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A NOTE ON THE EXTREMAL INDEX FOR SPACE–TIME PROCESSES

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Abstract

Let $\{X(\boldsymbol{s}, t),\ \boldsymbol{s} = (s_1, s_2) \in \mathbb{R}^2,\ t \in \mathbb{R}\}$ be a stationary random field defined over a discrete lattice. In this paper, we consider a set of domain of attraction criteria giving the notion of extremal index for random fields. Together with the extremal-types theorem given by Leadbetter and Rootzen (1997), this will give a characterization of the limiting distribution of the maximum of such random fields.

Keywords: Extremal index; random field

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1. Introduction

We consider the asymptotic distribution of the maximum of a stationary random field defined over a discrete lattice in $\mathbb{R}^3$. We will take the first two coordinates as space and the third coordinate as time; hence, we call this random field a spatiotemporal process. We are motivated by the extremal properties of linear spatiotemporal autoregressive moving average processes (Cliff and Ord (1975); see also Cressie (1993, pp. 449–450)) given by

$$\boldsymbol{X}(t) - \sum_{k=0}^{p} \boldsymbol{B}_k \boldsymbol{X}(t-k) = \boldsymbol{\varepsilon}(t) - \sum_{l=0}^{q} \boldsymbol{E}_l \boldsymbol{\varepsilon}(t-l),$$

where $\boldsymbol{X}(t) = (X(\boldsymbol{s}_i, t),\ i = 1, 2, \dots, n)^\top$, is a vector process defined at spatial locations $\boldsymbol{s}_i$, $i = 1, 2, \dots, n$, and at time points $t = 1, 2, \dots, T$; $\boldsymbol{B}_k$ and $\boldsymbol{E}_l$ are matrices of constants satisfying certain restrictions; and $\boldsymbol{\varepsilon}(t) = (\varepsilon(\boldsymbol{s}_i, t),\ i = 1, 2, \dots, n)^\top$, $t = 1, 2, \dots, T$, are independent and identically distributed random variables at space–time locations $(\boldsymbol{s}_i, t)$.

The traditional way of obtaining limiting results for the maximum of a stationary sequence is as follows.

1. Prove an extremal types-theorem, which shows that under a long-range dependence condition the maximum of the field is the maximum of an approximately independent sequence of submaxima.
2. Obtain domain of attraction criteria, which characterize the limiting distribution function of the maximum in terms of the tail of the common marginal distribution and local dependence behavior of the sequence, given in terms of the extremal index.

Leadbetter and Rootzen (1997) proved an extremal-types theorem for random fields in $\mathbb{R}^2$, under a weak coordinatewise-mixing (CW-mixing) condition. Although we will generalize

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this result to $\mathbb{R}^3$ (which is straightforward), our main contribution will be to obtain a set of domain of attraction criteria. We will show that the asymptotic distribution of the maximum of the spatiotemporal process defined over a lattice in $\mathbb{R}^3$ can be characterized in terms of the tail of the distribution function of the process, as well as three coordinatewise conditions that describe the propensity of consecutive large values of the process to cluster in each coordinate direction.

Hence, the outline of the paper is as follows. In Section 2, we give the CW-mixing condition of Leadbetter and Rootzen (1997), adapted for spatiotemporal processes. We also prove the extremal-types theorem under this condition. Although the proof is a straightforward extension of the extremal-types theorem given by Leadbetter and Rootzen (1997), we give the full proof for completeness and the reader's convenience. In Section 3, we define the notion of coordinatewise extremal indices, which resembles the definition of the extremal index given by O'Brien (1987) for stationary sequences, and prove the validity of a set of domain of attraction criteria based on these three coordinatewise conditions.

2. Extremal-types theorem

Let $X(\boldsymbol{s}, t)$ be a stationary spatiotemporal process defined over a discrete lattice

$$E_{\boldsymbol{n}} = \{(i, j, k),\ i = 0, 1, \dots, n_1,\ j = 0, 1, \dots, n_2,\ k = 0, 1, \dots, n_3\},$$

with $\boldsymbol{n} = (n_1, n_2, n_3)$. Here, we take $(i, j) = \boldsymbol{s}$ and $k = t$ respectively to be the position in the (x, y)-plane and the time at which the random field $X(\boldsymbol{s}, t)$ is evaluated. Let $F(x) = \mathrm{P}(X(0, 0, 0) \le x)$ be the marginal distribution function of the process.

For any subset $B \in E_{\boldsymbol{n}}$, define

$$M(B) = \max_{(i,j,k)\in B} X(i, j, k).$$

We are interested in the asymptotic distribution of $M(E_{\boldsymbol{n}})$, when suitably normalized, as $n_1 \to \infty$, $n_2 \to \infty$, and $n_3 \to \infty$ (we write $\lim_{\boldsymbol{n}\to\infty}$ for $\lim_{n_1\to\infty,\, n_2\to\infty,\, n_3\to\infty}$); that is, in

$$\lim_{\boldsymbol{n}\to\infty} \mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}(x)),$$

for some suitably chosen normalizing constants $b_{\boldsymbol{n}}$ and $a_{\boldsymbol{n}}$ such that $u_{\boldsymbol{n}} = a_{\boldsymbol{n}}x + b_{\boldsymbol{n}}$.

The following extension of the CW-mixing condition of Leadbetter and Rootzen (1997) yields the extremal-types theorem needed to characterize the limiting distribution of $M(E_{\boldsymbol{n}})$.

Let r_1, r_2, and r_3 be integers defining the lengths of blocks of cubes

$$B_{ijk} = [(i-1)r_1, ir_1] \times [(j-1)r_2, jr_2] \times [(k-1)r_3, kr_3],$$

which will be used for subdivision of $E_{\boldsymbol{n}}$. Assume that as $n_1 \to \infty$, $n_2 \to \infty$, and $n_3 \to \infty$, r_1, r_2, and r_3 all tend to ∞ in such a way that $r_i = o(n_i)$, $i = 1, 2, 3$. Let $m_i = [n_i/r_i]$, $i = 1, 2, 3$, be integers such that $E_{\boldsymbol{n}}$ contains $m_{\boldsymbol{n}} = m_1m_2m_3$ complete blocks and no more than $(m_1 + m_2 + m_3 + 1)$ incomplete blocks. (Note that $[\cdot]$ denotes the integer-part function.) Without loss of generality, we assume that, for every i, $m_ir_i = n_i$, in order that there be no incomplete blocks. This assumption eases notational difficulties and should not effect the asymptotic results. Let $l_i \equiv l_i(n_i)$, $i = 1, 2, 3$, be integers tending to ∞ in such a way that $l_i = o(r_i)$. With this notation, the random field is said to satisfy the CW-mixing condition (Leadbetter and Rootzen (1997)) for a given family of levels $u_{\boldsymbol{n}}$ and separation constants l_i if the following conditions are satisfied.

