AN ANALOGUE OF A CONJECTURE OF SATO AND TATE FOR A HILBERT MODULAR FORM

by H. L. RESNIKOFF† and R. L. SALDAÑA

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1. Introduction. If k denotes a number field and ε^m is the product of an elliptic curve ε with itself m times over k, then for each prime π where ε has non-degenerate reduction, the zeta factor $\zeta(\varepsilon_\pi, s)$ can be expressed as

$$\zeta(\varepsilon_\pi, s) = \frac{(1-\varepsilon_\pi|\pi|^{\frac{1}{2}-s})(1-\bar{\varepsilon}_\pi|\pi|^{\frac{1}{2}-s})}{(1-|\pi|^{-s})(1-|\pi|^{1-s})},$$

where $|\pi|$ denotes the norm of π. It is a consequence of a conjecture of Tate [16] that if ε does not have complex multiplications, then the numbers $x_\pi = \frac{1}{2}(\varepsilon_\pi+\bar{\varepsilon}_\pi)$ are distributed according to the density function

$$\frac{2}{\pi}(1-x^2)^{\frac{1}{2}}; \tag{1}$$

that is, the density of the set of primes π such that $-1 \leqq a \leqq \frac{1}{2}(\varepsilon_\pi+\bar{\varepsilon}_\pi) \leqq b \leqq 1$ is

$$\frac{2}{\pi}\int_a^b (1-x^2)^{\frac{1}{2}}\,dx.‡$$

According to Tate [16], machine calculations conducted by M. Sato support this conclusion.

Serre [13] (cf. [14], [15]) posed an analogous question for Ramanujan's function τ. Put

$$\Delta(z) = e^{2\pi iz}\left\{\prod_{n=1}^{\infty}(1-e^{2\pi inz})\right\}^{24}, \ \mathrm{Im}\, z > 0.$$

Δ is a cusp form of weight 12 for the classical modular group $SL(2, \mathbb{Z})/\{\pm Id\}$, hence an eigenfunction of the ring of Hecke operators. If the Fourier expansion of Δ be written

$$\Delta(z) = \sum_{n=1}^{\infty}\tau(n)e^{2\pi inz},$$

then $\tau(1) = 1$, the function $\tau : \mathbb{Z}^+ \to R$, which is Ramanujan's function, is multiplicative, and the associated Dirichlet series

$$\zeta_\Delta(s) = \sum_{n=1}^{\infty}\tau(n)n^{-s}$$

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‡ The reader is not likely to confuse use of the common symbol π to denote primes in a number field and the ratio of circumference to diameter of a circle.

admits the Euler product expansion

$$\zeta_\Delta(s) = \prod_p (1-\tau(p)p^{-s}+p^{11-2s})^{-1} \tag{2}$$

where p runs through the rational primes. A celebrated conjecture of Ramanujan [**10**] asserted that

$$abs\, \tau(p) \leqq 2p^{11/2}, \tag{3}$$

where *abs* denotes the absolute value. Deligne [**2**] showed that (3) is a consequence of Weil's conjectures, and Deligne [**3**] has recently proved the latter; hence (3) is valid and therefore, writing $x_p = \tau(p)/2p^{11/2}$, one has $-1 \leqq x_p \leqq 1$ and the factor $(1-\tau(p)p^{-s}+p^{11-2s})$ in (2) can be written as $(1-\varepsilon_p p^{11/2-s})(1-\bar{\varepsilon}_p p^{11/2-s})$, where $abs\ \varepsilon_p = 1$ and $2x_p = \varepsilon_p+\bar{\varepsilon}_p$. Serre's question ([**13**], pp. 14–15) is whether x_p is also distributed according to (1) as p runs through the rational primes. Lehmer [**9**] calculated the distribution of x_p for the 1229 primes less than 10^4 and found, in his words, "pretty reassuring" agreement with the Sato–Tate distribution (1).

The principal purpose of this paper is to provide numerical evidence which supports the Sato–Tate distribution for an analogue of Serre's question for Hilbert modular forms. The graded ring of Hilbert modular forms associated with the number field $\mathbb{Q}(\sqrt{5})$ is generated by an Eisenstein series of weight 2, two cusp forms of respective weights 6 and 15 which are unchanged when the argument variables are interchanged, and a "skew" cusp form χ_5 of weight 5 which changes sign when its argument variables are interchanged [**4**], [**11**]. Since the dimension of the $\mathbb{C}$-linear space of cusp forms of weight 5 is 1, χ_5 is an eigenfunction of the ring of Hecke operators associated with the Hilbert modular group [**7**], and it follows that the Fourier coefficients of a suitably normalized χ_5 are multiplicative functions of the algebraic integers in $\mathbb{Q}(\sqrt{5})$. Write the Fourier expansion of χ_5 in the form

$$\chi_5(\zeta) = \sum c(\nu)\, e^{2\pi i\sigma(\nu\zeta/\sqrt{5})} \tag{4}$$

where $\zeta = (\zeta_1, \zeta_2)$, Im $\zeta_i > 0$, σ denotes the trace (cf. §2) and ν runs through those integers in $\mathbb{Q}(\sqrt{5})$ such that $\nu/\sqrt{5}$ is totally positive.

Select $\varepsilon = (1+\sqrt{5})/2$ as fundamental unit of $\mathbb{Q}(\sqrt{5})$, and introduce the "Grössencharakter"

$$\lambda(\mu) = \exp\left\{\frac{i\pi}{4}\,\sigma\left(\frac{\log abs\ \mu}{\log abs\ \varepsilon}\right)\right\} \quad \text{for } \mu \in \mathbb{Q}(\sqrt{5}),$$

where the principal branch of the logarithm is chosen. A Dirichlet series associated with χ_5 is

$$\zeta_{\chi_5}(s, \lambda) = \sum_{(\nu)} -ic(\nu)\lambda(\nu)\, abs\, |\nu|^{-s}, \tag{5}$$

where the sum runs over the ideals in the ring of integers in $\mathbb{Q}(\sqrt{5})$; it is readily checked that the individual summands are independent of the choice of representative generator ν of (ν). A special case of a remarkable theorem of Hermann [7] asserts that ζ_{χ_5} can be expressed by the Euler product

$$\zeta_{\chi_5}(s, \lambda) = \prod_{(\pi)} (1+ic(\pi)\lambda(\pi)\, abs\, |\pi|^{-s}+abs\, |\pi|^{4-2s})^{-1} \tag{6}$$

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if χ_5 is so normalized that $c(\varepsilon) = 1$ (the product runs over the prime ideals in the ring of integers of $\mathbb{Q}(\sqrt{5})$). Moreover, ζ_{χ_5} satisfies the functional equation (Hermann [7], Satz 15):

$$G(5-s, \bar{\lambda}) = -G(s, \lambda),$$

where

$$G(s, \lambda) = \left(\frac{5}{4\pi^2}\right)^s \Gamma\left(s - \frac{i\pi}{4 \log abs\,|\varepsilon|}\right) \Gamma\left(s + \frac{i\pi}{4 \log abs\,|\varepsilon|}\right) \zeta_{\chi_5}(s, \lambda).$$

Now suppose that the following analogue of Petersson's extension of Ramanujan's conjecture is valid: if $a(\nu)$ denotes the νth Fourier coefficient of a normalized cusp form of weight w for Hilbert's modular group associated with some number field, then for prime ideals π,

$$abs\, a(\pi) \leqq 2(abs\,|\pi|)^{(w-1)/2}. \tag{7}$$

For χ_5 this becomes

$$abs\, c(\pi) \leqq 2(abs\,|\pi|)^2. \tag{8}$$

Set

$$x_{(\pi)} = c(\pi)/2(abs\,|\pi|)^2;$$

then (8) implies that $-1 \leqq x_{(\pi)} \leqq 1$ and, in analogy to the above for the classical modular form Δ, the factor $(1 + ic(\pi)\lambda(\pi)\, abs\,|\pi|^{-s} + abs\,|\pi|^{4-2s})$ can be written as $(1 - \varepsilon_{(\pi)} abs\,|\pi|^{2-s})$ $(1 - \bar{\varepsilon}_{(\pi)} abs\,|\pi|^{2-s})$ where $abs\, \varepsilon_{(\pi)} = 1$ and $(-i\lambda(\pi))x_\pi = (\varepsilon_{(\pi)} + \bar{\varepsilon}_{(\pi)})/2$. We are therefore led to inquire whether the $x_{(\pi)}$ are distributed according to the Sato–Tate distribution (1) as (π) runs through the prime ideals. The numerical evidence presented in the next section strongly suggests an affirmative answer.

