Colored isomorphism of classifiable C*-algebras

George A. Elliott and Jeffrey Im

Abstract. It is shown that the colored isomorphism class of a unital, simple, $\mathcal{Z}$-stable, separable amenable C*-algebra satisfying the universal coefficient theorem is determined by its tracial simplex.

1 Introduction

Approximate intertwinings have played a significant role in the classification theory of C*-algebras (e.g., [7, 20, 24, 25, 27, 28, 30, 32, 34–37, 40, 42, 43, 50, 51]). A related notion called *n*-colored isomorphism, which specializes to an approximate intertwining in the case of a single color, was considered by Castillejos in [10]. There it is shown that any two (unital) classifiable C*-algebras (i.e., those satisfying the hypotheses of Theorem 4.5) with at most one trace are two-colored isomorphic, and it is posed as a question whether any two such C*-algebras with isomorphic tracial simplices are *n*-colored isomorphic. We show that this question has an affirmative answer using a somewhat modified notion, which we shall simply refer to as colored isomorphism.

Theorem 4.5 *Any two unital, simple, $\mathcal{Z}$-stable, separable amenable C*-algebras satisfying the universal coefficient theorem (UCT) with isomorphic tracial simplices are colored isomorphic.*

Let us begin by giving context for the notion from [10] and comparing the approaches taken there and here. In fact, much of the original strategy is retained in this paper, so let us outline the proof of the main result, the finite case considered in Theorem 4.5, and mention the differences with [10] along the way. Throughout, let ω be a fixed free ultrafilter on the natural numbers. Two unital C*-algebras A and B were said in [10] to be *n-colored isomorphic* if there exist c.p.c. order zero maps $\varphi \colon A \to B$ and $\psi \colon B \to A$, and unitaries $u_1, \ldots, u_n \in A_\omega$ and $v_1, \ldots, v_n \in B_\omega$, such that

$$\sum_{k=1}^{n} u_k \psi\varphi(a) u_k^* = a \qquad \text{and} \qquad \sum_{k=1}^{n} v_k \varphi\psi(b) v_k^* = b$$

for all $a \in A$ and all $b \in B$. The present notion, colored isomorphism, on the other hand, has order zero maps at the level of ultrapowers A_ω and B_ω (and so includes the case that the ultrapower maps are induced by a sequence of order zero maps at the level of

the algebras A and B); and the u_i (resp. v_i) are contractions (not just unitaries) such that the absolute values squared add up to the identity of A_ω (resp. B_ω) (rather than a multiple of this). (We also assume that traces on the ultrapower determined by a single trace on the algebra are preserved.)

While the earlier notion, n-colored isomorphism, preserves the tracial cone, up to isomorphism, the present notion preserves the tracial simplex, up to isomorphism (Theorem 4.1). We note that isomorphism of tracial cones coincides with (a multiple of) tracial simplex isomorphism in the cases considered in [10] (at most one trace). Our definition is formulated in terms of colored equivalence for order zero maps, which has its origins in the $\mathcal{Z}$-stable implies finite nuclear dimension direction of the Toms–Winter conjecture, having first appeared in [45] and later more definitively in [6, 59]. (That of [10] is not explicitly based on colored equivalence of maps.)

Now let A and B be finite classifiable C*-algebras with isomorphic tracial simplices. In order to show that A and B are colored isomorphic, we must construct order zero maps $\varphi \colon A_\omega \to B_\omega$ and $\psi \colon B_\omega \to A_\omega$ such that $\psi\varphi\iota_A$ is colored equivalent to ι_A, where ι_A is the canonical embedding of A into A_ω, and likewise for $\varphi\psi\iota_B$ and ι_B. By the finite order zero uniqueness theorem of [6], which was later extended to remove restrictions on the tracial simplices in [11], it is sufficient to show that (in the notation of Corollary 2.3) $\tau(\psi\varphi)^n = \tau$ for all $n \in \mathbb{N}$ and all $\tau \in T(A_\omega)$, and that $\tau(\varphi\psi)^n = \tau$ for all $n \in \mathbb{N}$ and all $\tau \in T(B_\omega)$. Since $(\psi\varphi)^n = \psi^n\varphi^n$ and $(\varphi\psi)^n = \varphi^n\psi^n$, by Corollary 2.4, these tracial identities imply that φ^n and ψ^n also (for each n) induce mutually inverse isomorphisms of $T(A_\omega)$ and $T(B_\omega)$. In fact, the order zero maps we construct will induce the same mutually inverse isomorphisms of tracial simplices for each n.