Condition 1. (t-direction condition.) *For each i and j with $0 < i \le n_1$ and $0 < j \le n_2$, and for cubes*

$$B_1 = [0, i] \times [0, j] \times [0, a] \quad \textit{and} \quad B_2 = [0, i] \times [0, j] \times [b, c]$$

with $0 < a < n_3$, $a + l_3 \le b < c \le n_3$, and $c - b \le r_3$, we have

$$|\mathrm{P}(M(B_1) \le u_{\boldsymbol{n}},\ M(B_2) \le u_{\boldsymbol{n}}) - \mathrm{P}(M(B_1) \le u_{\boldsymbol{n}})\,\mathrm{P}(M(B_2) \le u_{\boldsymbol{n}})| \le \alpha_3(r_3, l_3), \tag{1}$$

where the t-coordinate mixing function α_3 satisfies $m_3\alpha_3(r_3, l_3) \to 0$ as $n_3 \to \infty$. Note that, with this condition, maxima defined over the cubes of size $n_1 \times n_2 \times r_3$ (or smaller in the (x, y)-plane) separated along the time direction by cubes of size $n_1 \times n_2 \times l_3$ are asymptotically independent as $n_3 \to \infty$.

Condition 2. (x-direction condition.) *For each j, $0 < j \le n_2$, and cubes*

$$B_1 = [0, a] \times [0, j] \times [0, r_3] \quad \textit{and} \quad B_2 = [b, c] \times [0, j] \times [0, r_3]$$

with $0 < a < n_1$, $a + l_1 \le b < c \le n_1$, and $c - b \le r_1$, we have

$$|\mathrm{P}(M(B_1) \le u_{\boldsymbol{n}},\ M(B_2) \le u_{\boldsymbol{n}}) - \mathrm{P}(M(B_1) \le u_{\boldsymbol{n}})\,\mathrm{P}(M(B_2) \le u_{\boldsymbol{n}})| \le \alpha_1(r_1, r_3, l_1), \tag{2}$$

where the x-coordinate mixing function α_1 satisfies $m_1 m_3 \alpha_1(r_1, r_3, l_1) \to 0$ as $n_3 \to \infty$ and $n_1 \to \infty$. The x-coordinate mixing condition says that maxima defined over strips of cubes of size $r_1 \times n_2 \times r_3$ (or smaller) are asymptotically independent, provided that they are separated along the x-direction by strips of cubes of size $l_1 \times n_2 \times r_3$.

Condition 3. (y-direction condition.) *For cubes*

$$B_1 = [0, r_1] \times [0, a] \times [0, r_3] \quad \textit{and} \quad B_2 = [0, r_1] \times [b, c] \times [0, r_3]$$

with $0 < a < n_2$, $a + l_2 \le b < c \le n_2$, and $c - b \le r_2$, we have

$$|\mathrm{P}(M(B_1) \le u_{\boldsymbol{n}},\ M(B_2) \le u_{\boldsymbol{n}}) - \mathrm{P}(M(B_1) \le u_{\boldsymbol{n}})\,\mathrm{P}(M(B_2) \le u_{\boldsymbol{n}})| \le \alpha_2(r_1, r_2, r_3, l_2),$$

where the y-direction mixing function α_2 satisfies

$$m_1 m_2 m_3 \alpha_2(r_1, r_2, r_3, l_2) \to 0 \tag{3}$$

as $n_1 \to \infty$, $n_2 \to \infty$, and $n_3 \to \infty$. Again, under the y-direction mixing condition, the maxima of the process over cubes of size $r_1 \times r_2 \times r_3$ are asymptotically independent, provided that these cubes are separated along the y-direction by cubes of size $r_1 \times l_2 \times r_3$. Note that, as we cut out smaller cubes, we need stricter conditions on the corresponding mixing functions and, hence, stricter conditions on the asymptotic independence of the maxima over smaller cubes.

The following result, due to Leadbetter and Rootzen (1997), essentially says that, under the above separate mixing conditions on each of the three directions, for $i = 1, \dots, m_1$, $j = 1, \dots, m_2$, and $k = 1, \dots, m_3$ the $m_1 m_2 m_3$ processes $X(i, j, k)$ defined over the blocks

$$B_{ijk} = [(i-1)r_1, ir_1] \times [(j-1)r_2, jr_2] \times [(k-1)r_3, kr_3],$$

each of size $r_1 r_2 r_3$, are asymptotically independent.

Lemma 1. *Assume that the stationary random field $X(i, j, k)$ satisfies the CW-mixing condition given above, for an appropriately chosen level $u_{\boldsymbol{n}}$. Then, for*

$$\begin{aligned} E_{\boldsymbol{n}} &= [0, n_1] \times [0, n_2] \times [0, n_3] \\ &= [0, m_1 r_1] \times [0, m_2 r_2] \times [0, m_3 r_3] \\ &= \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} \bigcup_{k=1}^{m_3} B_{ijk}, \end{aligned}$$

we find that

$$\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) = \mathrm{P}^{m_1 m_2 m_3}(M(J) \le u_{\boldsymbol{n}}) + o_{\boldsymbol{n}}(1),$$

where $J = B_{111} = [0, r_1] \times [0, r_2] \times [0, r_3]$, as n_1, n_2, and n_3 all tend to ∞.