The authors thank F. J. Dyson for bringing Tate's conjecture to our attention. We are indebted to J.-P. Serre for many helpful remarks and particularly for informing us of certain omissions in a previous version of the work reported here.

The remainder of this paper is organized as follows. Section 2 presents a table of the Fourier coefficients $c(\pi)$ of χ_5 for the 305 prime ideals of norm $\leqq 4391$, and also shows how the $c(\nu)$ can be expressed in terms of $c(\pi)$ for arbitrary integral ideals (ν); this interdependence provides a check on the validity of the tabulated entries. Section 2 also presents several statistical measures of the compatibility of the distribution of the $x_{(\pi)}$ for the tabulated values with the Sato–Tate distribution. Calculation of the coefficients $c(\nu)$ is not a straightforward matter. Section 3 describes a connection between Siegel modular forms of degree two and Hilbert modular forms associated with certain quadratic number fields which enables the Fourier coefficients of particular Hilbert modular forms to be expressed as linear combinations of the Fourier coefficients of Siegel modular forms. Specialization of this relationship yields the formula by means of which the values tabulated in §2 were calculated. Section 4 applies some of the results of §3 to Hilbert modular forms associated with quadratic fields other than $\mathbb{Q}(\sqrt{5})$.

2. The Fourier coefficients of χ_5. Define a multiplication on $\mathbb{C}^2$ by the bilinear pairing $(\alpha, \beta) = ((\alpha_1, \alpha_2), (\beta_1, \beta_2)) \mapsto (\alpha_1\beta_1, \alpha_2\beta_2) = \alpha\beta$ and introduce the *trace* $\sigma(\alpha) = \alpha_1 + \alpha_2$ and the *norm* $|\alpha| = \alpha_1\alpha_2$ for $\alpha \in \mathbb{C}^2$. If k is a real quadratic number field and $\alpha_1, \alpha_2 = \alpha_1^*$ are conjugates in k, put $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \subset \mathbb{C}^2$; in this way k is embedded in $\mathbb{C}^2$ and the restriction to the image of k in $\mathbb{C}^2$ of the trace and norm defined above are, respectively, the trace and norm in k. Extend the conjugation map to $\mathbb{C}^2$ by defining ${}^*\alpha = {}^*(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = \alpha^*$ for $\alpha \in \mathbb{C}^2$. If $\alpha \in \mathbb{R}^2$, write $\alpha > 0$ if $\alpha_1 > 0$, $\alpha_2 > 0$, and set $\mathscr{H} = \{\zeta \in \mathbb{C}^2 : \operatorname{Im} \zeta > 0\}$.

Let $k = \mathbb{Q}(\sqrt{5})$ and introduce the fundamental unit $\varepsilon_1 = (1+\sqrt{5})/2$; set $\varepsilon = (\varepsilon_1, \varepsilon_1^*)$. Since χ_5 is a skew Hilbert modular form of weight 5, $\chi_5(\varepsilon^{2n}\zeta) = (-)^n\chi_5(\zeta)$ and $\chi_5(\zeta^*) = -\chi_5(\zeta)$ for $\zeta \in \mathscr{H}$; these conditions are equivalent to the following two conditions for the Fourier coefficients $c(\nu)$ of χ_5:

$$c(\varepsilon^{2n}\nu) = (-)^n c(\nu), \tag{9}$$

and

$$c(-\nu^*) = -c(\nu). \tag{10}$$

These equations have the consequence that it is only necessary to calculate the $c(\nu)$ for those ν which lie in a fundamental domain for the action of the group Ξ^* generated by the transformations $\eta \mapsto \varepsilon^2\eta$ and $\eta \mapsto -\eta^*$ on the set of $\eta \in \mathbb{R}^2$ such that $\eta/\sqrt{5} > 0$. An element $\eta \in \mathbb{R}^2$ such that $\eta/\sqrt{5} > 0$ is said to be reduced with respect to the group of units if

$$\frac{\varepsilon_2}{\varepsilon_1} > \frac{\eta_1}{\eta_2} \geqq \frac{\varepsilon_1}{\varepsilon_2};$$

thus ε itself is reduced with respect to the group of units. The set $\mathfrak{F}_\varepsilon$ of elements reduced with respect to the group of units is easily seen to be a fundamental domain for the action of the group of units defined above. Furthermore, $\mathfrak{F}_* = \{\eta : \eta_2 \geqq \eta_1 \text{ and } \eta/\sqrt{5} > 0\}$ is a fundamental domain for the involution $\eta \mapsto -\eta^*$. Then

$$\mathfrak{F}_\varepsilon \cap \mathfrak{F}_* \equiv \mathfrak{F} = \left\{\eta\colon -1 \geqq \frac{\eta_1}{\eta_2} \geqq \frac{\varepsilon_1}{\varepsilon_2} \text{ and } \frac{\eta}{\sqrt{5}} > 0\right\}$$

is a fundamental domain for the jointly generated group Ξ^*. Elements of $\mathfrak{F}$ will be said to be Ξ^*-reduced; $c(\nu)$ can be calculated from (9) and (10) if $c(\mu)$ is known, where μ is reduced and equivalent to ν under the action of Ξ^*.

If π is a prime in $\mathbb{Q}(\sqrt{5})$, then there is a bijective correspondence between $abs\,|\pi|$ and the Ξ^*-reduced element of the Ξ^*-orbit of $\pi/\sqrt{5}$; we may therefore write $c(abs\,|\pi|)$ to unambiguously denote $c(\tilde{\pi})$ where $\tilde{\pi}$ is the Ξ^*-reduced element of the Ξ^*-orbit of $\pi/\sqrt{5}$ and $\pi/\sqrt{5} > 0$. The reader will remark that this bijective correspondence cannot be extended to norms of non prime integers of $\mathbb{Q}(\sqrt{5})$; for instance, $abs\,|\nu\nu^*| = abs\,|\nu^2|$ but ν^2 and $\nu\nu^*$ need not belong to the same Ξ^*-orbit.

The absolute values of the norms of primes, that is the norms of prime ideals $(\pi) \subset \mathbb{Q}(\sqrt{5})$, are those primes congruent to 1 or 9 mod 10, those squares of primes congruent to 3 or 7

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mod 10, and 5, according as the norm is of the form *abs* $\pi\pi^*$ with $(\pi) \neq (\pi^*)$, π is a rational prime, or $(\pi) = (\pi^*)$ respectively. Table I below lists the 305 Fourier coefficients $c(\pi)$ in the form $c(q)$ where $q = abs\,|\pi| \leqq 4391$, for primes $\pi \in \mathbb{Q}(\sqrt{5})$ such that $\pi/\sqrt{5}$ is totally real and Ξ^*-reduced.

