)$. Now either (a) $a = -1$, W is extremal and $C \in G_6$ by [20, 3.2] or (b) $a = 0$ and $C \in G_{5,0}$ by [20, 3.3] or (c) $a > 0$ and $C \in G_{9,a}$.

Case 4. deg B = 1

If C is thick, then $C \in G_4$. If C is a quasiprimitive 4-line, then C has type (a, b, c) for some integers $a \geq -1$ and $c \geq b \geq 0$. If $a = -1$, then the underlying double line Z is planar and C lies in a double plane, hence $C \in G_5$ by [14, 8.1 and 8.2]. If $a = 0$, then C is in $G_{10,m}$ for $m = -1 - b$ by Proposition 3.3. Finally, if $a > 0$, then $C \in G_{11,a,b}$.

EXAMPLE 6.3. As the restrictions in Table III imply, some of the components do not show up if the genus g is not small enough. The *five* irreducible components of $H_{4,-2}$ appear in Table IV:

Table IV. Irreducible components of $H_{4,-2}$

Label	Description	Dimension
G_1	Conic meeting a double line in a scheme of length 4	19
G_2	$L_1 \cup_{2P} Z \cup_{2Q} L_2$ L_i lines, Z a double line of genus -4	17
G_3	Conic meeting a double line in a double point	17
G_5	Double conic	17
$G_{10,0}$	Disjoint union of a conic and two lines	16

We can now prove $H_{4,g}$ is connected:

THEOREM 6.4. *The Hilbert scheme $H_{4,g}$ is connected if nonempty.*

Proof. We may assume $g \leq -2$, and it suffices to show that all the irreducible components can be connected to the component G_1 of extremal curves. The families G_2 and G_3 meet the family G_4 of thick 4-lines by Proposition 4.3, and G_2 meets G_1 by [1, 14], or [23]. In particular, thick 4-lines belong to the connected component of extremal curves, and it is enough to show that all other components can be connected to either G_1 or G_4 . G_5 meets G_1 by [14, 5.1 and 8.2]. That G_6 meets G_1 follows immediately from [11, 2.1] and [20]. $G_{7,a}$ meets G_1 by Corollary 5.5 and $G_{8,a}$ meets G_1 by Proposition 5.4 and [11, 2.5]. One can connect $G_{9,a}$ to G_1 by applying [20, 3.8] and [11, 2.1]. The families $G_{11,a,b}$ meet G_4 by Proposition 2.4. Applying Proposition 3.3, we see $G_{10,m}$ contains $G_{11,0,m-1}$ for $m > 0$ and, hence, meets G_4 as well. Lastly, we consider $G_{10,0}$. By definition this family contains curves $C = Z \cup L_1 \cup_P L_2$ where Z has degree two and L_i are meeting lines. Since Z is extremal, C specializes to an extremal curve $E \in G_1$ by [11, 2.1 and 2.5]. \square

As an application of our results, we can now give a counterexample to a conjecture of Ait-Amrane and Perrin [2]. The conjecture regards the following question, which has been a recurring theme of this paper:

QUESTION 6.5. *Let X and X_0 be two irreducible families of curves in $H_{d,g}$ having constant cohomology. Under what conditions do we have a nonempty intersection $\bar{X} \cap X_0 \neq \emptyset$ in $H_{d,g}$?*

We have been lucky in that whenever we suspected the existence of such a deformation, we could actually prove it. In general, this question is difficult. Semicontinuity [8, III, Section 12] gives a necessary condition: If $\tau^i(n) = h^i(\mathbb{P}^3, \mathcal{I}_C(n))$ for all $C \in X$ and $\tau_0^i(n) = h^i(\mathbb{P}^3, \mathcal{I}_{C_0}(n))$ for all $C_0 \in X_0$, then whenever $\bar{X} \cap X_0 \neq \emptyset$ we must

have $\tau^i(n) \leq \tau_0^i(n)$ for all i and n (we write $\tau \leq \tau_0$ for short). This condition is not sufficient, even when X (resp. X_0) is an irreducible component of the Hilbert scheme H_τ (resp. H_{τ_0}) of curves with fixed cohomology. This has been shown by a recent example of Ait-Amrane and Perrin [2].