Proof. With the above notation, write

$$\begin{aligned} J_{1,i} &= [0, m_1 r_1] \times [0, m_2 r_2] \times [(i-1) r_3, i r_3], \\ J_{2,i} &= [0, m_1 r_1] \times [0, m_2 r_2] \times [(i-1) r_3 + l_3, i r_3 - l_3], \\ J_i^* &= J_{1,i} - J_{2,i}. \end{aligned}$$

Then $E_{\boldsymbol{n}} = \bigcup_{i=1}^{m_3} J_{1,i}$ and, using stationarity, for $2 \le m \le m_3$,

$$\begin{aligned} 0 &\le \mathrm{P}\bigg(M\bigg(\bigcup_{i=1}^{m-1} J_{1,i}\bigg) \le u_{\boldsymbol{n}},\ M(J_{2,m}) \le u_{\boldsymbol{n}}\bigg) - \mathrm{P}\bigg(M\bigg(\bigcup_{i=1}^{m} J_{1,i}\bigg) \le u_{\boldsymbol{n}}\bigg) \\ &\le \mathrm{P}(M(J_m^*) > u_{\boldsymbol{n}}) \\ &= \mathrm{P}(M(J_1^*) > u_{\boldsymbol{n}}). \end{aligned} \tag{4}$$

Furthermore,

$$\begin{aligned} &\bigg|\mathrm{P}\bigg(M\bigg(\bigcup_{i=1}^{m-1} J_{1,i}\bigg) \le u_{\boldsymbol{n}},\ M(J_{2,m}) \le u_{\boldsymbol{n}}\bigg) - \mathrm{P}\bigg(M\bigg(\bigcup_{i=1}^{m-1} J_{1,i}\bigg) \le u_{\boldsymbol{n}}\bigg)\mathrm{P}(M(J_{2,m}) \le u_{\boldsymbol{n}})\bigg| \\ &\qquad \le \alpha_3(r_3, l_3) \end{aligned} \tag{5}$$

and

$$0 \le \mathrm{P}(M(J_{2,m}) \le u_{\boldsymbol{n}}) - \mathrm{P}(M(J_{1,m}) \le u_{\boldsymbol{n}}) \le \mathrm{P}(M(J_m^*) > u_{\boldsymbol{n}}); \tag{6}$$

hence, it follows, using stationarity, that

$$\begin{aligned} &\bigg|\mathrm{P}\bigg(M\bigg(\bigcup_{i=1}^{m} J_{1,i}\bigg) \le u_{\boldsymbol{n}}\bigg) - \mathrm{P}\bigg(M\bigg(\bigcup_{i=1}^{m-1} J_{1,i}\bigg) \le u_{\boldsymbol{n}}\bigg)\mathrm{P}(M(J_{1,1}) \le u_{\boldsymbol{n}})\bigg| \\ &\qquad \le \alpha_3(r_3, l_3) + 2\,\mathrm{P}(M(J_1^*) > u_{\boldsymbol{n}}). \end{aligned} \tag{7}$$

By applying (6) repeatedly, from CW-mixing we obtain

$$|\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) - \mathrm{P}^{m_3}(M(J_{1,1}) \le u_{\boldsymbol{n}})| \le 2 m_3\, \mathrm{P}(M(J_1^*) > u_{\boldsymbol{n}}) + o_{\boldsymbol{n}}(1). \tag{8}$$

We will now show that

$$\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) - \mathrm{P}^{m_3}(M(J_{1,1}) \le u_{\boldsymbol{n}}) \to 0. \tag{9}$$

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It is sufficient to show that this holds as $\boldsymbol{n} \to \infty$ in a manner such that $\mathrm{P}^{m_3}(M(J_1^*) \leq u_{\boldsymbol{n}})$ converges to some ρ, $0 \leq \rho \leq 1$. If $\rho = 1$ then $\mathrm{P}(M(J_1^*) > u_{\boldsymbol{n}}) \to 0$, and since $m_3 \log \mathrm{P}(M(J_1^*) \leq u_{\boldsymbol{n}}) \to 0$ it follows that $m_1 \mathrm{P}(M(J_1^*) > u_{\boldsymbol{n}}) \to 0$ and that (9) is a consequence of (8).

On the other hand, if $\rho < 1$ then, since $m_3\alpha_3(r_3, l_3) \to 0$ and $l_3 = o(r_3)$, there exists a $\beta_{\boldsymbol{n}} \to \infty$ such that $m_3\beta_{\boldsymbol{n}}\alpha_3(r_3, l_3) \to 0$ and $\beta_{\boldsymbol{n}} l_3 = o(r_3)$. Hence, for sufficiently large $\boldsymbol{n}$, $\beta_{\boldsymbol{n}}$ cubes congruent to J_1^* and mutually separated by at least a distance l_3 in the t-direction may be chosen in $J_{2,1}$. Arguments parallel to those yielding (4)–(8) then imply that

$$\mathrm{P}(M(J_{1,1}) \leq u_{\boldsymbol{n}}) \leq \mathrm{P}^{\beta_{\boldsymbol{n}}}(M(J_1^*) \leq u_{\boldsymbol{n}}) + \beta_{\boldsymbol{n}}\alpha_3(l_3, r_3), \tag{10}$$

whence

$$\begin{aligned}\mathrm{P}^{m_3}(M(J_{1,1}) \leq u_{\boldsymbol{n}}) &\leq \mathrm{P}^{m_3\beta_{\boldsymbol{n}}}(M(J_1^*) \leq u_{\boldsymbol{n}}) + \beta_{\boldsymbol{n}} m_3\alpha_3(l_3, r_3) \\ &= (\rho + o(1))^{\beta_{\boldsymbol{n}}} \\ &\to 0. \end{aligned} \tag{11}$$

Hence, the second term of the difference in (9) tends to 0. Finally, it follows similarly that

$$\mathrm{P}(M(E_{\boldsymbol{n}}) \leq u_{\boldsymbol{n}}) \leq \mathrm{P}^{m_3}(M(J_{2,1}) \leq u_{\boldsymbol{n}}) + m_3\alpha_3(l_3, r_3),$$

which tends to 0 since (10) and, hence, (11) apply with $M(J_{2,1})$ in place of $M(J_{1,1})$. Both terms on the left-hand side of (9) therefore tend to 0 if $\rho < 1$, and the convergence again holds.

To complete the proof of the lemma it is sufficient to establish that

$$\mathrm{P}^{m_3}(M(J_{1,1}) \leq u_{\boldsymbol{n}}) - \mathrm{P}^{m_1m_3}(M([0, r_1) \times [0, n_2] \times [0, r_3]) \leq u_{\boldsymbol{n}}) \to 0$$

and

$$\mathrm{P}^{m_1m_3}(M([0, r_1) \times [0, n_2] \times [0, r_3]) \leq u_{\boldsymbol{n}}) - \mathrm{P}^{m_1m_2m_3}(M(J) \leq u_{\boldsymbol{n}}) \to 0.$$

We do this by splitting first the cube $J_{1,1}$ into cubes $[(i-1)r_1, ir_1] \times [0, n_2] \times [0, r_3]$, $1 \leq i \leq m_1$, and repeating the above argument, then splitting the cube $[0, r_1) \times [0, n_2] \times [0, r_3]$ into cubes $[0, r_1) \times [(j-1)r_2, jr_2] \times [0, r_3]$, $j = 1, \ldots, m_2$, and repeating the argument a final time.