TABLE I. FOURIER COEFFICIENTS OF THE NORMALIZED CUSP FORM OF WEIGHT 5 FOR HILBERT'S MODULAR GROUP ASSOCIATED WITH $\mathbb{Q}(5^{\frac{1}{2}})$ FOR REDUCED PRIME CLASSES AND FOR THE REDUCED UNIT CLASS

abs $\lvert\pi\rvert$	$c(\pi)$	*abs* $\lvert\pi\rvert$	$c(\pi)$
1	1	491	−379188
4	−10	499	−32500
5	0	509	−504630
9	120	521	328482
11	−108	529	297560
19	−140	541	−205738
29	810	569	374760
31	−728	571	−165932
41	−1512	599	−421200
49	1400	601	−26152
59	−3780	619	−23660
61	4592	631	233672
71	432	641	182088
79	8840	659	579420
89	−13230	661	−519008
101	3402	691	−304612
109	8320	701	−123552
131	9828	709	266230
139	7420	719	−446040
149	10800	739	−315220
151	27352	751	−291248
169	2690	761	−600642
179	−58860	769	348110
181	−14378	809	−407970
191	−50112	811	1121092
199	5600	821	−1026432
211	−26308	829	−1057840
229	−17290	839	−249480
239	44280	859	908180
241	93688	881	273672
251	−62748	911	−1126008
269	45360	919	21440
271	−106232	929	−143640
281	25272	941	407862
289	166370	971	−153468
311	137592	991	432688
331	−90028	1009	25870
349	−96950	1019	−1102140
359	−184680	1021	−1022518
379	−69260	1031	946728
389	28080	1039	547120
401	−33102	1049	1256850
409	150920	1051	1787452
419	86940	1061	1804842
421	102832	1069	−1453760
431	223128	1091	1003212
439	285320	1109	−69930
449	59400	1129	−418760
461	−165942	1151	1971648
479	257040	1171	−408668

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TABLE I—*Continued*

abs $\|\pi\|$	$c(\pi)$	*abs* $\|\pi\|$	$c(\pi)$
1181	1276128	2081	7889778
1201	−2307448	2089	−1917230
1229	−2113290	2099	1568700
1231	−2305072	2111	558792
1249	−2889250	2129	543240
1259	2566620	2131	646828
1279	−833560	2141	−4043088
1289	−2259630	2161	4796008
1291	−1492988	2179	5579860
1301	1140048	2209	9695480
1319	−347760	2221	−6242032
1321	−1825432	2239	4780720
1361	−2593458	2251	−1563748
1369	2877410	2269	−8527360
1381	1196672	2281	−42728
1399	1773800	2309	1602720
1409	3274830	2311	3019592
1429	1699360	2339	−1411020
1439	−1597320	2341	5017712
1451	−753948	2351	−3634848
1459	−1532020	2371	−2680132
1471	−179968	2381	6395328
1481	306072	2389	−3855830
1489	504280	2399	−10168200
1499	1849500	2411	7905492
1511	−2632392	2441	4222638
1531	−11228	2459	937980
1549	1628150	2521	−2159768
1559	4256280	2531	1874772
1571	4198068	2539	−9041620
1579	1027660	2549	7581600
1601	−2453598	2551	5837048
1609	3197320	2579	−6603660
1619	−2495340	2591	9918288
1621	−3843382	2609	−7915320
1669	1555190	2621	−9135882
1699	3607100	2659	12907580
1709	−1754730	2671	13555168
1721	−555282	2689	−8042680
1741	−1712438	2699	7055100
1759	−968120	2711	−6239808
1789	1928630	2719	13725040
1801	−3562952	2729	1302210
1811	−2689092	2731	2145572
1831	1581472	2741	9680688
1849	6834200	2749	−11606000
1861	2579542	2789	−9087120
1871	2311632	2791	−4740512
1879	1472240	2801	−2550798
1889	52920	2809	−3412030
1901	−189648	2819	−502740
1931	−5055372	2851	142652
1949	−2948400	2861	5911542
1951	−5921552	2879	6483240
1979	−5885460	2909	10450080
1999	−3566000	2939	−3519180
2011	4330108	2969	6303960
2029	4831190	2971	−2876468
2039	−4146120	2999	3780000
2069	279990	3001	−10270498

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TABLE I—*Continued*

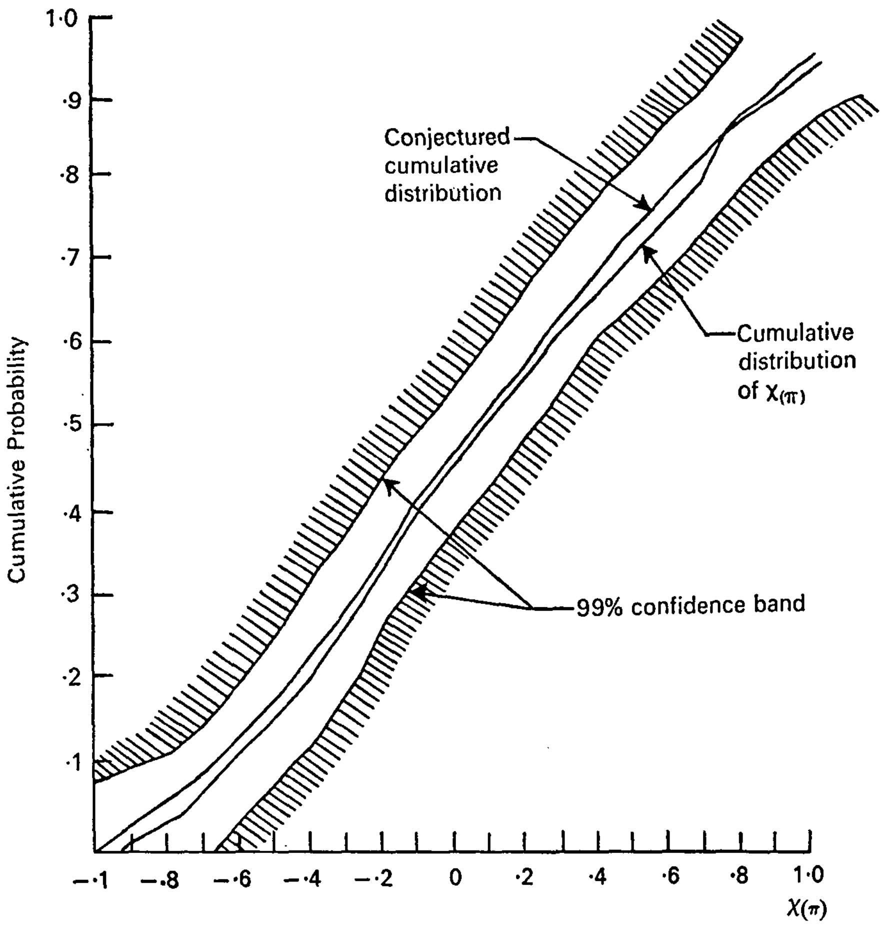
abs $|\pi|$	$c(\pi)$	abs $|\pi|$	$c(\pi)$
3011	13931892	3851	−3776652
3019	11381140	3881	4195422
3041	−11341512	3889	167080
3049	−5711150	3911	13858992
3061	−12755408	3919	14889560
3079	9204160	3929	−5509890
3089	−2537730	3931	−24795628
3109	−10265680	3989	−6571530
3119	6261840	4001	21233502
3121	−2020382	4019	4720140
3169	3757810	4021	−17402518
3181	−6563872	4049	9799650
3191	5025888	4051	−18743452
3209	−10455480	4079	−16548840
3221	−18067968	4091	1831788
3229	−5432960	4099	15955300
3251	−1385748	4111	−14608208
3259	−11894620	4129	−26422760
3271	15234232	4139	19859580
3299	12428100	4159	−13959920
3301	−19591702	4201	19793198
3319	−13575760	4211	18133308
3329	−5408910	4219	−23343460
3331	−1280972	4229	−27560790
3359	−7960680	4231	14947072
3361	−4417208	4241	4008312
3371	−6422868	4259	33562620
3389	−3720330	4261	−63392
3391	−235088	4271	−10622232
3449	−5556600	4289	1449630
3461	−13490442	4339	−32613980
3469	−3150710	4349	−19666800
3491	−9961812	4391	6527088
3499	21668500		
3511	3707392		
3529	14734190		
3539	−23407380		
3541	−16595488		
3559	−7434280		
3571	12551068		
3581	17827722		
3631	−21644672		
3659	13755420		
3671	−11361168		
3691	15754388		
3701	−17696448		
3709	−14947520		
3719	6708960		
3739	−1104220		
3761	−22450392		
3769	−7647640		
3779	22404060		
3821	14560182		