A more subtle necessary condition is afforded by the Rao modules of the curves. Let A be a discrete valuation ring with fraction field K and residue field k and let $\mathcal{C} \subset \mathbb{P}_A^3$ be a family of locally Cohen–Macaulay curves over A . Then the Rao module M_{C_K} of the generic curve is a flat deformation of a subquotient M of the Rao module M_{C_k} of the special curve ([13, Proposition 5.9], [1, § 4.2.2], [2, Proposition 13]). This means that there are submodules $M_1 \subset J \subset M_{C_k}$ such that $M = J/M_1$ and M is a flat deformation of M_{C_k} . Moreover,

$$\dim(M_1)_n = h^0\mathcal{I}_{C_k}(n) - h^0\mathcal{I}_{C_K}(n), \tag{13}$$

$$\dim(M_{C_k}/J)_n = h^2\mathcal{I}_{C_k}(n) - h^2\mathcal{I}_{C_K}(n). \tag{14}$$

In view of this result, the following conjecture of Ait-Amrane and Perrin is natural:

CONJECTURE 6.6 ([2], Conjecture 14). *Let X and X_0 be irreducible components of H_τ and H_{τ_0} respectively. Suppose that*

- (1) $\tau \leq \tau_0$
- (2) *The Rao module of the generic curve C_ξ of X is a flat deformation of a subquotient of the Rao module of a curve C_0 in X_0 and that the numerical conditions (13) and (14) hold.*

Then $\bar{X} \cap X_0 \neq \emptyset$ in $H_{d,g}$.

As it turns out, the conditions of the conjecture are still insufficient, as we note in the following example.

EXAMPLE 6.7. For $g \leq -3$, let $X \subset H_{4,g}$ denote the irreducible family $H_{n(g)}$ from Proposition 4.3. The closure \bar{X} is precisely the family of thick 4-lines. Let $X_0 = G_1$ denote irreducible family consisting of extremal curves. We claim that $\bar{X} \cap X_0 = \emptyset$. Indeed, an extremal curve cannot specialize to a thick 4-line because this would violate semicontinuity, while a thick 4-line has everywhere embedding dimension 3, and so cannot specialize to an extremal curve that has generic embedding dimension 2 [18]. On the other hand, we will now show that the conditions of the conjecture hold.

First we compare the Rao modules. Let C be a general thick 4-line with support L and set $S = S_L$. Proposition 4.3 shows that $M_C \cong S/(a, b, c)(-g - 1)$ where a, b, c are general forms of degree $-g + 1$ in S . Choose a linear form $l \in S$ so that (a, l) is a regular sequence and a form $f \in (b, c)$ of degree $-g + 2$ so that (a, f) is a regular sequence. We consider the extremal Koszul module $M = S/(a, lf)$. Since the multiplication $S/(a) \xrightarrow{\cdot l} S/(a)$ is injective and the image of the submodule (\bar{f}) is

(\bar{f}) , we see that the submodule $J = \bar{I}M$ is isomorphic to $S/(a, f)$. Since $f \in (b, c)$ by choice, $S/(a, b, c)$ is a quotient of J by $M_1 = (lb, lc)$. If E is an extremal curve corresponding to M , then $\deg E = 4$, $p_a(E) = g$ and we have just shown that M_C is a subquotient of M_E . It is clear that $\dim(M/J)_n = 1$ for $g \leq n \leq 0$ and zero otherwise; this is seen to be precisely $h^2\mathcal{I}_E(n) - h^2\mathcal{I}_C(n)$ in comparing the spectra of these curves. Finally, since the Euler characteristics of \mathcal{I}_E and \mathcal{I}_C are the same, the exact sequences relating the Rao modules shows that $\dim(M_1)_n = h^0\mathcal{I}_E(n) - h^0\mathcal{I}_C(n)$. In particular, the semicontinuity conditions are immediate.

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