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FIVE-DIMENSIONAL WEAKLY EXCEPTIONAL QUOTIENT SINGULARITIES

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Abstract A singularity is said to be weakly exceptional if it has a unique purely log terminal blow-up. This is a natural generalization of the surface singularities of types D_n, E_6, E_7 and E_8. Since this idea was introduced, quotient singularities of this type have been classified in dimensions up to at most 4. This note extends that classification to dimension 5.

Keywords: singularities; quotient singularities; weakly exceptional

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1. Introduction

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite group. It makes sense to study the quotient singularities on the varieties of the form $\mathbb{C}^n/G$ (from now on, these are referred to as the singularities induced by G). When studying singularities (and, in particular, quotient singularities), one may consider the following type of birational morphisms.

Theorem 1.1 (Cheltsov and Shramov [4, Theorem 3.7]). *Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity. There then exists a birational morphism $\pi\colon W \to V$ such that the following hypotheses are satisfied:*

- *the exceptional locus of π consists of one irreducible divisor E such that $O \in \pi(E)$,*
- *the log pair (W, E) has purely log terminal singularities,*
- *the divisor $-E$ is a π-ample $\mathbb{Q}$-Cartier divisor.*

Definition 1.2 (Kudryavtsev [8]). Let $(V \ni O)$ be a germ of a Kawamata log terminal singularity, and let $\pi\colon W \to V$ be a birational morphism satisfying the conditions of Theorem 1.1. Then, π is a *purely log terminal (plt) blow-up* of the singularity.

This naturally leads to the following definition.

Definition 1.3 (Kudryavtsev [8]). We say that the singularity $(V \ni O)$ is *weakly exceptional* if it has a unique plt blow-up.

Remark 1.4. This definition naturally generalizes the properties of quotients of $\mathbb{C}^2$ by the action of binary dihedral (also known as dicyclic), tetrahedral, octahedral and icosahedral groups into higher dimensions.

One has the following criterion for a singularity to be weakly exceptional.

Theorem 1.5 (Cheltsov and Shramov [4, Theorem 3.15]). *Take $G \subset \mathrm{GL}_N(\mathbb{C})$ with no quasi-reflections, and let $\bar{G}$ be its natural projection into $\mathrm{PGL}_N(\mathbb{C})$. The singularity $\mathbb{C}^N/G$ is then weakly exceptional if and only if the pair $(\mathbb{P}^{N-1}, \Delta)$ is log canonical for any $\bar{G}$-invariant effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} -\mathrm{K}_{\mathbb{P}^{N-1}}$.*

Corollary 1.6. *If $G \subset \mathrm{GL}_N(\mathbb{C})$ with no quasi-reflections has a semi-invariant of degree at most $N-1$, then the singularity induced by it is not weakly exceptional.*

Remark 1.7. The reverse implication does not hold in general, for example, for $N = 4$ (see [**10**]).

Theorem 1.8 (Cheltsov and Shramov [4, Theorem 1.30]). *Let $G \subset \mathrm{GL}_N(\mathbb{C})$ be a finite subgroup containing no quasi-reflections that induces a weakly exceptional singularity. Then G is irreducible.*

In fact, in dimension 2, the induced quotient singularity is weakly exceptional exactly when the group action is irreducible. Unfortunately, this fails already in dimension 3.

It follows from the Chevalley–Shephard–Todd theorem (see [**12**, Theorem 4.2.5]) that, to study the weak exceptionality of $\mathbb{C}^n/G$, one can always assume that G contains no quasi-reflections. Moreover, it follows from Theorem 1.5 that the weak exceptionality only depends on the image of G under the natural projection to $\mathrm{PGL}_n(\mathbb{C})$. So, to study the weak exceptionality of $\mathbb{C}^n/G$, it is enough to consider the case of $G \subset \mathrm{SL}_n(\mathbb{C})$.

The classification of the groups giving rise to weakly exceptional singularities in dimension 2 is well known.

Theorem 1.9 (a rephrasing of [11, Section 5.2.3]). *Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite group. Then, G induces a weakly exceptional singularity if and only if it is a non-abelian binary dihedral, tetrahedral, octahedral or icosahedral group.*

The groups giving rise to weakly exceptional singularities in dimensions 3 and 4 have recently been classified (see [**10**]). Due to the large number of irreducible groups in dimensions higher than 2, it makes more sense to look at the irreducible groups that give rise to non-weakly exceptional singularities. In particular, in dimension 3 only finitely many conjugacy classes do so. Unfortunately, the same is not true in dimension 4.