We have calculated the $x_{(\pi)}$ which correspond to the tabulated values and tested the Sato–Tate conjecture in several ways. The first four moments of the distribution (1) and the corresponding values obtained from the 305 values in Table I are:

TABLE II

Moment	Distribution (1)	Data from Table I
1	0	−0·0106
2	1/4 = 0·25	0·2494
3	0	0·0088
4	1/8 = 0·125	0·1231

The value of chi-squared for 19 degrees of freedom is 14·61. Moreover, application of the Kolmogorov–Smirnov test leads to Figure 1. These three tests each signal a high degree of compatibility of the tabulated data with the distribution (1). Figure 2 displays a histogram of the $\chi_{(\pi)}$-distribution and the Sato–Tate distribution which it approximates.



Kolomogorov–Smirnov test at 99% confidence level

FIGURE 1

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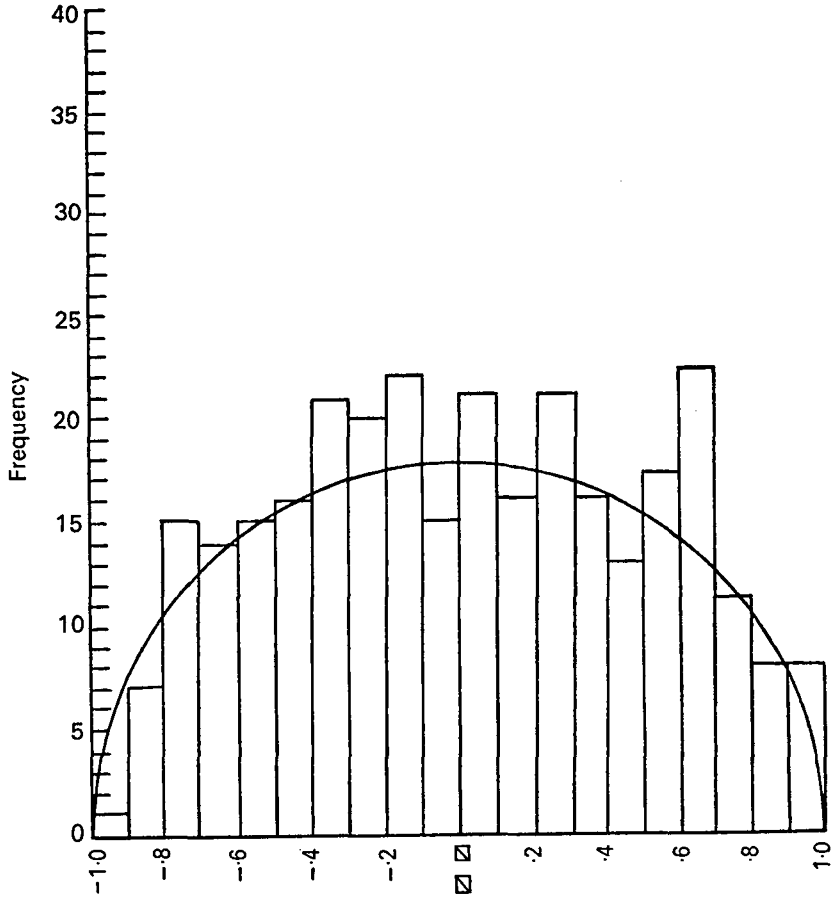
The reader will naturally want to know how the numbers in Table I were calculated, and what checks were employed to verify that they are correct. The first question will be answered in §3; some brief remarks concerning the second question will be made here.



Histogram for $\chi_{(\pi)}$ and conjectured density function

FIGURE 2

The procedure used for calculating Table I actually produced all the $c(\nu)$ such that $\sigma(\nu) < 60$, and in addition, certain values for ν with larger trace; the trace limitations were due to the limited precision of computation. The values which correspond to non prime ν make possible checks of the validity of the $c(\pi)$ for prime π as a consequence of the following considerations. Because χ_5 is an eigenfunction of the ring of Hecke operators associated with the Hilbert modular group for $\mathbb{Q}(\sqrt{5})$, the results of Hermann [7] can be applied. Now Satz 11 of [7] can be interpreted in the following way in the present situation. Define the quartic character θ on the multiplicative subgroup of $\mathbb{Q}(\sqrt{5})$ by the prescription

$$\theta(\varepsilon) = \theta(\pi) = \theta(-\pi^*) = i \tag{11}$$

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if π is an Ξ^*-reduced prime, and extend multiplicatively. Further define $c(\mu)$ for non totally positive $\mu/\sqrt{5}$ by $c(\varepsilon\mu) = \bar{\theta}(\varepsilon)c(\mu)$. The eigenvalues of the Hecke operator $T(v)$ which corresponds to v are just $\theta(v)a(v)/a(1)$, where $a(v)$ are the Fourier coefficients of eigenfunctions of the Hecke ring; since $c(\varepsilon) = 1$ for the Fourier coefficient of χ_5 (cf. Table I), it follows that $-i\theta(v)c(v)$ is the eigenvalue of $T(v)$. Then Satz 11 of [7] asserts that

$$c(\mu)c(v) = \sum_{(\delta)|(\mu,v)} \bar{\theta}(\delta^2/\sqrt{5})\,|\delta|^4\, c(\mu v/\delta^2). \tag{12}$$

In particular, if μ and v are relatively prime, then

$$\begin{aligned} c(\mu)c(v) &= \bar{\theta}\left(\frac{1}{\sqrt{5}}\right)c(\mu v) \\ &= +ic(\mu v) \\ &= -c(\varepsilon\mu v). \end{aligned} \tag{13}$$

This multiplicative condition supplied a convenient test of the calculations which was applicable to all primes π such that 2π was within the range of our calculations. For instance, let π be the Ξ^*-reduced prime of absolute norm 1009; then Table I gives $c(\pi) = 25870$, and $c(2\varepsilon) = -10$. Our calculations showed that the Ξ^*-reduced integer of absolute norm $2^2(1009) = 4036$ is $\mu = 13+29\sqrt{5}$ and $c(\mu) = -258700$. Since $\pi = \dfrac{13+29\sqrt{5}}{2} = \frac{1}{2}\mu$, we find that $c(\mu) = c(2\pi) = -c(\varepsilon(2\varepsilon)(\pi))$ by (9), $= +c(2\varepsilon)c(\pi)$, in agreement with (13).