3. Domain of attraction criteria: the notion of extremal index in space–time

We now approximate $\mathrm{P}(M(J) \leq u_{\boldsymbol{n}})$, where

$$J = \{(i, j, k),\ 0 \leq i \leq r_1,\ 0 \leq j \leq r_2,\ 0 \leq k \leq r_3\},$$

for some normalizing constants $a_{\boldsymbol{n}}$ and $b_{\boldsymbol{n}}$ such that $u_{\boldsymbol{n}} = a_{\boldsymbol{n}}x + b_{\boldsymbol{n}}$ with

$$\lim_{\boldsymbol{n}\to\infty} n_1n_2n_3\,\mathrm{P}(X(0, 0, 0) > u_{\boldsymbol{n}}) = \tau(x) > 0.$$

Note that if the random field is independent as well as stationary, then

$$\mathrm{P}(M(J) \leq u_{\boldsymbol{n}}) = \mathrm{P}^{r_1r_2r_3}(X(0, 0, 0) \leq u_{\boldsymbol{n}}).$$

Our aim is to study the asymptotic effect on $\mathrm{P}(M(J) \leq u_{\boldsymbol{n}})$ of the local dependence structure of the random field.

For $k = 0, 1, \dots, r_3$ let

$$\zeta_k = \max_{0 \le i \le r_1,\, 0 \le j \le r_2} X(i, j, k),$$

and for $0 \le i \le r_1$ let

$$\nu_i = \max_{0 \le j \le r_2} X(i, j, 0).$$

Note that the ζ_k are the maxima over the (spatial) coordinates $\{(i, j),\ 0 \le i \le r_1,\ 0 \le j \le r_2\}$ at each time point $k = 1, \dots, r_3$. We will call them the maxima of time plates. For each fixed $i = 0, 1, \dots, r_1$, the ν_i are the maxima over the coordinates $\{(i, j),\ 0 \le j \le r_2\}$ at fixed $k = 0$. We will call them the maxima of y-arrays.

The local dependence structure of the stationary spatiotemporal process will be characterized by three conditions, which represent the propensity of the large values of the process to cluster along the t- , x-, and y-coordinates.

Assume that the following limits exist.

1. t-coordinate clustering propensity:

$$\lim_{\boldsymbol{n} \to \infty} \mathrm{P}(\zeta_1 \le u_{\boldsymbol{n}},\ \zeta_2 \le u_{\boldsymbol{n}}, \dots, \zeta_{r_3} \le u_{\boldsymbol{n}} \mid \zeta_0 > u_{\boldsymbol{n}}) = \theta_1 \tag{12}$$

for some $\theta_1,\ 0 < \theta_1 \le 1$. This condition indicates how the largest values over the (x, y)-plane cluster at consecutive time points.

2. x-coordinate clustering propensity:

$$\lim_{\boldsymbol{n} \to \infty} \mathrm{P}(\nu_1 \le u_{\boldsymbol{n}},\ \nu_2 \le u_{\boldsymbol{n}}, \dots, \nu_{r_1} \le u_{\boldsymbol{n}} \mid \nu_0 > u_{\boldsymbol{n}}) = \theta_2 \tag{13}$$

for some $\theta_2,\ 0 < \theta_2 \le 1$. This condition indicates how largest values in the y-columns cluster along the x-direction at a fixed point of time.

3. y-coordinate clustering propensity:

$$\lim_{\boldsymbol{n} \to \infty} \mathrm{P}(X(0, 1, 0) \le u_{\boldsymbol{n}},\ X(0, 2, 0) \le u_{\boldsymbol{n}}, \dots, X(0, r_2, 0) \le u_{\boldsymbol{n}} \mid X(0, 0, 0) > u_{\boldsymbol{n}}) = \theta_3 \tag{14}$$

for some $\theta_3,\ 0 < \theta_3 \le 1$. This condition indicates how the large values of the process cluster along the y-direction at a fixed x-coordinate and a fixed time.

We will show, by extending the results of O'Brien (1987), that these coordinatewise clustering conditions uniquely characterize how the process clusters in time and space and give us domain of attraction criteria.

Theorem 1. *Assume that $u_{\boldsymbol{n}} = u_{\boldsymbol{n}} x + b_{\boldsymbol{n}}$ is chosen such that*

$$\lim_{\boldsymbol{n} \to \infty} n_1 n_2 n_3 (1 - F(u_{\boldsymbol{n}})) \to \tau(x).$$

Assume that the process $\{X(i, j, k),\ (i, j, k) \in E_{\boldsymbol{n}}\}$ satisfies the CW-mixing condition given in (1), (2), and (3) for l_i, $i = 1, 2, 3$ and $u_{\boldsymbol{n}}$ chosen as above. Assume further that the limits in (12), (13), and (14) exist. Then

$$\lim_{\boldsymbol{n} \to \infty} |\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) - \exp(-\theta_1 \theta_2 \theta_3 \tau(x))| = 0.$$

Proof. We extend O'Brien's (1987) technique to random fields. Let

$$M^{\zeta}_{r_3} = \max_{0\le k\le r_3} \zeta_k \quad \text{and} \quad M^{\zeta}_{k_1,k_2} = \max_{k_1+1\le k\le k_2} \zeta_k;$$

then

$$M^{\zeta}_{0,r_3} = \max_{1\le k\le r_3} \zeta_k.$$

Similarly, define

$$M^{\nu}_{r_1} = \max_{0\le i\le r_1} \nu_i \quad \text{and} \quad M^{\nu}_{i_1,i_2} = \max_{i_1+1\le i\le i_2} \nu_i;$$

then

$$M^{\nu}_{0,r_1} = \max_{1\le i\le r_1} \nu_i.$$

Finally, let

$$M^{X}_{r_2} = \max_{0\le j\le r_2} X(0, j, 0) \quad \text{and} \quad M^{X}_{j_1,j_2} = \max_{j_1+1\le j\le j_2} X(0, j, 0);$$

then

$$M^{X}_{0,r_2} = \max_{1\le j\le r_2} X(0, j, 0).$$

Thus,

$$\begin{aligned} \mathrm{P}(M(J) > u_{\boldsymbol{n}}) &= \mathrm{P}\Big(\max_{0\le i\le r_3} \zeta_i > u_{\boldsymbol{n}}\Big) \\ &= \mathrm{P}\Big(\bigcup_{i=0}^{r_3} (\zeta_i > u_{\boldsymbol{n}})\Big) \\ &= \sum_{i=0}^{r_3} \mathrm{P}(\zeta_i > u_{\boldsymbol{n}},\ M^{\zeta}_{i,r_3} \le u_{\boldsymbol{n}}), \end{aligned}$$

which follows from the fact that, for any events $A_1, \dots, A_n$,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n}\Big(A_i \cap \bigcap_{k=i+1}^{n} A^{\mathrm{c}}_j\Big)$$

and the events $A_i \cap \bigcap_{k=i+1}^{n} A^{\mathrm{c}}_j$, $i = 1, \dots, n$, are mutually exclusive.