Example 1.10. Write the coordinates of $\mathbb{C}^4$ as a 2×2 matrix, and act on it by left-hand and right-hand multiplication by the elements of the binary dihedral groups $\bar{\mathbb{D}}_{2k}, \bar{\mathbb{D}}_{2l} \subset \mathrm{SL}_2(\mathbb{C})$. Thus, one gets an irreducible action of an arbitrarily large finite group, which has a semi-invariant quadric defined by the determinant of the matrix. This in turn implies (by Corollary 1.6) that the induced singularity is not weakly exceptional. For details, see [**10**].

The purpose of this paper is to prove the following result.

Theorem 1.11. *If $G \subset \mathrm{SL}_5(\mathbb{C})$ is an irreducible monomial group that induces a non-weakly exceptional singularity, then $|G| \leqslant 5 \cdot 4^4 \cdot 5!$, with this bound attained.*

Keeping in mind Remark 1.4 and the results of [**10**], one gets the following corollaries.

Corollary 1.12. *Take $p \in \{2,3,5\}$. Suppose that $G \subset \mathrm{SL}_p(\mathbb{C})$ is a finite subgroup acting irreducibly and monomially, but the singularity induced by G is not weakly exceptional. Then $|G| \leqslant p(p-1)^{(p-1)}p!$.*

Corollary 1.13. *Take $p \in \{2,3,5\}$. There exist only finitely many finite groups $G \subset \mathrm{SL}_p(\mathbb{C})$ (up to conjugation) such that G acts irreducibly, but the singularity induced by G is not weakly exceptional.*

The proof of Theorem 1.11 relies on this paper's main technical result, which one can consider to be the structure theorem for the irreducible groups in $\mathrm{SL}_5(\mathbb{C})$ inducing non-weakly exceptional singularities (using the notation introduced in Definition 2.1 throughout).

Theorem 1.14. *Let $G \subset \mathrm{SL}_5(\mathbb{C})$ be a finite subgroup acting irreducibly. The singularity of $\mathbb{C}^5/G$ is then weakly exceptional exactly when the following hold.*

(1) *The action of G is primitive and G contains a subgroup isomorphic to the Heisenberg group of all unipotent 3×3 matrices over $\mathbb{F}_5$ (for a better classification of all such groups, see [**9**]).*

(2) *The action of G is monomial (giving $G \cong D \rtimes T$, with D an abelian group as above and T a transitive subgroup of $\mathbb{S}_5$), and (using notation from § 3) none of the following hold.*

- *D is central in $\mathrm{SL}_5(\mathbb{C})$. In this case, G can be isomorphic to $\mathbb{A}_5$, $\mathbb{S}_5$ or to their central extensions by $\mathbb{Z}_5$.*
- *$|G| = 55$ or $55 \cdot 5$ with $|D| = 11$ or $11 \cdot 5$, respectively, $T \cong \mathbb{Z}_5 \subset \mathbb{S}_5$, and there exists a $k \in \mathbb{Z}$, $1 \leqslant k \leqslant 4$, such that D is generated by $[11, 1, 4^k, 4^{2k}, 4^{3k}, 4^{4k}]$ and (in the latter case) also the scalar element $\zeta_5 \,\mathrm{Id}$. In this case, G is isomorphic to $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ or $(\mathbb{Z}_5 \times \mathbb{Z}_{11}) \rtimes \mathbb{Z}_5$.*
- *$|G| = 305$ or $305 \cdot 5$ with $|D| = 61$ or $61 \cdot 5$, respectively, $T \cong \mathbb{Z}_5 \subset \mathbb{S}_5$, and there exists a $k \in \mathbb{Z}$, $1 \leqslant k \leqslant 4$, such that D is generated by $[61, 1, 34^k, 34^{2k}, 34^{3k}, 34^{4k}]$ and (in the latter case) also the scalar element $\zeta_5 \,\mathrm{Id}$. In this case, G is isomorphic to $\mathbb{Z}_{61} \rtimes \mathbb{Z}_5$ or $(\mathbb{Z}_5 \times \mathbb{Z}_{61}) \rtimes \mathbb{Z}_5$.*
- *There exists some $d \in \{2,3,4\}$ and ω with $\omega^5 = 1$, such that*
  - *for all $g \in D$, g^d is a scalar,*
  - *$|D| \in \{d^k, 5d^k\}$ (depending on whether D contains any non-trivial scalar elements) with $1 \leqslant k \leqslant 4$,*
  - *the polynomial $x_1^d + \omega x_2^d + \omega^2 x_3^d + \omega^3 x_4^d + \omega^4 x_5^d$ is G-semi-invariant.*