The more complicated instances of (12), wherein powers of a prime occur, can be applied to few of the tabulated entries because the norms of the powers rapidly grow beyond the bounds of the data. But there are several opportunities. For example, $\pi = 1+2\sqrt{5}$ is an Ξ^*-reduced prime and $abs\,|\pi| = 19$, while $\mu = \dfrac{1+17\sqrt{5}}{2} = \varepsilon(\pi^*)^2$ is also reduced and evidently $abs|\mu| = 361$. Table I provides $c(\pi) = -140$ and the supplementary calculations show $c(\mu) = 110721$. Application of (12) yields

$$\begin{aligned} c(\pi)^2 &= \bar{\theta}\left(\frac{1}{\sqrt{5}}\right)c(\pi^2)+\bar{\theta}\left(\frac{\pi^2}{\sqrt{5}}\right)|\pi|^4c(1) \\ &= ic(\pi^2)-i|\pi|^4c(1) \\ &= -c(\varepsilon\pi^2)+|\pi|^4c(\varepsilon); \end{aligned}$$

from (9) and (10), $c(\varepsilon\pi^2) = -c(-\varepsilon^*\pi^{*2}) = +c(-\varepsilon^2\varepsilon^*\pi^{*2}) = c(\varepsilon\pi^{*2}) = c(\mu)$, and indeed, $(-140)^2 = -110721+(19)^4$.

3. On calculating certain Fourier coefficients. A Hilbert modular form proportional to χ_5 was first introduced by Gundlach [4] (cf. [11]) as the product of 10 theta functions associated with the principal congruence subgroup of level 2 of Hilbert's modular group for $\mathbb{Q}(\sqrt{5})$.

Although it would be possible to calculate the coefficients $c(v)$ directly from this definition, it is neither the simplest, nor an informative, way. Our efforts proceeded along a different path, thanks to an illuminating remark made to one of the authors (HLR) by A. Selberg in February 1973. Selberg remarked that automorphic forms in several complex variables give rise, in certain cases, to automorphic forms attached to submanifolds by restricting a type of normal derivative of the ambient form to the submanifold, and he adduced the following amusing example: let the product of classical upper half planes be embedded in the Siegel upper half plane of degree two as the diagonal matrices; the Cartesian product Γ^2 of the classical modular group with itself acts on the image as a subgroup of Siegel's modular group. If $z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ be a point in the Siegel upper half plane and $\tilde{\phi}$ is a Siegel modular form of weight w, then the restriction $\phi = \left.\dfrac{\partial\tilde{\phi}}{\partial z_3}\right|_{z_3=0}$ is a form of weight $w+1$ for Γ^2. If $w \equiv 0 \bmod 2$, then $\phi \equiv 0$, but if $w \equiv 1 \bmod 2$, then $\tilde{\phi} = \tilde{\psi}\tilde{\chi}_{35}$ where $\tilde{\psi}$ is a Siegel modular form of even weight and $\tilde{\chi}_{35}$ is a cusp form of weight 35. It follows that $\left.\dfrac{\partial\tilde{\phi}}{\partial z_3}\right|_{z_3=0} = \tilde{\psi}\Big|_{z_3=0} \cdot \left.\dfrac{\partial\tilde{\chi}_{35}}{\partial z_3}\right|_{z_3=0}$; the second factor is a skew cusp form of weight 36 for Γ^2 proportional, as one readily computes, to $\{\Delta(z_1)\Delta(z_2)\}^2\{g_{12}(z_1)\Delta(z_2)-g_{12}(z_2)\Delta(z_1)\}$, with $g_w(z_k)$ the Eisenstein series of weight w and $\Delta(z_k)$ the (normalized) cusp form of weight 12, both for the classical modular group.

The remainder of this section simply elaborates Selberg's remark in a restricted context which is nevertheless more general than that necessary for the applications described in this paper.

Let $\mathfrak{U}$ denote a compact real Jordan algebra of rank 2 with unit element c, reduced trace $a \mapsto \sigma(a)$ and reduced norm $a \mapsto |a|$ (cf. [1] for definitions and uncited results concerning Jordan algebras). Introduce $L(x)$ and $P(x)$ for $x \in \mathfrak{U}$ by $L(x)y = xy$, $P(x) = 2L(x)^2 - L(x^2)$ for all $y \in \mathfrak{U}$, and extend these endomorphisms to $\mathfrak{U} \otimes \mathbb{C}$ in the natural way. $\mathscr{X}(\mathfrak{U}) = \mathfrak{U} + i \exp \mathfrak{U}$ is biholomorphically equivalent to a bounded symmetric domain of rank 2 [8], where $\exp \mathfrak{U} = \{\exp a = \sum_{n=0}^{\infty} a^n/n! : a \in \mathfrak{U}\}$. Let Γ be a discrete subgroup of the group $\operatorname{Bih} \mathscr{X}(\mathfrak{U})$ of biholomorphic automorphisms of $\mathscr{X}(\mathfrak{U})$ and let (Γ, w) denote the $\mathbb{C}$-linear space of holomorphic functions $\tilde{\phi} : \mathscr{X}(\mathfrak{U}) \to \mathbb{C}$ such that

$$\tilde{\phi}(\gamma z) = \det\left(\frac{\partial \gamma z}{\partial z}\right)^{-w/2q} \tilde{\phi}(z) \tag{14}$$

where $\det\left(\dfrac{\partial \gamma z}{\partial z}\right)$ denotes the Jacobian determinant and $q = \dim_{\mathbb{R}} \mathfrak{U}/\operatorname{rank} \mathfrak{U}$. Let $\mathscr{M}_a \subset \mathscr{X}(\mathfrak{U})$ be the linear complex submanifold of codimension 1 in $\mathscr{X}(\mathfrak{U})$ defined by

$$\mathscr{M}_a = \{z \in \mathscr{X}(\mathfrak{U}) : \sigma(az) = 0\} \tag{15}$$

for a fixed $a \in \mathfrak{U} - \{0\}$, and let $\Gamma_a \subset \Gamma$ be the subgroup of biholomorphic automorphisms of $\mathscr{X}(\mathfrak{U})$ which preserve $\mathscr{M}_a$. The restriction of $\det\left(\dfrac{\partial \gamma z}{\partial z}\right)^{-w/2q}$ to $\mathscr{M}_a$ defines a factor of automorphy on $\mathcal{M}_a$, and we denote the $\mathbb{C}$-linear space of Γ_a-automorphic forms relative to this factor of automorphy by (Γ_a, w). Evidently $\phi \in (\Gamma, w)$ implies that $\phi \mid \mathcal{M}_a \in (\Gamma_a, w)$.

Introduce the σ-gradient ∇ by

$$\frac{\partial f}{\partial z}(b) = \sigma(b, \nabla_z f) = \sigma(b, \nabla f), \quad b \in \mathfrak{U} \otimes \mathbb{C},$$

for differentiable $f: \mathscr{X}(\mathfrak{U}) \to \mathbb{C}$; $\sigma(a, \nabla f)$ is the gradient of f in a direction normal to $\mathcal{M}_a$. We will investigate the restriction $\sigma(a, \nabla\phi)|_{\mathcal{M}_a}$ for $\phi \in (\Gamma, w)$. Recalling [8] that Bih $\mathscr{X}(\mathfrak{U})$ is generated by maps of the form

$$\begin{cases} z \mapsto z+t, \quad t \in \mathfrak{U}, \\ z \mapsto Az, \quad A \in \operatorname{Aut}\mathfrak{U} = \text{automorphisms of } \mathfrak{U}, \\ z \mapsto -z^{-1}, \end{cases} \tag{16}$$

we find that

$$\sigma(a, \nabla\phi)(z+t) = \sigma(a, \nabla\phi)(z). \tag{17}$$

Further,

$$\sigma(a, \nabla\phi)(Az) = \sigma(a, \nabla_{Az}\phi)(z); \tag{18}$$

now use $\nabla_{Az} = A^{-1}\nabla_z$ and the fact that automorphisms are self-adjoint with respect to σ to obtain

$$= \sigma(A^{-1}a, \nabla_z\phi)(z).$$

Last, from $\nabla_{-z^{-1}} = P(z)\nabla_z$ and $\phi(-z^{-1}) = |z|^w\phi(z)$, we find that

$$\begin{aligned} \sigma(a, \nabla\phi)(-z^{-1}) &= \sigma(a, P(z)\nabla_z|z|^w\phi(z)) \\ &= |z|^w\sigma(a, P(z)\nabla_z\phi)(z) + \phi(z)\sigma(a, P(z)\nabla|z|^w). \end{aligned} \tag{19}$$