Roughly speaking, φ is constructed with a sequence of maps at the level of the algebras A and B which factor through a fixed AF algebra D via a fixed embedding α_A. Furthermore, α_A is chosen so that it induces an isomorphism of the tracial simplices $T(D)$ and $T(A)$. This is obtained from the recently established homomorphism theorem of [37]. Such embeddings are now also known to exist outside the classification setting (i.e., without the assumption of $\mathcal{Z}$-stability), building on ideas from [60, 61]. Lastly, the maps φ_k into B from the AF algebra D are chosen to have prescribed tracial data. More specifically, given a faithful trace μ_k on $C_0(0,1]$ and an affine map $\Phi \colon T(B) \to T(D)$, φ_k is an order zero map satisfying the identity

$$(*)\qquad\qquad \tau\varphi_k^n = \mu_k(t^n)\Phi(\tau)$$

for each $k, n \in \mathbb{N}$ and each $\tau \in T(B)$, where t denotes the identity map on $(0,1]$. (But, for obvious reasons, with $\mu_k(t^n) \to 1$ for each n.) This is the content of Theorem 4.3.

Let us show how the desired order zero maps φ and ψ can be constructed from here. Let Φ be an isomorphism of the tracial simplices $T(B)$ and $T(D)$, which exists since $T(A)$ is assumed to be isomorphic to $T(B)$, by hypothesis, and D was chosen so that $T(D)$ is isomorphic to $T(A)$; and let μ_k be a sequence of faithful traces on $C_0(0,1]$ such that $\lim_{k\to\omega} \mu_k(t^n) = 1$ for each $n \in \mathbb{N}$. (It is enough that $\lim_{k\to\infty} \mu_k(t) = 1$.) Then with $\varphi_k \colon D \to B$ satisfying $(*)$, and with $\varphi \colon A_\omega \to B_\omega$ the order zero map induced by the sequence $(\varphi_k \alpha_A)$, we have

$$\tau\varphi^n = \tau((\varphi_k\alpha_A)^n) = \tau(\varphi_k^n\alpha_A)$$

for each $n \in \mathbb{N}$ and for each trace $\tau \in T(B_\omega)$ with Corollary 2.4 having been used at the last equality. In order to make use of the fact that μ_k levels out moments, we will need a reduction which was observed in [47]. The limit traces in $T(B_\omega)$ (i.e., those which are induced by a sequence of traces in $T(B)$) are weak* dense in $T(B_\omega)$ (Theorem 2.7). Therefore, it suffices to check that the desired tracial identities (see the next paragraph) hold for such traces. Let a trace τ of the form $\lim_{k \to \omega} \tau_k$ in $T(B_\omega)$ be given. Then

$$\tau(\varphi_k^n \alpha_A) = \lim_{k \to \omega} \tau_k \varphi_k^n \alpha_A = \lim_{k \to \omega} \alpha_A^* \tau_k \varphi_k^n = \alpha_A^* \lim_{k \to \omega} \tau_k \varphi_k^n$$

$$\overset{(*)}{=} \alpha_A^* \lim_{k \to \omega} \mu_k(t^n) \Phi(\tau_k) = \alpha_A^* \lim_{k \to \omega} \Phi(\tau_k) = \lim_{k \to \omega} \alpha_A^* \Phi(\tau_k)$$

for each $k, n \in \mathbb{N}$. The second to last equality uses the fact that $\lim_{k \to \omega} \mu_k(t^n) = 1$ and continuity of α_A^* is used for the third equality and the last two.