Now,

$$\begin{aligned} \mathrm{P}(\zeta_i > u_{\boldsymbol{n}},\ M^{\zeta}_{i,r_3} \le u_{\boldsymbol{n}}) &= \mathrm{P}(\zeta_i > u_{\boldsymbol{n}},\ \zeta_{i+1} \le u_{\boldsymbol{n}}, \dots, \zeta_{r_3} \le u_{\boldsymbol{n}}) \\ &= \mathrm{P}(\zeta_0 > u_{\boldsymbol{n}},\ \zeta_1 \le u_{\boldsymbol{n}}, \dots, \zeta_{r_3-i} \le u_{\boldsymbol{n}}) \\ &\ge \mathrm{P}(\zeta_0 > u_{\boldsymbol{n}},\ \zeta_1 \le u_{\boldsymbol{n}}, \dots, \zeta_{r_3} \le u_{\boldsymbol{n}}) \\ &= \mathrm{P}(\zeta_0 > u_{\boldsymbol{n}},\ M^{\zeta}_{0,r_3} \le u_{\boldsymbol{n}}), \end{aligned}$$

and we have

$$\mathrm{P}(M(J) > u_{\boldsymbol{n}}) \ge r_3\, \mathrm{P}(\zeta_0 > u_{\boldsymbol{n}},\ M^{\zeta}_{0,r_3} \le u_{\boldsymbol{n}}). \tag{15}$$

Hence,

$$\begin{aligned}
\mathrm{P}(M(J) \le u_{\boldsymbol{n}}) &= 1 - \mathrm{P}(M(J) > u_{\boldsymbol{n}}) \\
&\le 1 - r_3\,\mathrm{P}(\zeta_0 > u_{\boldsymbol{n}},\ M^{\zeta}_{0,r_3} \le u_{\boldsymbol{n}}) \\
&= 1 - r_3\,\mathrm{P}(M^{\zeta}_{0,r_3} \le u_{\boldsymbol{n}} \mid \zeta_0 > u_{\boldsymbol{n}})\,\mathrm{P}(\zeta_0 > u_{\boldsymbol{n}}).
\end{aligned} \tag{16}$$

Using the arguments leading to (15), we also have

$$\begin{aligned}
\mathrm{P}(\zeta_0 > u_{\boldsymbol{n}}) &= \mathrm{P}\Big(\max_{0\le i\le r_1} \nu_i > u_{\boldsymbol{n}}\Big) \\
&= \mathrm{P}(M^{\nu}_{r_1} > u_{\boldsymbol{n}}) \\
&\ge r_1\,\mathrm{P}(\nu_0 > u_{\boldsymbol{n}},\ M^{\nu}_{0,r_1} \le u_{\boldsymbol{n}}) \\
&= r_1\,\mathrm{P}(M^{\nu}_{0,r_1} \le u_{\boldsymbol{n}} \mid \nu_0 > u_{\boldsymbol{n}})\,\mathrm{P}(\nu_0 > u_{\boldsymbol{n}}),
\end{aligned} \tag{17}$$

and repeating the argument for $\mathrm{P}(\nu_0 > u_{\boldsymbol{n}})$ gives

$$\begin{aligned}
\mathrm{P}(\nu_0 > u_{\boldsymbol{n}}) &= \mathrm{P}\Big(\max_{0\le j\le r_2} X(0, j, 0) > u_{\boldsymbol{n}}\Big) \\
&= \mathrm{P}(M^{X}_{r_2} > u_{\boldsymbol{n}}) \\
&\ge r_2\,\mathrm{P}(X(0,0,0) > u_{\boldsymbol{n}},\ M^{X}_{0,r_2} \le u_{\boldsymbol{n}}) \\
&= r_2\,\mathrm{P}(M^{X}_{0,r_2} \le u_{\boldsymbol{n}} \mid X(0,0,0) > u_{\boldsymbol{n}})\,\mathrm{P}(X(0,0,0) > u_{\boldsymbol{n}}).
\end{aligned} \tag{18}$$

By combining (16), (17), and (18) we obtain

$$\begin{aligned}
\mathrm{P}(M(J) \le u_{\boldsymbol{n}}) \le 1 - r_1r_2r_3\,&\mathrm{P}(M^{\zeta}_{0,r_3} \le u_{\boldsymbol{n}} \mid \zeta_0 > u_{\boldsymbol{n}})\,\mathrm{P}(M^{\nu}_{0,r_1} \le u_{\boldsymbol{n}} \mid \nu_0 > u_{\boldsymbol{n}}) \\
&\times \mathrm{P}(M^{X}_{0,r_2} \le u_{\boldsymbol{n}} \mid X(0,0,0) > u_{\boldsymbol{n}})\,\mathrm{P}(X(0,0,0) > u_{\boldsymbol{n}}).
\end{aligned}$$

Hence, from the CW-mixing condition, as $\boldsymbol{n} \to \infty$,

$$\begin{aligned}
&\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) \\
&\quad = \mathrm{P}^{m_1m_2m_3}(M(J) \le u_{\boldsymbol{n}}) + o_{\boldsymbol{n}}(1), \\
&\quad \le (1 - r_1r_2r_3\,\mathrm{P}(M^{\zeta}_{0,r_3} \le u_{\boldsymbol{n}} \mid \zeta_0 > u_{\boldsymbol{n}})\,\mathrm{P}(M^{\nu}_{0,r_1} \le u_{\boldsymbol{n}} \mid \nu_0 > u_{\boldsymbol{n}}) \\
&\qquad\qquad \times \mathrm{P}(M^{X}_{0,r_2} \le u_{\boldsymbol{n}} \mid X(0,0,0) > u_{\boldsymbol{n}})\,\mathrm{P}(X(0,0,0) > u_{\boldsymbol{n}}))^{m_1m_2m_3} + o_{\boldsymbol{n}}(1) \\
&\quad \to \exp(-\theta_1\theta_2\theta_3\tau(x)).
\end{aligned}$$