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Proof. Let $G \subset \mathrm{SL}_5(\mathbb{C})$ be a finite group. Since 5 is a prime, G is either primitive or monomial. This means that the result follows immediately from Lemma 2.4 and the considerations in § 3. $\square$

Proof of Theorem 1.11. This proof follows directly from Theorem 1.14. The bound is attained by a group $G = D \rtimes T$ with $D = \mathbb{Z}_5 \times \mathbb{Z}_4^4$ acting by scalar multiplication of coordinates of $\mathbb{C}^5$, and $T \cong \mathbb{S}_5$ acting by permuting the basis. Here, $\mathrm{Z}(G) = \mathbb{Z}_5$. This group preserves the polynomial $\sum_{i=1}^{5} x_i^4$. $\square$

This leads to the following conjecture.

Conjecture 1.15. *For any prime p, there exist only finitely many finite groups $G \subset \mathrm{SL}_p(\mathbb{C})$ (up to conjugation) such that G acts irreducibly, but the singularity induced by G is not weakly exceptional.*

It seems that an even stronger result holds: take any prime p and suppose that $G \subset \mathrm{SL}_p(\mathbb{C})$ is a finite subgroup acting irreducibly and monomially, but the singularity induced by G is not weakly exceptional. Then $|G| \leqslant p(p-1)^{(p-1)}p!$.

Note that Conjecture 1.15 can easily be shown to fail for infinitely many composite dimensions, as the construction in Example 1.10 can easily be generalized to any dimension $n = k^2$.

2. General considerations

Definition 2.1 (Höfling [7, § 2]). Given a representation of a group G on a space V, a *system of imprimitivity* for the action is a set $\{V_1, \dots, V_k\}$ of distinct subspaces of $V = V_1 \oplus \dots \oplus V_k$ such that, for all i and for all $g \in G$, there exists j with $g(V_i) = V_j$. Clearly, $\{V\}$ will always be one such system. If this is the only system of imprimitivity for this action, this action is called *primitive*. If there exists a system where all the V_i are one dimensional, then the action is called *monomial*. If, for any system of imprimitivity $\{V_1, \dots, V_k\}$, and any $1 \leqslant i$, $j \leqslant k$, there exist $g_{i,j} \in G$ such that $g_{i,j}(V_i) = V_j$, then the action is called irreducible.

Since any group $G \subset \mathrm{GL}_5(\mathbb{C})$ comes with a canonical faithful representation, it makes sense to say that the group itself, rather than that representation, is primitive, monomial or irreducible.

Theorem 2.2 (Cheltsov and Shramov [3]). *Let G be a finite subgroup in $\mathrm{GL}_5(\mathbb{C})$ that does not contain reflections. The singularity $\mathbb{C}^5/G$ is then weakly exceptional if and only if the group G is irreducible and does not have semi-invariants of degree at most 4.*

It is worth noting that the property only depends on the projection of G into $\mathrm{PGL}_5(\mathbb{C})$. Therefore, from now on we assume that $G \subset \mathrm{SL}_5(\mathbb{C})$. If it does not, take instead a group $G' \subset \mathrm{SL}_5(\mathbb{C})$ that has the same projection into $\mathrm{PGL}_5(\mathbb{C})$.

This theorem provides two possible approaches to computing the list of irreducible groups giving rise to singularities in dimension 5 that are not weakly exceptional: either by obtaining a list of finite groups of automorphisms of projective 3-folds of low degrees and seeing which of their actions descend to actions on $\mathbb{P}^4$, or by directly computing which groups have semi-invariant polynomials of degree at most 4 in five variables. Since the finite subgroups of $\mathrm{SL}_5(\mathbb{C})$ fit into two small families, which are relatively easy to work with, we have chosen to follow the second approach.