The remaining map ψ is constructed in a similar way. Let $\alpha_B : B \to E$ be an AF embedding which induces a tracial simplex isomorphism $T(E) \to T(B)$, and let $\psi_k : E \to A$ be order zero maps satisfying the identity

$$\tau \psi_k^n = \mu_k(t^n) \Psi(\tau)$$

for each $k, n \in \mathbb{N}$ and each $\tau \in T(A)$, where Ψ is the tracial simplex isomorphism $(\alpha_A^* \Phi \alpha_B^*)^{-1}$. Let $\psi : B_\omega \to A_\omega$ denote the order zero map induced by the sequence $(\psi_k \alpha_B)$. Then for each limit trace $\tau = \lim_{k \to \omega} \tau_k$ in $T(A_\omega)$,

$$\tau \psi^n = \lim_{k \to \omega} \alpha_B^* \Psi(\tau_k)$$

for each $n \in \mathbb{N}$. Therefore (see the proof of Theorem 4.5 for more details),

$$\tau(\psi \varphi)^n = (\varphi^n)^* \left(\lim_{k \to \omega} \alpha_B^* \Psi(\tau_k) \right) = \lim_{k \to \omega} \alpha_A^* \Phi \alpha_B^* \Psi(\tau_k) = \tau$$

for each $n \in \mathbb{N}$ and each limit trace $\tau \in T(A_\omega)$. By Theorem 2.7, the above identity holds for all $\tau \in T(A_\omega)$. A symmetric argument shows that $\tau(\varphi \psi)^n = \tau$ for each $n \in \mathbb{N}$ and each $\tau \in T(B_\omega)$. Therefore, φ and ψ determine a colored isomorphism of A and B. (To simplify the discussion, we omit the question of preserving constant limit traces and defer to the proof of Theorem 4.5.)

Since the n-colored isomorphism of [10] requires unitaries rather than contractions, a different uniqueness theorem is developed for order zero maps in the unique trace case (see [10, Lemma 5.6.2]). It provides a stronger statement than [6, Theorem 5.5] (in the unique trace case) since it is applicable to pairs of order zero maps (rather than one $*$-homomorphism and one order zero map) and because it provides unitary equivalence of the order zero maps involved rather than after tensoring the order zero maps with a positive element $h \in \mathcal{Z}$ with full spectrum. Let h be such an element with the additional stipulation that $\tau_{\mathcal{Z}}(h^n) = \tau_{\mathcal{Z}}((1_{\mathcal{Z}} - h)^n) = 1/(n+1)$ for each $n \in \mathbb{N}$ where $\tau_{\mathcal{Z}}$ denotes the unique trace of $\mathcal{Z}$. Because the scaling factors introduced by the unitaries differ from those introduced by contractions under traces, the compositions of the order zero maps $\varphi : A \to B$ and $\psi : B \to A$ implementing the n-colored equivalence are compared with the contractive order zero maps $\rho_{A,h} := \sigma_1(\mathrm{id}_A \otimes h)$ and $\rho_{A,h} := \sigma_2(\mathrm{id}_B \otimes h)$ under traces where $\sigma_1 : A \otimes \mathcal{Z} \to A$ and $\sigma_2 : B \otimes \mathcal{Z} \to B$ are isomorphisms whose inverses are approximately unitarily equivalent to the first factor embeddings. The moment problem in this setting is

more complicated than the one in ours because the order zero maps involved in the uniqueness theorem are induced by constant sequences of maps, and therefore the moments need to match up on the dot rather than only approximately. Moreover, the target moments of $\psi\varphi$ and $\varphi\psi$ actually depend on $n \in \mathbb{N}$, as

$$\tau_A \rho_1^n = \frac{\tau_A(\cdot)}{n+1} \quad \text{and} \quad \tau_B \rho_2^n = \frac{\tau_B(\cdot)}{n+1},$$

where τ_A and τ_B denote the unique trace of A and B (see [10, Theorem 5.6.8]).