We will now prove the opposite inequality, namely

$$\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) \ge \exp(-\theta_1\theta_2\theta_3\tau(x)) \quad \text{as } \boldsymbol{n} \to \infty.$$

Let η_1, η_2, and η_3 be sequences of integers converging to ∞, as $\boldsymbol{n} \to \infty$, in such a manner that $r_1 = o(\eta_1)$, $\eta_1 = o(n_1)$, $r_2 = o(\eta_2)$, $\eta_2 = o(n_2)$, $r_3 = o(\eta_3)$, and $\eta_3 = o(n_3)$. Let υ_1, υ_2, and υ_3 be integers such that $\upsilon_1 = [n_1/\eta_1]$, $\upsilon_2 = [n_2/\eta_2]$, and $\upsilon_3 = [n_3/\eta_3]$. Furthermore, let

$$\hat{J} = \{0 \le i \le \eta_1,\ 0 \le j \le \eta_2,\ 0 \le k \le \eta_3\}.$$

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For such choices of integers, it follows from the CW-mixing condition that

$$\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) = \mathrm{P}^{\upsilon_1\upsilon_2\upsilon_3}(M(\hat{J}) \le u_{\boldsymbol{n}}) + o_{\boldsymbol{n}}(1).$$

Now assume that

$$\phi_k = \max_{0\le i\le\eta_1,\, 0\le j\le\eta_2} X(i, j, k),$$

$$\psi_i = \max_{0\le j\le\eta_2} X(i, j, 0).$$

Then

$$\begin{aligned}
\mathrm{P}(M(\hat{J}) > u_{\boldsymbol{n}}) &= \mathrm{P}\Big(\max_{0\le k\le\eta_3} \phi_k > u_{\boldsymbol{n}}\Big) \\
&= \mathrm{P}(M^{\phi}_{\eta_3} > u_{\boldsymbol{n}}) \\
&= \mathrm{P}(\{M^{\phi}_{r_3} > u_{\boldsymbol{n}}\} \cup \{M^{\phi}_{r_3} < u_{\boldsymbol{n}}\} \cap \{M^{\phi}_{r_3+1,\eta_3} > u_{\boldsymbol{n}}\}) \\
&= \mathrm{P}(M^{\phi}_{r_3} > u_{\boldsymbol{n}}) + \mathrm{P}(\{M^{\phi}_{r_3} < u_{\boldsymbol{n}}\} \cap \{M^{\phi}_{r_3+1,\eta_3} > u_{\boldsymbol{n}}\}) \\
&= \mathrm{P}(M^{\phi}_{r_3} > u_{\boldsymbol{n}}) + \mathrm{P}(M^{\phi}_{\eta_3-r_3} > u_{\boldsymbol{n}},\ M^{\phi}_{\eta_3-r_3+1,\eta_3} < u_{\boldsymbol{n}}) \quad \text{(by stationarity).}
\end{aligned}$$

Note that, since $r_3 = o(\eta_3)$,

$$\mathrm{P}(M(J) > u_{\boldsymbol{n}}) = \mathrm{P}(M^{\phi}_{r_3} > u_{\boldsymbol{n}}) = o(\mathrm{P}(M(\hat{J}) > u_{\boldsymbol{n}})) \quad \text{as } n_3 \to \infty.$$

Hence,

$$\mathrm{P}(M(\hat{J}) > u_{\boldsymbol{n}})(1 + o_{\boldsymbol{n}}(1)) = \mathrm{P}(M^{\phi}_{\eta_3-r_3} > u_{\boldsymbol{n}},\ M^{\phi}_{\eta_3-r_3+1,\eta_3} < u_{\boldsymbol{n}}).$$

By writing the event

$$\{M^{\phi}_{\eta_3-r_3} > u_{\boldsymbol{n}},\ M^{\phi}_{\eta_3-r_3+1,\eta_3} \le u_{\boldsymbol{n}}\} = \bigcup_{i=0}^{\eta_3-r_3} \{\phi_i > u_{\boldsymbol{n}}\} \cap B,$$

where $B = \{M^{\phi}_{\eta_3-r_3+1,\eta_3} \le u_{\boldsymbol{n}}\}$, in the 'mutually exclusive' form

$$\bigcup_{i=0}^{\eta_3-r_3} \Big((\{\phi_i > u_{\boldsymbol{n}}\} \cap B) \cap \bigcap_{j=i+1}^{\eta_3-r_3} (\{\phi_j \le u_{\boldsymbol{n}}\} \cup B^{\mathrm{c}}) \Big),$$

we obtain

$$\begin{aligned}
\mathrm{P}(M^{\phi}_{\eta_3-r_3} > u_{\boldsymbol{n}},\ M^{\phi}_{\eta_3-r_3+1,\eta_3} \le u_{\boldsymbol{n}}) &= \sum_{i=0}^{\eta_3-r_3} \mathrm{P}\Big(\{\phi_i > u_{\boldsymbol{n}}\} \cap \bigcap_{j=i+1}^{\eta_3-r_3} \{\phi_j \le u_{\boldsymbol{n}}\} \cap B \Big) \\
&= \sum_{i=0}^{\eta_3-r_3} \mathrm{P}(\phi_i > u_{\boldsymbol{n}},\ M^{\phi}_{i+1,\eta_3} \le u_{\boldsymbol{n}}).
\end{aligned}$$