One readily verifies that

$$\nabla|z|^w = w|z|^w z^{-1};$$

recalling that $P(z)$ is self-adjoint with respect to σ, (19) becomes

$$\sigma(a, \nabla\phi)(-z^{-1}) = |z|^w\sigma(P(z)a, \nabla\phi)(z) + w|z|^w\phi(z)\sigma(a, z). \tag{20}$$

Now restrict z in (19), (18), and (20) to $\mathcal{M}_a$ and A to $\operatorname{Aut}\mathfrak{U} \cap \Gamma_a$. Then $z \in \mathcal{M}_a$ implies that $A^{-1}z \in \mathcal{M}_a$; so $0 = \sigma(a, A^{-1}z) = \sigma(A^{-1}a, z)$, from which we conclude that $A^{-1}a = a$. The restriction of (20) is

$$\sigma(a, \nabla\phi)(-z^{-1})\Big|_{\mathcal{M}_a} = |z|^w\sigma(P(z)a, \nabla\phi)(z)\Big|_{\mathcal{M}_a}, \tag{21}$$

where we assume that $z \mapsto -z^{-1} \in \Gamma_a$. These relations show that $\sigma(a, \nabla\tilde{\varphi})|_{\mathscr{M}_a}$ will be a Γ_a-automorphic form if and only if there is a function $\lambda : \mathfrak{U}\otimes\mathbb{C} \to \mathbb{C}$ such that

$$\sigma(P(z)a, \nabla\tilde{\phi})(z) = \lambda(z)\sigma(a, \nabla\tilde{\phi})(z). \tag{22}$$

At this point we use the rank restriction on $\mathfrak{U}$: if $u \in \mathfrak{U}\otimes\mathbb{C}$, then

$$u^2 - \sigma(u)u + |u| c = 0. \tag{23}$$

Let $a^{\frac{1}{2}}$ denote a square root of $a \in \mathfrak{U} - \{0\}$ and set $u = P(a^{\frac{1}{2}})z$. Then substitution in (23) produces

$$(P(a^{\frac{1}{2}})z)^2 - \sigma(P(a^{\frac{1}{2}})z)P(a^{\frac{1}{2}})z + |P(a^{\frac{1}{2}})z| = 0\,;$$

now use the well-known properties [1]:

$$\begin{aligned} u^2 &= P(u)c, \\ P(P(u)v) &= P(u)P(v)P(u), \quad P^n(u) = P(u^n), \\ |P(u^{\frac{1}{2}})v| &= |u|\,|v| \end{aligned}$$

to conclude that

$$P(a^{\frac{1}{2}})P(z)P(a^{\frac{1}{2}})c - \sigma(a, z)P(a^{\frac{1}{2}})z + |a|\,|z|c = 0. \tag{24}$$

Restriction of (24) to $\mathscr{M}_a$ yields

$$P(z)a = -|a|\,|z|P^{-1}(a^{\frac{1}{2}})c = -|a|\,|z|a^{-1}, \tag{25}$$

and substitution of (25) in (22) provides the equality

$$\begin{array}{ccc} \sigma(P(z)a, \nabla\tilde{\phi})\Big|_{\mathscr{M}_a} & = & \lambda(z)\sigma(a, \nabla\tilde{\phi})\Big|_{\mathscr{M}_a} \\ \| & & \\ -|a|\,|z|\sigma(a^{-1}, \nabla\tilde{\phi})\Big|_{\mathscr{M}_a}. & & \end{array}$$

Hence, $\sigma(a, \nabla\tilde{\phi})|_{\mathscr{M}_a}$ will be an automorphic form if $a^{-1} = \mu a$ for some $\mu \in \mathbb{R}$, that is

$$a^2 \in \mathbb{R}c. \tag{26}$$

We have proved

THEOREM 1. *If $a^2 \in \mathbb{R}c$ and $a \neq 0$, then*

$$\sigma(a, \nabla)\Big|_{\mathscr{M}_a} : (\Gamma_a, w) \to (\Gamma_a, w+1).$$

The theorem can be applied to Hammond's embedding [6] of certain Hilbert modular groups in Siegel's modular group of degree two. Let k denote a real quadratic number field with discriminant Δ which is a sum of two squares:

$$\Delta = u^2 + v^2, \quad v \equiv 0 \bmod 2. \tag{27}$$

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Let Γ_Δ denote the corresponding Hilbert modular group acting on the product $\mathscr{H}$ of halfplanes in the usual way, and write $\zeta = (\zeta_1, \zeta_2) \in \mathscr{H}$ as in §1. Put

$$\eta_1 = \frac{u+\sqrt{\Delta}}{2}, \eta_2 = \frac{u-\sqrt{\Delta}}{2}. \tag{28}$$

The map

$$\begin{aligned} \sqrt{\Delta}\, z_1 &= \eta_1\zeta_1 - \eta_2\zeta_2 \\ \sqrt{\Delta}\, z_2 &= -\eta_2\zeta_1 + \eta_1\zeta_2 \\ \sqrt{\Delta}\, z_3 &= +\sqrt{-|\eta|}\,(\zeta_1 - \zeta_2) \end{aligned} \tag{29}$$

embeds $\mathscr{H}$ onto the submanifold of codimension 1 in Siegel's upper half plane $\mathscr{Z}$ of degree two which is given by

$$\mathscr{M}_{u,v} = \left\{z \in \mathscr{Z} : \frac{v}{2}(z_1 - z_2) - uz_3 = 0\right\}. \tag{30}$$

Identify $\mathscr{Z}$ with the half space $\mathscr{Z}(\mathfrak{U})$ corresponding to the Jordan algebra $\mathfrak{U}$ of 2×2 symmetric real matrices; then σ denotes matrix trace and $|\cdot|$ denotes the determinant of a matrix. Set

$$a = \tfrac{1}{2}\begin{pmatrix} v & -u \\ -u & -v \end{pmatrix}; \tag{31}$$

then (30) can be written

$$\mathscr{M}_{u,v} = \{z \in \mathscr{Z}(\mathfrak{U}) : \sigma(az) = 0\}. \tag{32}$$

Observe that $a^2 = -\dfrac{\Delta}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; hence the theorem can be applied. It is known [6] that the embedding (29) induces an isomorphism of Γ_Δ into a subgroup of Siegel's modular group of degree two Γ. Write $\partial_k = \dfrac{\partial}{\partial z_k}$; then we have

THEOREM 2. *If* $\tilde{\phi} \in (\Gamma, w)$, *then*

$$\phi \equiv \left\{\frac{v}{2}(\partial_1 - \partial_2) - \frac{u}{2}\partial_3\right\}\tilde{\phi}\Big|_{\mathscr{M}_{u,v}} \in (\Gamma_\Delta, w+1).$$

Moreover, ϕ is *skew*, *i.e.*

$$\phi(\zeta_1, \zeta_2) = -\phi(\zeta_2, \zeta_1).$$

Only the last assertion has not been proved. But note that

$$z \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z_2 & -z_3 \\ -z_3 & z_1 \end{pmatrix} = z^{\#} \in \Gamma \text{ and}$$

$$\tilde{\phi}(z^{\#}) = \tilde{\phi}(z); \text{ hence } \phi(\zeta_1, \zeta_2) = \tilde{\phi}(z)\Big|_{\mathscr{M}_{u,v}} = \tilde{\phi}(z^{\#})\Big|_{\mathscr{M}_{u,v}} = \phi(\zeta_2, \zeta_1)$$

by (29), whereas the differential operator $\sigma(a\nabla)$ changes sign, under $z \mapsto z^{\#}$. This completes the proof.