The main step in constructing φ and ψ (in [10]) is the construction of maps out of unital, simple, separable AF algebras with unique trace into $\mathcal{Z}$ realizing specific moments. Let us outline the construction of φ using this result. By [60, Theorem A], A embeds into a simple, separable, unital AF algebra D with unique trace via a map α_A which induces an isomorphism of tracial simplices. Then an order zero map $\phi : D \to \mathcal{Z}$ is constructed, using a measure μ on the unit interval with the very specific moments $\mu(t^n) = 1/\sqrt{n+1}$, where t denotes the identity map on $(0,1]$, such that

$$\tau_{\mathcal{Z}} \phi^n = \frac{\tau_D}{\sqrt{n+1}}$$

for each $n \in \mathbb{N}$ (see [10, Lemma 5.6.5]). The order zero map φ is then given by the composition

$$A \xhookrightarrow{\alpha_A} D \xrightarrow{\phi} \mathcal{Z} \xrightarrow{1_B \otimes \mathrm{id}_{\mathcal{Z}}} B \otimes \mathcal{Z} \xrightarrow{\sigma_2} B.$$

Apart from ϕ, each of the above maps is a trace-preserving $*$-homomorphism, and so by Corollary 2.4, the moments of φ are

$$\tau_B \varphi^n = \tau \sigma_2 (1_B \otimes 1_{\mathcal{Z}}) \phi^n \alpha_A = \frac{\tau_B}{\sqrt{n+1}}$$

for each $n \in \mathbb{N}$. The second order zero map ψ is constructed in a similar way so that

$$\tau_A (\psi\varphi)^n = \frac{\tau_A}{n+1} \quad \text{and} \quad \tau_B (\varphi\psi)^n = \frac{\tau_B}{n+1}$$

for each $n \in \mathbb{N}$. The appropriate uniqueness theorem (see [10, Lemma 5.6.2]) is then applied twice with $\psi\varphi$: once with $\rho_{A,h}$, and once more with $\rho_{A,1-h}$, and similarly for $\varphi\psi$, in order to obtain a two-colored isomorphism of A and B.

All that remains to be outlined in the construction of order zero maps, either in our setting or in that of [10], is how to ensure the prescribed tracial data. Let us begin with the approach taken in [10]. Let D be a simple, separable, unital AF algebra with unique trace τ and let ν be a fully supported Borel measure on $[0,1]$. Recall (e.g., from [33]) that τ induces a functional $d_\tau : \mathrm{Cu}(D) \to [0,\infty]$. It is stated in [10, Proposition 1.10.12] that the map $\sigma : \mathrm{Lsc}([0,1], \mathrm{Cu}(D)) \to \mathrm{Cu}(\mathcal{Z})$ determined by the rule

$$f \mapsto \int_0^1 d_\tau(f(t)) \, d\nu(t)$$

for each $f \in \mathrm{Lsc}([0,1], \mathrm{Cu}(D))$ is a Cuntz category morphism. Since $\mathcal{Z}$ has unique trace, σ maps into $\mathrm{Cu}(\mathcal{Z})$ (which is naturally isomorphic to $V(D) \sqcup (0,\infty]$, by [33, Corollary 6.8]). By [4, Theorem 2.6], the natural map $\mathrm{Cu}(C([0,1], D)) \to$

Lsc([0,1], Cu(D)) is a Cuntz category isomorphism. Combining this isomorphism with σ, one has a Cuntz category morphism from Cu(C([0,1], D)) $\to$ Cu($\mathcal{Z}$). This induces a Cuntz category morphism between $C_0(0,1] \otimes D$ and $\mathcal{Z}$ of the augmented invariant of [54] and by [54, Theorem 1.0.1], there exists a ∗-homomorphism $\pi: C_0(0,1] \otimes D \to \mathcal{Z}$ which then induces an order zero map $\rho: D \to \mathcal{Z}$, by [68, Corollary 4.1]. It is then checked that ρ satisfies the desired tracial identity in [10, Lemma 5.6.5].