Since, for every $i = 0, 1, \ldots, \eta_3 - r_3$, we have $\eta_3 \ge i + r_3$ and

$$\{M^{\phi}_{i+1,\eta_3} < u_{\boldsymbol{n}}\} \subseteq \{M^{\phi}_{i+1,r_3} < u_{\boldsymbol{n}}\},$$

we then obtain

$$\begin{aligned}
\mathrm{P}(M^{\phi}_{\eta_3-r_3} > u_{\boldsymbol{n}},\ M^{\phi}_{\eta_3-r_3+1,\eta_3} \le u_{\boldsymbol{n}}) &\le \sum_{i=0}^{\eta_3-r_3} \mathrm{P}(\phi_i > u_{\boldsymbol{n}},\ M^{\phi}_{i+1,r_3+i} \le u_{\boldsymbol{n}}) \\
&= \sum_{i=0}^{\eta_3-r_3} \mathrm{P}(\phi_0 > u_{\boldsymbol{n}},\ M^{\phi}_{1,r_3} \le u_{\boldsymbol{n}}) \\
&\le \eta_3\, \mathrm{P}(\phi_0 > u_{\boldsymbol{n}},\ M^{\phi}_{1,r_3} \le u_{\boldsymbol{n}}) \\
&= \eta_3\, \mathrm{P}(M^{\phi}_{1,r_3} \le u_{\boldsymbol{n}} \mid \phi_0 > u_{\boldsymbol{n}})\, \mathrm{P}(\phi_0 > u_{\boldsymbol{n}}). \\
&\le \eta_3\, \mathrm{P}(M^{\zeta}_{1,r_3} \le u_{\boldsymbol{n}} \mid \zeta_0 > u_{\boldsymbol{n}})\, \mathrm{P}(\phi_0 > u_{\boldsymbol{n}}). \qquad (19)
\end{aligned}$$

The final inequality in (19) follows from the fact that, for every $k = 1, \dots, r_3$, $\phi_k \ge \zeta_k$ almost surely.

If we now start with

$$\mathrm{P}(\phi_0 > u_{\boldsymbol{n}}) = \mathrm{P}\Big(\max_{0\le i\le \eta_1} \psi_i > u_{\boldsymbol{n}}\Big)$$

and apply the steps leading to (19), we find that

$$\mathrm{P}(\phi_0 > u_{\boldsymbol{n}}) \le \eta_1\, \mathrm{P}(M^{\nu}_{0,r_1} \le u_{\boldsymbol{n}} \mid \nu_0 > u_{\boldsymbol{n}})\, \mathrm{P}(\nu_0 > u_{\boldsymbol{n}}). \qquad (20)$$

By applying the steps yet again, to $\mathrm{P}(\nu_0 > u_{\boldsymbol{n}}) = \mathrm{P}(M^X_{\eta_2} > u_{\boldsymbol{n}})$, we obtain

$$\mathrm{P}(M^X_{\eta_2} > u_{\boldsymbol{n}}) \le \eta_2\, \mathrm{P}(M^X_{0,r_2} \le u_{\boldsymbol{n}} \mid X(0,0,0) > u_{\boldsymbol{n}})\, \mathrm{P}(X(0,0,0) > u_{\boldsymbol{n}}). \qquad (21)$$

From Lemma 1 and the CW-mixing condition, with η_1, η_2, η_3, υ_1, υ_2, and υ_3 as defined above, as $\boldsymbol{n} \to \infty$ we find that

$$\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) = \mathrm{P}^{\upsilon_1\upsilon_2\upsilon_3}(M(\hat{J}) \le u_{\boldsymbol{n}}) + o_{\boldsymbol{n}}(1),$$

and from (19), (20), and (21) we have

$$\begin{aligned}
\mathrm{P}(M(E_{\boldsymbol{n}}) \le u_{\boldsymbol{n}}) \ge{}& [1 - \eta_1\eta_2\eta_3\, \mathrm{P}(X(0,0,0) > u_{\boldsymbol{n}})\, \mathrm{P}(M^{\zeta}_{1,r_3} \le u_{\boldsymbol{n}} \mid \zeta_0 > u_{\boldsymbol{n}}) \\
&\times \mathrm{P}(M^{\nu}_{1,r_1} \le u_{\boldsymbol{n}} \mid \nu_0 > u_{\boldsymbol{n}})\, \mathrm{P}(M^X_{1,r_2} \le u_{\boldsymbol{n}} \mid X(0,0,0) > u_{\boldsymbol{n}}) \\
&\times (1 + o_{\boldsymbol{n}}(1))]^{\upsilon_1\upsilon_2\upsilon_3} + o_{\boldsymbol{n}}(1) \\
\to{}& \exp(-\theta_1\theta_2\theta_3\tau(x)).
\end{aligned}$$

The choice of the indices θ_1, θ_2, and θ_3 is not unique. For example, we could first have sliced the cube J along the x-direction, into plates parameterized by j and k, and looked at how the largest values over these plates cluster along the x-direction; then looked at how large values of the process cluster along the y-direction; and finally considered how the large values cluster in time.

Let $\gamma_i = \max_{0\le j\le r_2,\, 0\le k\le r_3} X(i,j,k)$ and $\tau_j = \max_{0\le k\le r_3} X(0,j,k)$, and assume that the following limits exist, where $0 < \theta_1^*, \theta_2^*, \theta_3^* \le 1$.

1. x-coordinate clustering propensity:

$$\lim_{\boldsymbol{n}\to\infty} \mathrm{P}(\gamma_1 \le u_{\boldsymbol{n}},\ \gamma_2 \le u_{\boldsymbol{n}}, \dots, \gamma_{r_1} \le u_{\boldsymbol{n}} \mid \gamma_0 > u_{\boldsymbol{n}}) = \theta_1^*.$$

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2. y-coordinate clustering propensity:

$$\lim_{n\to\infty} \mathrm{P}(\tau_1 \le u_n,\ \tau_2 \le u_n, \dots, \tau_{r_2} \le u_n \mid \tau_0 > u_n) = \theta_2^*.$$

3. t-coordinate clustering propensity:

$$\lim_{n\to\infty} \mathrm{P}(X(0,0,1) \le u_n,\ X(0,0,2) \le u_n, \dots, X(0,0,r_3) \le u_n \mid X(0,0,0) > u_n) \to \theta_3^*.$$

If the process is stationary but not isotropic (in the sense that the process $X(i, j, k)$ is a subsampled version of an isotropic, continuous-parameter random field), then, in general, $\theta_i^* \ne \theta_i$, $i = 1, 2, 3$. However, provided that the limits exist, the extremal index of the stationary process θ will be unique, i.e. $\theta_1^*\theta_2^*\theta_3^* = \theta_1\theta_2\theta_3$.