To begin with, it is easier to deal with the case of G being a primitive group, and then to look into the monomial case.

Theorem 2.3 (Feit [5, § 8.5]). *If $G \subset \mathrm{SL}_5(\mathbb{C})$ is a finite group acting primitively, then either G is one of $\mathbb{A}_5$, $\mathbb{A}_6$, $\mathbb{S}_5$, $\mathbb{S}_6$, $\mathrm{PSL}_2(11)$ and $\mathrm{Sp}_4(\mathbb{F}_3)$, or G is a subgroup of the normalizer $\mathbb{HM}$ of the Heisenberg group $\mathbb{H}$ of all unipotent 3×3 matrices over $\mathbb{F}_5$, such that $\mathbb{H} \subset G \subseteq \mathbb{HM}$.*

Lemma 2.4. *Let $G \subset \mathrm{SL}_5(\mathbb{C})$ be a finite primitive subgroup. Then G gives rise to a weakly exceptional singularity if and only if it contains a subgroup isomorphic to the Heisenberg group $\mathbb{H}$.*

Proof. Since there exists a very small number of such groups (see Theorem 2.3), one can simply look at the low symmetric powers of their five-dimensional irreducible representations. This gives the following.

- The actions of $\mathbb{A}_5$, $\mathbb{S}_5$, $\mathbb{A}_6$, $\mathbb{S}_6$ have semi-invariants of degree 2, since they are conjugate to subgroups of $\mathrm{GL}_5(\mathbb{R})$.
- The action of $\mathrm{PSL}_2(11)$ has a semi-invariant of degree 3: the Klein cubic 3-fold (see [**1**]).
- The action of $\mathrm{Sp}_4(\mathbb{F}_3)$ has a semi-invariant of degree 4: the Burkhardt quartic 3-fold (see [**2**]).
- If G contains the Heisenberg group $\mathbb{H}$, then G cannot have any semi-invariants of degree at most 4 (either apply Theorem 2.2 to [**3**, Theorem 1.15] or apply Lemma 3.5 to the (monomial) representations of H of dimension at most 5).

3. Monomial groups

Throughout this section, ζ_n is used to denote a primitive nth root of unity. This is chosen consistently for different n, i.e. so $\zeta_{mn}^m = \zeta_n$.

If $G \subset \mathrm{SL}_5(\mathbb{C})$ is a finite irreducible monomial group, then take its system of imprimitivity consisting of one-dimensional subspaces. Let D be the normal subgroup of G preserving these subspaces. Then, clearly, D is abelian, and $G = D \rtimes T$, where T is a transitive subgroup of $\mathbb{S}_5$ permuting the spaces. Moreover, there exists a basis for $\mathbb{C}^5$ in which D acts by multiplication by diagonal matrices, and there exists an element $\tau \in G \setminus D$ acting by $(x_1, x_2, x_3, x_4, x_5) \mapsto (x_2, x_3, x_4, x_5, x_1)$.

To establish non-ambiguous notation, one needs to mention that in this paper the notation $\mathbb{D}_{2n}$ means the dihedral group of $2n$ elements, and

$$\mathbb{GA}(1,5) = \langle (1\ 2\ 3\ 4\ 5), (2\ 3\ 5\ 4) \rangle \subset \mathbb{S}_5$$

is the general affine group with parameters $(1, 5)$. Furthermore, for any $g \in G$ and any polynomial f, write $g(f) = f \circ g$.

Remark 3.1 (see, for example, the appendix of [13]). If G is not generated by D and τ, then $\mathbb{Z}_5 \subsetneq T \subseteq \mathbb{S}_5$, so it is a well-known fact that T must be one of $\mathbb{D}_{10}$, $\mathbb{GA}(1, 5)$, $\mathbb{A}_5$ and $\mathbb{S}_5$, (up to choosing τ) generated by (1 2 3 4 5) (corresponding to τ) and (2 5)(3 4), (2 3 5 4), (1 2 3) or (1 2), respectively.