In order to move beyond the unique trace case in Theorem 4.5, the order zero maps implementing a colored equivalence cannot be made with a sequence of maps that factor through $\mathcal{Z}$, as that would collapse the trace space to a single point (such maps necessarily preserve the trace space up to isomorphism, by Theorem 4.1). Hence, a replacement for [10, Lemma 5.6.5] is needed. Our formulation of the order zero map realizing prescribed tracial data maps directly into a unital, simple, separable, exact, $\mathcal{Z}$-stable C∗-algebra B with stable rank one whose tracial simplex is isomorphic to that of an AF algebra D (no longer assumed to have unique trace) instead of $\mathcal{Z}$. This order zero map is constructed by showing that the map $\sigma: \mathrm{Cu}(C_0(0,1] \otimes D) \to \mathrm{LAff}_+(TB)$ determined by the rule

$$[d] \mapsto (\mu \otimes \Phi(\cdot))[d]$$

for each $[d] \in \mathrm{Cu}(C_0(0,1] \otimes D)$ is a Cuntz category morphism, where μ is a faithful densely defined lower semicontinuous trace on $C_0(0,1]$. Since $\mathrm{LAff}_+(TB)$ is a subobject of Cu(B) (Lemma 2.12), σ extends to a Cuntz category morphism into Cu(B).

In fact, we could not figure out how to use the characterization of compact containment given in [4] to work directly at the level of the cone over the AF algebra and so it is established that σ is a Cuntz category morphism by showing that it is the inductive limit of Cuntz category morphisms σ_i at the finite stages of the AF algebra inductive limit decomposition. That σ_i is a generalized Cuntz morphism follows from [33]. To show that compact containment is preserved by σ_i, we show (in the proof of Theorem 4.3) that σ_i is a weighted direct sum of copies of the functional $d_\mu: \mathrm{Cu}(C_0(0,1]) \to [0,\infty]$. This reduces the problem of showing that σ_i preserves compact containment to showing that the functional d_μ preserves compact containment. This is done in Section 3, where we give a topological characterization of when functionals arising from a faithful densely defined lower semicontinuous trace preserve compact containment. In particular, an essential property of the half open interval is that it does not contain nonempty compact open sets. (At the end of the section, we give a sufficient condition for when nonfaithful densely defined lower semicontinuous traces induce functionals which preserve compact containment.) In Theorem 4.3, it is shown that the morphisms σ_i give rise to a one-sided intertwining at the level of the Cuntz category. By the classification theorem (of [14]) for ∗-homomorphisms out of cones, this then gives rise to an approximate one-sided intertwining at the level of C∗-algebras. The inductive limit ∗-homomorphism $\pi: C_0(0,1] \otimes D \to B$ then induces the desired order zero map.

The outline of the paper is as follows: In Section 2, we introduce basic notions and terminology along with important results that are essential for the main results. We also include statements which are likely known to experts, but possibly not spelled out in literature. In Section 3, we give a characterization for when a trace τ on a

C*-algebra A induces a functional d_τ on $\mathrm{Cu}(A)$ which preserves compact containment. In Section 4, we prove the main results and discuss some consequences and questions.

2 Preliminaries

2.1 Order zero maps

Let A and B be C*-algebras, and let $\varphi : A \to B$ be a completely positive (c.p.) map (c.p.c. if the map is contractive). The map φ is said to have *order zero* if it preserves orthogonality (see [68, Definition 1.3]). Examples of such maps include $*$-homomorphisms and more generally, the product of a $*$-homomorphism $\pi : A \to B$ with a positive element $h \in B$ which commutes with the image of π. The following structure theorem shows that all order zero maps admit such a decomposition. The $*$-homomorphism π_φ which appears below is called the support $*$-homomorphism of φ.

Theorem 2.1 [68, Theorem 2.3] *Let A and B be C*-algebras, and let $\varphi : A \to B$ be a c.p. order zero map. Define $C := C^*(\varphi(A)) \subseteq B$ (and denote the multiplier algebra of C by $\mathcal{M}(C)$). Then there exist a unique positive element h_φ in the center of $\mathcal{M}(C)$ and a $*$-homomorphism $\pi_\varphi : A \to \mathcal{M}(C)$ such that*

$$\varphi(a) = h_\varphi \pi_\varphi(a) \tag{2.1}$$

for all $a \in A$. Necessarily, $\|h_\varphi\| = \|\varphi\|$, and if A is unital, then $h_\varphi = \varphi(1_A) \in B$.