Let the Fourier expansion of $\tilde{\phi} \in (\Gamma, w)$ be

$$\tilde{\phi}(z) = \sum_{n \geqq 0} \tilde{c}(n) e^{2\pi i \sigma(nz)}; \tag{33}$$

here the sum runs over all semi-integral semi-positive definite real symmetric matrices. Let

$$\phi(\zeta) = \sigma(a, \nabla\tilde{\phi})(z)\Big|_{\mathcal{M}_{u,v}} \tag{34}$$

with a given by (31), and let the Fourier expansion of ϕ be

$$\phi(\zeta) = 2\pi i \sum_{v/\sqrt{\Delta} \geqq 0} c(v) \exp 2\pi i \sigma\left(\frac{v\zeta}{\sqrt{\Delta}}\right), \tag{35}$$

where v runs through the integral elements of k such that $v/\sqrt{\Delta}$ is 0 or totally positive. Then comparison of (33) with (34) yields

THEOREM 3.

(a) $c(v) = \sum_{n \geqq 0} \left\{\frac{v}{2}(n_1 - n_2) - un_3\right\} \tilde{c}(n)$

$n_3 v + n_1 \eta_1 - n_2 \eta_2 = v$

where v runs through the integral elements of k such that $v/\sqrt{\Delta} > 0$;

(b) $c(0) = 0$;

(c) $c(\sqrt{\Delta}) = 0$.

COROLLARY 4. *If the discriminant Δ of a quadratic number field is a sum of two squares, then there is a skew non-identically zero Γ_{Δ}-cusp form of weight w for every odd $w \geqq 5$.*

Proof. There is an Eisenstein series $\tilde{\phi}_{\tilde{w}}$ of weight $\tilde{w} \equiv 0 \bmod 2$ for $\tilde{w} \geqq 4$; the corresponding ϕ_w is a skew cusp form of weight $w = \tilde{w}+1$. ϕ_w is not identically zero because

$$c\left(\frac{u+\sqrt{\Delta}}{2}\right) = c(\eta_1) = \frac{v}{2}\tilde{c}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \tag{36}$$

but $\tilde{c}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = (-)^{w/2}\dfrac{2w}{B_{w/2}}$, where $B_{w/2}$ is the Bernoulli number, $\neq 0$; this latter assertion is valid because the Siegel Φ-operator carries Eisenstein series to Eisenstein series of the same weight.

If $u = v/2 = 1$, then $\Delta = 5$ and the normalized cusp form χ_5 introduced in (4) turns out to be

$$\frac{1}{2\pi i}\cdot\frac{1}{240}\sigma(a, \nabla\tilde{\phi}_4)\Big|_{\mathcal{M}_{1,2}} = \chi_5, \tag{37}$$

where $\tilde{\phi}_4 \in (\Gamma, 4)$ is the normalized Eisenstein series of weight 4 and $a = \frac{1}{2}\begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$; thus, knowledge of the Fourier coefficients of $\tilde{\phi}_4$ enables one to calculate the Fourier coefficients of χ_5 by using Theorem 3. The entries in Table I were calculated in this manner, making use of the linear recursions given in [**12**] to calculate the coefficients of $\tilde{\phi}_4$.

It is evident from Theorem 3 that the values of the Fourier coefficients of $\tilde{\phi}_4$ also enable one to calculate the coefficients of a non-identically zero skew cusp form in $(\Gamma_\Delta, 5)$, but it is not known whether this form spans the space $(\Gamma_\Delta, 5)$ except for $\Delta = 5, 8$, in which cases the answer is affirmative [**4**], [**5**], [**12**]. Nevertheless, it may be of some interest to tabulate some of these coefficients for fields with class number greater than 1. This question is taken up in §4 below.

Equation (36) shows that the sum in Theorem 3 telescopes to one term if $v = (u+\sqrt{\Delta})/2$. This example is a special case of a general identity theorem which it may not be out of place to include here. Set $\mathcal{D}(r) = \{x \in \mathfrak{U} : \sigma(x) \leqq r\} \cap \exp \mathfrak{U}$; $\mathcal{D}(r)$ is a closed disk. The semi-integral matrices $n \geqq 0$ such that $\sigma(n) = r$ constitute the intersection of a lattice in $\mathfrak{U}$ with $\mathcal{D}(r)$. If we put $\mathfrak{U} \in x = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$, $X = x_1 - x_2$, $Y = 2x_3$, then $\mathcal{D}(r) = \{x : (x_1-x_2)^2 + 4x_3^2 \leqq r^2 = (x_1+x_2)^2\}$ and the semi-integral matrices $n \geqq 0$ in $\mathcal{D}(r)$ correspond to coordinate pairs $(X, Y) \in \mathbb{Z}^2$ such that $X \equiv r \pmod 2$ and $X^2 + Y^2 \leqq r^2$. Moreover, we find from (30) that $\mathcal{D}(r) \cap \mathcal{M}_{u,v} = \{x \in \mathcal{D}(r) : vX - uY = 0\}$. Now suppose that $\tilde{\phi} \in (\Gamma, w)$; then $\tilde{\phi}|_{\mathcal{M}_{u,v}} \in (\Gamma_\Delta, w)$ and the Fourier coefficients $c(\nu)$ of $\tilde{\phi}|_{\mathcal{M}_{u,v}}$ can be expressed in terms of the coefficients $\tilde{c}(n)$ of $\tilde{\phi}$ by the formula

$$c(\nu) = \sum_{\substack{n \geqq 0 \\ n_3 v + n_1\eta_1 - n_2\eta_2 = \nu}} \tilde{c}(n); \tag{38}$$

(cf. the analogous formula in Theorem 3). The condition $n_3 v + n_1\eta_1 - n_2\eta_2 = \nu$ is equivalent to the pair

$$\sigma(n) = \sigma(\nu/\sqrt{\Delta})$$

and

$$uM + vN = \nu + \nu^*$$

where $M = n_1 - n_2$, $N = 2n_3$, and ν^* is the conjugate of ν. The lattice points (M, N) in $\mathcal{D}(r)$ consequently lie on a line perpendicular to $\mathcal{D}(r) \cap \mathcal{M}_{u,v}$. It follows that if n is fixed, then for all but a finite number of discriminants Δ, the line through n perpendicular to $\mathcal{M}_{u,v}$ will not contain any other semi-integral matrix $n \geqq 0$ and therefore

$$c(\nu) = \tilde{c}(n) \text{ with } \nu = n_3 v + n_1\eta_1 - n_2\eta_2 \tag{39}$$

for all but a finite number of discriminants. This observation leads to

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THEOREM 5. *There is an integer K depending only on w such that if $\tilde{\phi}$, $\tilde{\psi} \in (\Gamma, w)$ and $c(v)$ are the Fourier coefficients of $(\tilde{\phi}-\tilde{\psi})|_{\mathcal{M}_{u,v}}$, then, for all but a finite number of discriminants, $c(v) = 0$ for $\sigma(v/\sqrt{\Delta}) < K$ implies that $\tilde{\phi} \equiv \tilde{\psi}$.*

Proof. $\dim_{\mathbf{C}}(\Gamma, w) < \infty$ implies the existence of a K which depends only on w such that $\tilde{c}(n) = 0$ for $\sigma(n) < K$ gives $\tilde{\phi} \equiv 0$, where the $\tilde{c}(n)$ are the Fourier coefficients of $\tilde{\phi} \in (\Gamma, w)$. Let $\mathscr{S}_K$ denote the set of slopes of the lines determined by pairs of semi-integral matrices $n' \geqq 0$, $n'' \geqq 0$ such that $\sigma(n') = \sigma(n'') < K$. If $\Delta = u^2+v^2$ is the discriminant of the field generated by a positive square free integer, then u and v can have at most the factor 2 in common and therefore the set of slopes of the line segments $\{\mathscr{D}(r) \cap \mathcal{M}_{u,v} : \Delta = u^2+v^2$ is a discriminant and $v \equiv 0 \pmod 2\}$ is the set of distinct numbers $\{v/(2u) : u^2+v^2$ is a discriminant and $v \equiv 0 \pmod 2\}$. Hence, for all but a finite number of discriminants, $v/(2u) \notin \mathscr{S}_K$ and consequently (39) is valid for $\sigma(n) < K$. It follows that $c(v) = 0$ for $\sigma(v/\sqrt{\Delta}) < K$ implies that $\tilde{\phi} \equiv 0$. Now replace $\tilde{\phi}$ by $(\tilde{\phi}-\tilde{\psi})$ to obtain the statement of the Theorem.