Since G is a finite group, any $g \in D$ must multiply the coordinates by roots of unity. From now on, write $[n, a_1, a_2, a_3, a_4, a_5]$ for the element acting as

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (\zeta_n^{a_1} x_1, \zeta_n^{a_2} x_2, \zeta_n^{a_3} x_3, \zeta_n^{a_4} x_4, \zeta_n^{a_5} x_5).$$

It is clear that

$$[n, a_1, a_2, a_3, a_4, a_5] = [kn, ka_1, ka_2, ka_3, ka_4, ka_5]$$

for any $k \in \mathbb{Z}_{>0}$, so it will always be assumed that the presentation has the minimal possible $n \in \mathbb{Z}_{>0}$. Note that, since $g \in \mathrm{SL}_5(\mathbb{C})$, it must be true that $\sum_i a_i = nk$ for some $k \in \mathbb{Z}$. Also note that replacing a_i by $a_i \pm n$ gives the same element.

Lemma 3.2. *If all the elements of D are scalar, then the singularity induced by G is not weakly exceptional.*

Proof. In this case, G must be either one of the groups mentioned in Remark 3.1 or a central extension of one of them by $\mathbb{Z}_5$. On this list, the only groups that have irreducible five-dimensional representations are $\mathbb{A}_5$, $\mathbb{S}_5$ and their central extensions by $\mathbb{Z}_5$. It is easy to see that all of these have semi-invariants of degree 2. □

From now on, one can assume that D contains a non-scalar element.

Lemma 3.3. *Let $g \in D$ be a non-scalar element of order pq for some integers $p, q > 1$. Then either $p = 5$, or there exists $g' \in D$, a non-scalar element of order p.*

Proof. Set $g' = g^q$. Scalar elements in $\mathrm{SL}_5(\mathbb{C})$ have order 1 or order 5, so either $p = 5$ or g' is not a scalar. □

Proposition 3.4. *Define the following monomials in five variables $x_1, \dots, x_5$, as in Table 1.*

Any polynomial f of degree at most 4 that is semi-invariant under the action of τ must then be one of

$$A_1 \sum_{j=0}^{4} \omega^j \tau^j(m_{1,1}), \qquad \sum_{i=1}^{3} \left[B_i \sum_{j=0}^{4} \omega^j \tau^j(m_{2,i}) \right],$$
$$\sum_{i=1}^{7} \left[C_i \sum_{j=0}^{4} \omega^j \tau^j(m_{3,i}) \right], \qquad \sum_{i=1}^{14} \left[D_i \sum_{j=0}^{4} \omega^j \tau^j(m_{4,i}) \right],$$

where $A_1, B_i, C_i, D_i \in \mathbb{C}$ and ω is some (not necessarily primitive) fifth root of 1.

Table 1. *Representatives of τ-orbits of monomials.*

$m_{1,1}=x_1$			
$m_{2,1}=x_1^2$	$m_{2,2}=x_1x_2$	$m_{2,3}=x_1x_3$	
$m_{3,1}=x_1^3$	$m_{3,2}=x_1^2x_2$	$m_{3,3}=x_1^2x_3$	$m_{3,4}=x_1^2x_4$
$m_{3,5}=x_1^2x_5$	$m_{3,6}=x_1x_2x_3$	$m_{3,7}=x_1x_2x_4$	
$m_{4,1}=x_1^4$	$m_{4,2}=x_1^3x_2$	$m_{4,3}=x_1^3x_3$	$m_{4,4}=x_1^3x_4$
$m_{4,5}=x_1^3x_5$	$m_{4,6}=x_1^2x_2^2$	$m_{4,7}=x_1^2x_3^2$	$m_{4,8}=x_1^2x_2x_3$
$m_{4,9}=x_1^2x_2x_4$	$m_{4,10}=x_1^2x_2x_5$	$m_{4,11}=x_1^2x_3x_4$	$m_{4,12}=x_1^2x_3x_5$
$m_{4,13}=x_1^2x_4x_5$	$m_{4,14}=x_1x_2x_3x_4$		

Proof. The polynomial f is semi-invariant under the action of τ, so set $\omega = f/\tau(f)$. We have that $\tau^5 = \mathrm{id}$, so $\omega^5 = 1$. □

Any polynomial that is τ-semi-invariant and contains a monomial m must contain all the monomials from the τ-orbit of m. It is easy to check that the $m_{d,i}$ above are representatives of all orbits of monomials of degree $d \leqslant 4$ in 5 variables; the result follows. □

Now look at how the elements of D act on these polynomials. Since D preserves the basis of $\mathbb{C}^5$, all the monomials are D-semi-invariant, so every τ-invariant polynomial must be preserved. Applying $g = [p, a_1, \dots, a_5]$ (p prime, $0 \leqslant a_i < p$, a_i not all equal), we get the following.