Corollary 2.2 [6, Proposition 1.4] *Let $\varphi : A \to B$ be a c.p.c. order zero map between C*-algebras where A is unital. Then φ is a $*$-homomorphism if, and only if, $\varphi(1_A)$ is a projection.*

Proof This follows immediately from Theorem 2.1. ∎

The next corollary is known as the order zero functional calculus. To avoid ambiguity with the notation introduced below, we will never use φ^n to mean iterated composition.

Corollary 2.3 [68, Corollary 3.2] *Let $\varphi : A \to B$ be a c.p.c. order zero map, and let the notation be as in Theorem 2.1. Then, for any positive function f in $C_0((0,1])$, the map*

$$f(\varphi) : A \to B$$

defined by

$$f(\varphi)(\cdot) := f(h_\varphi)\pi_\varphi(\cdot) \tag{2.2}$$

is a well-defined c.p. order zero map taking values in C. If f has norm at most one, then the map $f(\varphi)$ is contractive.

Let $\varphi : A \to B$ and $\psi : B \to C$ be c.p.c. order zero maps between unital C*-algebras, and let the notation be as in Theorem 2.1. The following statement concerning the composition $\psi\varphi$ is established in [10] by proving that the support $*$-homomorphism of a composition is essentially the composition of the the individual support $*$-homomorphisms (more precisely, it is shown that $h_{\psi\varphi}\pi_{\psi\varphi} = h_{\psi\varphi}\pi_\psi\pi_\varphi$ [10, Corollary 1.4.14]). We give a slightly more direct proof.

Corollary 2.4 [10, Section 5.6]   *Suppose $\varphi: A \to B$ and $\psi: B \to C$ are c.p.c. order zero maps between C^*-algebras. Then*

$$(2.3) \qquad\qquad (\psi\varphi)^n = \psi^n \varphi^n$$

for each $n \in \mathbb{N}$ (excluding $n = 0$). If φ is a $*$-homomorphism, then $\varphi^n = \varphi$.

Proof   By [68, Proposition 2.2], we may suppose, without loss of generality, that A is unital. Let the notation be as in Theorem 2.1 for φ, ψ and $\psi\varphi$. Then we have

$$(2.4) \qquad\qquad h_{\psi\varphi} = \psi\varphi(1_A) = \psi(h_\varphi) \overset{(2.1)}{=} h_\psi \pi_\psi(h_\varphi),$$

using that A is unital for the first two equalities. Now, for fixed $n \in \mathbb{N}$, we have

$$(\psi\varphi)^{n+1} \overset{(2.2)}{=} h_{\psi\varphi}^{n+1} \pi_{\psi\varphi} = h_{\psi\varphi}^n h_{\psi\varphi} \pi_{\psi\varphi} \overset{(2.4)}{=} h_\psi^n \pi_\psi(h_\varphi^n) h_{\psi\varphi} \pi_{\psi\varphi}$$

$$\overset{(2.1)}{=} h_\psi^n \pi_\psi(h_\varphi^n) \psi\varphi \overset{(2.1)}{=} (h_\psi^{n+1} \pi_\psi(h_\varphi^n) \pi_\psi)(h_\varphi \pi_\varphi)$$

$$= (h_\psi^{n+1} \pi_\psi)(h_\varphi^{n+1} \pi_\varphi) \overset{(2.2)}{=} \psi^{n+1} \varphi^{n+1}.$$

If φ is a $*$-homomorphism, then $h_\varphi = \varphi(1_A)$ is a projection and so $\varphi^n = \varphi$. ∎

The following correspondence between c.p.c. order zero maps and $*$-homomorphisms out of cones is from [68], but the formulation presented here is that of [6, Proposition 1.3]. Both Corollary 2.5 and Lemma 2.6 will be used in the proof of Theorem 4.5.

Corollary 2.5 [68, Corollary 3.1]   *Let A and B be C^*-algebras. There is a one-to-one correspondence between c.p.c. order zero maps $\varphi: A \to B$ and $*$-homomorphisms $\pi: C_0(0,1] \otimes A \to B$ where φ and π are related by the commuting diagram*

$$(2.5) \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\ a \,\mapsto\, t \otimes a\ } & C_0(0,1] \otimes A \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\pi} \\ & & B \end{array}$$

and $t \in C_0(0,1]$ denotes the identity function.