Example 1. Consider the spatial process

$$X(i, j) = \phi X(i-1, j) + \varepsilon(i, j), \qquad i, j = 1, 2, \dots, n, \tag{22}$$

where $|\phi| < 1$ and $\{\varepsilon(i, j)\}$ are independent, identically distributed random variables with regularly varying tails such that, as $x \to \infty$,

$$\mathrm{P}(\varepsilon(0,0) > x) \sim x^{-\alpha} L(x)$$

for some $\alpha \in (0, 2)$ and a slowly varying function $L(x)$. Furthermore, we assume that

$$\lim_{x\to\infty} \frac{\mathrm{P}(\varepsilon(0,0) > x)}{\mathrm{P}(|\varepsilon(0,0)| > x)} = 1.$$

The stationary solution to (22) is given by

$$X(i, j) = \sum_{k=0}^{\infty} \phi^k \varepsilon(i-k, j).$$

From Resnick (1987), $X(0, 0)$ and $\varepsilon(0, 0)$ are tail equivalent; that is, as $x \to \infty$,

$$\mathrm{P}(X(0,0) > x) \sim c x^{-\alpha} L(x)$$

for some constant c. Let $a_n = n^{2/\alpha}$ and $u_{n^2} \equiv u_{n^2}(x) = a_n x$; then, for $x > 0$,

$$n^2\, \mathrm{P}(X(0,0) > u_{n^2}) \sim \tau(x) \quad \text{as } n \to \infty.$$

Note that $X(i, j)$ is an independent sequence along the y-direction. Let

$$v_i = \max_{0\le j\le r_n} X(i, j), \qquad i = 0, 1, \dots, r_1,$$

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and let $r_i = r_n$ and $m_i = m_n$, $i = 1, 2$. Then $n = r_n m_n$ and, as $n \to \infty$,

$$\begin{aligned}
n \sum_{i=1}^{r_n} \mathrm{P}(\nu_1 > u_{n^2},\ \nu_i > u_{n^2})
&= n \sum_{i=1}^{r_n} \mathrm{P}\biggl(\bigcup_{j=0}^{r_n} \{X(1, j) > u_{n^2}\},\ \bigcup_{k=0}^{r_n} \{X(i, k) > u_{n^2}\}\biggr) \\
&\le n \sum_{i=1}^{r_n} \sum_{j=0}^{r_n} \sum_{k=0}^{r_n} \mathrm{P}(X(1, j) > u_{n^2},\ X(i, k) > u_{n^2}) \\
&= \begin{cases} n \displaystyle\sum_{i=1}^{r_n} \sum_{j=0}^{r_n} \sum_{k=0}^{r_n} \mathrm{P}(X(1, j) > u_{n^2})\, \mathrm{P}(X(i, k) > u_{n^2}) & \text{if } j \neq k, \\ n \displaystyle\sum_{i=1}^{r_n} \sum_{j=0}^{r_n} \mathrm{P}(X(1, j) > u_{n^2},\ X(i, j) > u_{n^2}) & \text{if } j = k. \end{cases}
\end{aligned} \tag{23}$$

The first term in (23) (for $j \neq k$) is of order $O(1/m_n^3)$, whereas the second term (for $j = k$) can be written as

$$n r_n \sum_{i=1}^{r_n} \mathrm{P}(X(1, 1) > u_{n^2},\ X(i, 1) > u_{n^2}).$$

If $u_n = n^{1/\alpha} x$ (or $u_{n^2} = n^{2/\alpha} x$) then

$$n \sum_{i=1}^{r_n} \mathrm{P}(X(1, 1) > u_{n^2},\ X(i, 1) > u_{n^2}) = n r_n \sum_{i=1}^{r_n} \mathrm{P}(X(1, 1) > u_n,\ X(i, 1) > u_n) g_n(i),$$

where

$$\begin{aligned}
g_n(i) &= r_n \frac{\mathrm{P}(X(1, 1) > u_{n^2},\ X(i, 1) > u_{n^2})}{\mathrm{P}(X(1, 1) > u_n,\ X(i, 1) > u_n)} \\
&= r_n \frac{\mathrm{P}(\sum_{k=0}^{i-2} \phi^k \varepsilon(i-k, 1) > u_{n^2})}{\mathrm{P}(\sum_{k=0}^{i-2} \phi^k \varepsilon(i-k, 1) > u_n)} \\
&= O\biggl(\frac{1}{m_n}\biggr),
\end{aligned}$$

uniformly in $i = 1, 2, \dots, r_n$. Since $X(i, 1)$ is an (autoregressive) AR(1) process, we have

$$n \sum_{i=1}^{r_n} \mathrm{P}(X(1, 1) > u_n,\ X(i, 1) > u_n) = O\biggl(\frac{1}{m_n}\biggr).$$

Hence, it follows from (23) that ν_i satisfies the $D'(u_{n^2})$ condition and $\theta_1 = 1$.

Note that $(X(0, j),\ j = 0, 1, \dots, r_n)$ is an independent sequence. It is easy to see that $\theta_2 = 1$ and $\theta := \theta_1 \theta_2 = 1$.

On the other hand, with $\zeta_j = \max_{0 \le i \le r_n} X(i, j)$, we see that $\{\zeta_j\}$ is an independent sequence, and that

$$\theta_1^* = \lim_{n \to \infty} \mathrm{P}(\zeta_1 < u_{n^2}, \dots, \zeta_{r_n} < u_{n^2} \mid \zeta > u_{n^2}) = 1.$$

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Similarly,

$$\theta_2^* = \lim_{n\to\infty} \mathrm{P}(X(1,0) < u_{n^2}, \ldots, X(r_n, 0) < u_{n^2} \mid X(0,0) > u_{n^2}) = 1,$$

since the maximum of the array $X(1,0), X(2,0), \ldots, X(n,0)$ is $O_p(n^{1/\alpha})$ and cannot cluster above the level $n^{2/\alpha}$. Hence, again $\theta := \theta_1^*\theta_2^* = 1$.

In general, the calculation of the extremal index for STARMA processes using this characterization would not be easy. However, it permits the adaptation of the runs method (see, for example, Embrechts *et al.* (1997, pp. 422–423)) to estimate the extremal index for these processes.

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References

Cliff, A. D. and Ord, J. K. (1975). Model building and the analysis of spatial pattern in human geography. *J. R. Statist. Soc. B* **37,** 297–348.

Cressie, N. A. C. (1993). *Statistics for Spatial Data*. John Wiley, New York.

Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events*. Springer, Berlin.

O'Brien, G. L. (1987). Extreme values for stationary and Markov sequences. *Ann. Prob.* **15,** 281–291.

Leadbetter, M. R. and Rootzen, H. (1997). On extreme values in stationary random fields. Res. Rep. 507, Center for Stochastic Processes, The University of North Carolina and Chapel Hill.

Resnick, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.