Lemma 3.5. *For any $g = [n, a_1, \dots, a_5] \in D$, a_i not all equal (i.e. g is not scalar), the expressions in Table 2 hold (replacing g by its scalar multiple if necessary) for some parameter $a \in \mathbb{Z}$ ($0 \leqslant a \leqslant n$).*

Proof. The proof relies on fairly straightforward algebra and using the fact that $\sum_i a_i = 0 \pmod n$. All these calculations are almost identical, so only one of them (for $D_2 \neq 0$) is shown here.

If $D_2 \neq 0$, then the semi-invariance suggests that

$$3a_1 + a_2 \equiv 3a_2 + a_3 \equiv 3a_3 + a_4 \equiv 3a_4 + a_5 \equiv 3a_5 + a_1 \pmod n.$$

This immediately implies that $n \neq 3$ (otherwise, we get that $a_1 \equiv \dots \equiv a_5 \pmod n$), making g a scalar), and, hence, by Lemma 3.3, n is not divisible by 3. Furthermore, it is easy to see that

$$3a_1 \equiv 2a_2 + a_3, \qquad 3a_2 \equiv 2a_3 + a_4, \qquad 3a_3 \equiv 2a_4 + a_5,$$
$$3a_4 \equiv 2a_5 + a_1, \qquad 3a_5 \equiv 2a_1 + a_2 \pmod n.$$

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Table 2. *Conditions for semi-invariance.*

$A_1 = 0$	
$B_1 = 0$ or $n = 2$	$B_2 = B_3 = 0$
$C_1 = 0$ or $n = 3$	$C_2 = 0$ or $g = [11, a, 4^3a, 4^6a, 4^9a, 4^{12}a]$
$C_3 = 0$ or $g = [11, a, 4^4a, 4^8a, 4^{12}a, 4^{16}a]$	$C_4 = 0$ or $g = [11, a, 4^1a, 4^2a, 4^3a, 4^4a]$
$C_5 = 0$ or $g = [11, a, 4^2a, 4^4a, 4^6a, 4^8a]$	$C_6 = C_7 = 0$
$D_1 = 0$ or $n \in \{2, 4\}$	$D_2 = 0$ or $g = [61, a, 34^2a, 34^4a, 34^6a, 34^8a]$
$D_3 = 0$ or $g = [61, a, 34^1a, 34^2a, 34^3a, 34^4a]$	$D_4 = 0$ or $g = [61, a, 34^4a, 34^8a, 34^{12}a, 34^{16}a]$
$D_5 = 0$ or $g = [61, a, 34^3a, 34^6a, 34^9a, 34^{12}a]$	$D_6 = D_7 = 0$ or $n = 2$
$D_8 = 0$ or $g = [11, a, 4^1a, 4^2a, 4^3a, 4^4a]$	$D_9 = 0$ or $g = [11, a, 4^2a, 4^4a, 4^6a, 4^8a]$
$D_{10} = D_{11} = 0$	$D_{12} = 0$ or $g = [11, a, 4^3a, 4^6a, 4^9a, 4^{12}a]$
$D_{13} = 0$ or $g = [11, a, 4^4a, 4^8a, 4^{12}a, 4^{16}a]$	$D_{14} = 0$

Since $a_1 + \cdots + a_5 \equiv 0 \pmod n$, we get that

$$\begin{aligned} 0 &\equiv 2(a_1 + \cdots + a_5) \equiv 2a_1 + (2a_2 + a_3) + a_3 + (2a_4 + a_5) + a_5 \pmod n \\ &\equiv 2a_1 + 3a_1 + a_3 + 3a_3 + a_5 \equiv 5a_1 + 4a_3 + a_5 \pmod n \\ &\equiv 5a_1 + 4a_3 + (3a_3 - 2a_4) \equiv 5a_1 + 7a_3 - 2(3a_2 - 2a_3) \pmod n \\ &\equiv 5a_1 + 11a_3 - 3(2a_2) \equiv 5a_1 + 11a_3 - 3(3a_1 - a_3) \equiv 14a_3 - 4a_1 \pmod n, \end{aligned}$$

giving that $4a_1 \equiv 14a_3 \pmod n$. Similarly, we get that

$$4a_1 \equiv 14a_3, \qquad 4a_2 \equiv 14a_4, \qquad 4a_3 \equiv 14a_5, \qquad 4a_4 \equiv 14a_1, \qquad 4a_5 \equiv 14a_2.$$