Lemma 2.6   *The order zero functional calculus can be recovered from the identity*

$$(2.6) \qquad\qquad f(\varphi)(a) = \pi(f \otimes a),$$

where f is a positive function in $C_0(0,1]$, $a \in A$, and φ and π are as in the corollary above.

Proof   We may again, by [68, Proposition 2.2], suppose without loss of generality that A is unital. Let the notation be as in Theorem 2.1, and let $f \in C_0(0,1]_+$ be given. By Theorem 2.1 and Corollary 2.5, $h_\varphi = \varphi(1_A) = \pi(t \otimes 1_A)$. So for fixed $n \in \mathbb{N}$ and $a \in A$,

$$\varphi^{n+1}(a) \overset{(2.2)}{=} h_\varphi^{n+1} \pi_\varphi(a) = h_\varphi^n(h_\varphi \pi_\varphi(a)) \overset{(2.1)}{=} h_\varphi^n \varphi(a)$$

$$\overset{(2.5)}{=} \pi(t^n \otimes 1_A) \pi(t \otimes a) = \pi(t^{n+1} \otimes a).$$

It is readily seen from this calculation and approximating f by polynomials in $C_0(0,1]$ that $f(\varphi)(a) = \pi(f \otimes a)$. ∎

2.2 Ultrapowers

Let A be a C*-algebra. The bounded sequence algebra of A, denoted by $l^\infty(A)$, is defined to be the collection of all norm-bounded sequences of elements from A. The ideal of elements in $l^\infty(A)$ which go to zero along the ultrafilter ω is denoted by $c_\omega(A)$. The *ultrapower* of A is the quotient C*-algebra $A_\omega := l^\infty(A)/c_\omega(A)$. We shall write $(a_n)_{n=1}^\infty$ for an element in A_ω rather than the class it belongs to. We will occasionally write a to refer to the image of a under the canonical embedding of A into A_ω. A sequence $\varphi_n : A \to B$ of c.p.c. order zero maps between C*-algebras induces a c.p.c. order zero map $\varphi : A_\omega \to B_\omega$ between ultrapowers. We denote the induced map by $\varphi := (\varphi_n)_{n=1}^\infty$. If A is unital, then, with notation as in Theorem 2.1, we see that $h_\varphi = (h_{\varphi_n})_{n=1}^\infty$ and so the order zero functional calculus for φ can be realized by applying positive functions in $C_0(0,1]$ componentwise to φ_n.

2.3 Traces and finiteness

Let A be a C*-algebra. By a *trace* on A, we will mean a lower semicontinuous function $\tau : A_+ \to [0,\infty]$ that is additive, preserves zero, is positively homogeneous, and satisfies the trace identity $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$. We will denote by N_τ the ideal of elements $a \in A$ such that $\tau(a^*a) = 0$. A trace τ is said to be *faithful* if $\tau(a^*a) = 0$ implies $a = 0$. We will use the notation $T(A)$ (or simply TA) to denote the collection of all tracial states on A—a Choquet simplex if A is unital. The *limit traces* of A_ω are the tracial states on A_ω which are equal to $\lim_{n\to\omega} \tau_n$ where (τ_n) is some sequence of tracial states on A. We will denote the collection of limit traces on A_ω by $T_\omega(A_\omega)$. We will occasionally not make a notational distinction between a trace in $T(A)$ and (what we will refer to as) the constant limit trace in $T(A_\omega)$ induced by it.