Let $c(v)$ denote the vth Fourier coefficient of $\phi = \sigma(a, \nabla\tilde{\phi})|_{\mathcal{M}_{u,v}}$, $a = \frac{1}{2}\begin{pmatrix} v & -u \\ -u & -v \end{pmatrix}$. A similar argument shows that there is a constant K such that $c(v) = 0$ for $\sigma(v/\sqrt{\Delta}) < K$ implies

TABLE III. FOURIER COEFFICIENTS OF THE NORMALIZED CUSP FORM χ_5 FOR HILBERT'S MODULAR GROUP ASSOCIATED WITH $\mathbf{Q}(\sqrt{10})$ FOR REDUCED $v/\sqrt{40}$.
$v = k+l(20+\sqrt{10})$.

abs $\lvert v\rvert$	l	$-k$	$c(v)$	abs $\lvert v\rvert$	l	$-k$	$c(v)$
1	1	17	1	186	5	92	24192
4	2	34	18	201	5	93	10332
36	2	38	−6	214	5	94	−25344
39	2	39	−168	225	5	95	−37785
40	2	40	0	234	5	96	−24192
9	3	51	84	241	5	97	11592
65	3	55	−630	246	5	98	48384
74	3	56	−1152	249	5	99	43092
81	3	57	756	250	5	100	0
86	3	58	2304	36	6	102	252
89	3	59	882	104	6	104	−1008
16	4	68	292	135	6	105	−7560
79	4	71	−392	164	6	106	−8316
96	4	72	−3024	191	6	107	5544
111	4	73	−1512	216	6	108	30240
124	4	74	4144	239	6	109	31320
135	4	75	7560	260	6	110	−10080
144	4	76	5940	279	6	111	−65016
151	4	77	−4568	296	6	112	−83232
156	4	78	−15120	311	6	113	−24192
159	4	79	−12096	324	6	114	61488
160	4	80	0	335	6	115	98280
25	5	85	630	344	6	116	79776
106	5	88	−3456	351	6	117	−3528
129	5	89	−6216	356	6	118	−77364
150	5	90	0	359	6	119	−71856
169	5	91	12474	360	6	120	0

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that $\tilde{\phi} = 0$. Indeed, according to Theorem 3 and the argument given above, for all but finitely many discriminants, $\sigma(\nu/\sqrt{\Delta}) < K$ implies that

$$c(\nu) = \left\{\frac{v}{2}(n_1 - n_2) - un_3\right\}\tilde{c}(n)$$

with $\nu = n_3 v + n_1\eta_1 - n_2\eta_2$. Thus $c(\nu) = 0$ but $\tilde{c}(n) \neq 0$ implies that $(v/2)(n_1 - n_2) = un_3$ in this range. But one readily proves that a Siegel modular form cannot have all its non-zero Fourier coefficients $\tilde{c}(n)$, $\sigma(n) < K$ supported on a linear submanifold of positive codimension.

4. Applications with $\Delta = 40$ and $\Delta = 229$. In this section Theorem 3 is applied to obtain the leading Fourier coefficients of $\chi_5 \equiv \sigma(a, \nabla\tilde{\phi}_4)|_{\mathscr{M}_{u,v}}$ for $\Delta = 40$ and $\Delta = 229$. These cases are of special interest because the class number of the corresponding fields are greater

TABLE IV. FOURIER COEFFICIENTS OF THE NORMALIZED CUSP FORM χ_5 FOR HILBERT'S MODULAR GROUP ASSOCIATED WITH $\mathbb{Q}(229^{\frac{1}{2}})$ FOR REDUCED $\nu/\sqrt{229}$.

$$\nu = k + l\left(\frac{229+\sqrt{229}}{2}\right).$$

abs $\lvert\nu\rvert$	l	$-k$	$c(\nu)$	*abs* $\lvert\nu\rvert$	l	$-k$	$c(\nu)$
1	1	107	1	1321	5	562	−86688
4	2	214	18	1341	5	563	−105840
225	2	227	−15	1359	5	564	−78624
228	2	228	−420	1375	5	565	11592
229	2	229	0	1389	5	566	102816
9	3	321	84	1401	5	567	120960
425	3	334	−1764	1411	5	568	94752
443	3	335	−3744	1419	5	569	23436
459	3	336	756	36	6	642	1512
473	3	337	4896	905	6	653	−3276
485	3	338	2016	972	6	654	−27972
16	4	428	292	1037	6	655	−45738
592	4	440	−1148	1100	6	656	−19404
627	4	441	−9828	1161	6	657	30240
660	4	442	−8316	1220	6	658	63756
691	4	443	4144	1277	6	659	79002
720	4	444	14364	1332	6	660	40068
747	4	445	12852	1385	6	661	4284
772	4	446	1372	1661	6	667	−20448
900	4	454	−270	1700	6	668	−144648
907	4	455	−12960	1737	6	669	−247968
912	4	456	−37800	1772	6	670	−277056
915	4	457	−30240	1805	6	671	−133056
916	4	458	0	1836	6	672	61992
25	5	535	630	1865	6	673	229824
729	5	546	−27	1892	6	674	362304
781	5	547	−11232	1917	6	675	296352
831	5	548	−24864	1940	6	676	165312
879	5	549	−18144	1961	6	677	22752
925	5	550	12474	2025	6	681	−1260
969	5	551	42336	2036	6	682	−56700
1011	5	552	37296	2045	6	683	−226800
1051	5	553	14688	2052	6	684	−341460
1089	5	554	33	2057	6	685	−374220
1299	5	561	−21924	2060	6	686	−204120
				2061	6	687	0

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than 1. In fact, 40 is the least discriminant corresponding to a field of class number 2 which is a sum of two squares, and 229 is the least discriminant of a field of class number 3 which is prime and a sum of two squares. It is not known whether the Hilbert modular form χ_5 is an eigenfunction of the Hecke ring in either of these cases.

A basis for the ring of integers of the field k_Δ of discriminant Δ is, in any case, given by 1 and $\omega = (\Delta+\sqrt{\Delta})/2$. If $v = k+l\omega$, then

$$\sigma\left(\frac{v}{\sqrt{\Delta}}\right) = l.$$

Tables III and IV above present some Fourier coefficients $c(v)$ of χ_5 for $\Delta = 40$ and $\Delta = 229$ respectively. In both cases, v is constrained to lie in the domain $\left\{\alpha : -1 \geqq \frac{\alpha_1}{\alpha_2} \geqq \frac{\eta_1}{\eta_2}\right\}$, where $\eta_1 = (u+\sqrt{\Delta})/2$, and $(u, v) = (6, 4)$ for $\Delta = 40$ and $(u, v) = (15, 4)$ for $\Delta = 229$. Note that η_1 is a fundamental unit in each case.

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Rice University
Houston, Texas 77001
U.S.A.

Present address:
University of California
Irvine
California 92664, U.S.A.