Since n is not a multiple of 3, 3 is invertible $(\operatorname{mod} n)$, and so, writing

$$\begin{aligned} 9a_1 &\equiv 2(3a_2) + 3a_3 \equiv 7a_3 + 2a_4 \pmod n, \\ 27a_1 &\equiv 20a_4 + 7a_5 \pmod n, \\ 81a_1 &\equiv 61a_5 + 20a_1 \pmod n, \end{aligned}$$

one deduces that either $61|n$ or $a_1 \equiv a_5 \pmod n$. By symmetry (or repeating the calculation for $a_2, \dots, a_5$) one sees that

$$61a_1 = 61a_2 = 61a_3 = 61a_4 = 61a_5 \pmod n,$$

and since $n, a_1, \dots, a_5$ are assumed not to all have a common divisor, one sees that either $n = 61$ or g is a scalar. Since $14 \equiv 34 \cdot 4 \pmod{61}$, the result follows. □

Corollary 3.6. *Let $G \subset \mathrm{SL}_5(\mathbb{C})$ be a finite irreducible monomial group that induces a non-weakly exceptional singularity. Then either $|D|$ or $|D|/5$ is in $\{2^k, 3^k, 11^k, 61^k\}$ for some positive integer k.*

One now needs to look at the possible isomorphism classes of T. The remainder of this section completes the proof of the main technical theorem by excluding most of the

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possibilities for T. In particular, Corollary 3.8 deals with the case where the size of D is divisible by 11 or 61, and Proposition 3.9 shows that the remaining groups only need to be checked against the diagonal hypersurfaces.

Corollary 3.7. *Let $G \subset \mathrm{SL}_5(\mathbb{C})$ be a finite irreducible monomial group that induces a non-weakly exceptional singularity, and there exists $g \in G$, an element of order 11 or 61. Then $G = D \rtimes \mathbb{Z}_5$ (with D as above).*

Proof. It is easy to see that $D \rtimes \mathbb{Z}_5 \subseteq G$. Assume that the inequality is strict. By looking at the action of G on the polynomials, it is then clear that $C_i, C_j \neq 0$ for some $2 \leqslant i \neq j \leqslant 5$. Any elements of D must then be of the form specified in Lemma 3.5. However, it is easy to see that an element being in two of the forms at the same time means (in the notation of Lemma 3.5) that $a = 0$, and so this is the identity element, leading to a contradiction. A similar argument works for the relevant D_i. □

Corollary 3.8. *If G contains an element of order 11 or 61 but induces a singularity that is not weakly exceptional, then G belongs to one of 16 conjugacy classes given in Theorem 1.14 (2) (defined by the choice of a primitive root of unity modulo 11 or 61, respectively, and by whether or not G contains non-trivial scalars).*

In Corollary 3.8, the groups with elements of order 11 are automorphisms of the well-known Klein cubic 3-fold (see [**1**]). Similarly, the groups with elements of order 61 are automorphisms of the Klein quartic 3-fold (see [**6**, § 4.3]).

Proposition 3.9. *Let G be a finite monomial group as described above, preserving the polynomial*

$$\begin{aligned} h(x_1, \dots, x_5) = D_6(&x_1^2x_2^2 + \omega x_2^2x_3^2 + \omega^2 x_3^2x_4^2 + \omega^3 x_4^2x_5^2 + \omega^4 x_5^2x_1^2) \\ &+ D_7(x_1^2x_3^2 + \omega x_2^2x_4^2 + \omega^2 x_3^2x_5^2 + \omega^3 x_4^2x_1^2 + \omega^4 x_5^2x_2^2), \end{aligned}$$

semi-invariant for some values of D_6, D_7 not both 0, and some ω a fifth root of 1. Then $\omega = 1$, and the polynomial $f(x_1, \dots, x_5) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$ is also G-semi-invariant.

Proof. Decompose $G = D \rtimes T$, $\tau \in T$ as above. Lemma 3.5 implies that, for any $g \in D$, g^2 is a scalar, and so such a group D also leaves f semi-invariant. Therefore, it remains to check that the representatives of generators of T leave f semi-invariant. This is obviously true if $T \cong \mathbb{Z}_5$ (then T is generated by the image of τ).

Therefore, it remains to show that the proposition holds for $\mathbb{Z}_5 \subsetneq T \subseteq \mathbb{S}_5$. Looking at the subgroups of $\mathbb{S}_5$, this means that $\mathbb{D}_{10} \subseteq T \subseteq \mathbb{S}_5$. In particular, there exists $\delta \in G \setminus D$, such that the image of δ is (up to conjugation and choosing τ appropriately) $(2\ 5)(3\ 4) \in \mathbb{D}_{10} \subseteq T \subseteq \mathbb{S}_5$. Therefore, there exist $\lambda_i \in \mathbb{C} \setminus 0$ such that g is defined by $(x_1, x_2, x_3, x_4, x_5) \mapsto (\lambda_1 x_1, \lambda_5 x_5, \lambda_4 x_4, \lambda_3 x_3, \lambda_2 x_2)$.

Applying this to h and solving the resulting equations, we get that $\lambda_2^2 = \lambda_1^2\omega^4$, $\lambda_3^2 = \lambda_1^2\omega^3$, $\lambda_4^2 = \lambda_1^2\omega^3$, $\lambda_5^2 = \lambda_1^2\omega$. By the definition of the semi-direct product, we have $\delta^2 \in D$, so $\lambda_1^2 = C(-1)^{a_1}$, $\lambda_3\lambda_4 = C(-1)^{a_3}$. This, and the fact that (by construction) $\omega^5 = 1$, implies that $\omega = 1$, and, hence, $\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2 = \lambda_5^2$, making f semi-invariant under the action of δ.

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Hence, the proposition holds unless $\mathbb{D}_{10} \subsetneq T \subseteq \mathbb{S}_5$. Doing the same calculation (simplified, as $\omega = 1$) for the elements of $G \setminus D$ that are preimages of $(1\ 2\ 3) \in \mathbb{A}_5 \subset \mathbb{S}_5$ and $(2\ 3\ 5\ 4) \in \mathbb{GA}(1,5) \subset \mathbb{S}_5$ excludes the remaining three possibilities for T. □

This concludes the proof of the main technical result of this note, showing that a group whose size is not divisible by 11 or 61 need only be checked against the diagonal hypersurfaces. The final result estimates the maximal size and the number of groups preserving such hypersurfaces, thus completing the proof of the paper's main result.

Proposition 3.10. *Assume that G is a monomial group leaving the polynomial*

$$f(x_1, \ldots, x_5) = x_1^d + \omega x_2^d + \omega^2 x_3^d + \omega^3 x_4^d + \omega^4 x_5^d$$

(for some $d \leqslant 4$) semi-invariant. Then, G belongs to one of finitely many conjugacy classes.

Proof. Since for all $g_D \in D$, $g_D^{5d} = \mathrm{id}$ (as g_D^d is a scalar), there exist only finitely many possibilities for D up to choice of basis (in fact, at most $5d^4$). The element τ has been chosen explicitly, so one need only worry about elements of G not generated by D and τ. But, since $\mathbb{Z}_5 \subseteq T \subseteq \mathbb{S}_5$, by Remark 3.1, G must be generated by D, τ and one more element δ, with the projection of δ into $\mathbb{S}_5$ being one of the four known elements. Set $\delta(f) = \psi f$.

Since $\delta \in \mathbb{S}_5$, $\delta^k = \mathrm{id}$ for some $k \leqslant 6$, and so $\psi^k = 1$. Furthermore, for all $i \leqslant 5$, there exist $j, l \leqslant 5$ such that $\delta(x_i^d) = \psi\omega^l x_j^d$ (as f is preserved), so any non-zero entries in the matrix of δ must be roots of 1 of degree at most $5kd \leqslant 30d \leqslant 120$.

Therefore, there exist only finitely many possible conjugacy classes for G. □

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