By [68, Corollary 3.4], a c.p.c. order zero map $\varphi : A \to B$ induces a mapping of bounded traces $\varphi^* : \mathbb{R}_+ T(B) \to \mathbb{R}_+ T(A)$. We would be be interested in the case that φ^* is an isomorphism of tracial simplices, but this hope turns out not to be realistic. Rather, considering a sequence of c.p.c. order zero maps $\varphi_k : A \to B$, we will be interested in the case that the c.p.c. order zero map $\varphi : A_\omega \to B_\omega$ induced by (φ_k) preserves constant limit traces (i.e., takes constant limit traces on B_ω to constant limit traces on A_ω), and furthermore, the map $\tau \mapsto \lim_{k\to\omega} \tau\varphi_k, \tau \in T(B)$, is an isomorphism of the simplices $T(B)$ and $T(A)$. (Note that requiring the sequence (φ_k) to preserve constant limit traces is equivalent to requiring the sequence $(\tau\varphi_k)$ to be norm convergent for every $\tau \in T(B)$.) More generally, we shall be interested in the condition that a constant limit trace preserving c.p.c. order zero map $\varphi : A_\omega \to B_\omega$ (not necessarily arising from a sequence (φ_k) as above) induces an isomorphism $T(B) \to T(A)$ of tracial simplices via the composed map $\iota_A^* \varphi^* c_B$ where $c_B : T(B) \to T(B_\omega)$ denotes the embedding of $T(B)$ as constant limit traces on $T(B_\omega)$. (One might also consider the tracial cones, instead, and ask when they are isomorphic.)

Calculations involving traces on A_ω will often be reduced to the case of limit traces by using the following fact about weak* density of limit traces. Generalizations of the statement presented here can be found in [46, 47, 57].

Theorem 2.7 [47, Theorem 8] *Let A be a separable, exact, and $\mathcal{Z}$-stable C*-algebra (where $\mathcal{Z}$ denotes the Jiang–Su algebra [39]). Then $T_\omega(A)$ is weak* dense in $T(A_\omega)$.*

Let A be a C*-algebra with nonempty tracial state space. We define the seminorm $\|\cdot\|_{2,T(A)}$ by

$$\|a\|_{2,T(A)} := \sup_{\tau \in T(A)} (\tau(a^*a))^{1/2},$$

for each $a \in A$. The *trace-kernel ideal* of A_ω is the set

$$J_A := \{(a_n) \in A_\omega : \lim_{n \to \omega} \|a_n\|_{2,T(A)} = 0\},$$

and the quotient of A_ω by J_A is called the *uniform tracial ultrapower* of A, and it is denoted by A^ω. More details about these notions can be found in [11, Section 1] and [41, Section 4].

Warning: In literature, A^ω has sometimes been used for the ultrapower of A, while A_ω has sometimes been used for the central sequence algebra of A. We remind the reader that A_ω denotes the ultrapower of A and that A^ω denotes the uniform tracial ultrapower of A.

Lemma 2.8   *Let A be a C*-algebra, and let B be a C*-algebra with $T(B)$ nonempty such that the limit traces of B_ω are dense in $T(B_\omega)$ (e.g., C*-algebras as in Theorem 2.7), and let $\varphi, \psi : A \to B_\omega$ be c.p.c. order zero maps. Then $\tau\varphi = \tau\psi$ for all $\tau \in T(B_\omega)$ if, and only if, φ agrees with ψ in the uniform tracial ultrapower B^ω.*

Proof   Let $\pi : B_\omega \to B^\omega$ denote the canonical quotient map. It is enough to show that an element of B_ω is in the kernel of every trace in $T(B_\omega)$ exactly when it belongs to the trace-kernel ideal of B_ω. Suppose $(b_n) \in \ker \tau$ for every $\tau \in T(B_\omega)$. For each $n \in \mathbb{N}$, there exists a trace $\tau_n \in T(B)$ such that $\tau_n(b_n^*b_n)$ is within $1/n$ of $\|b_n\|_{2,T(B)}^2$. By assumption, (b_n) is in the kernel of the limit trace $\lim_{n \to \omega} \tau_n$ on $T(B_\omega)$. Since the kernel of a trace is a left ideal, it follows that

$$\|\pi(b_n)\|^2 = \lim_{n \to \omega} \|b_n\|_{2,T(B)}^2 = \lim_{n \to \omega} \sup_{\tau \in T(B)} \tau(b_n^*b_n) = \lim_{n \to \omega} \tau_n(b_n^*b_n) = 0.$$

Conversely, suppose $b = (b_n)$ is in the trace-kernel ideal of B_ω. Since the limit traces of B_ω are weak* dense in $T(B_\omega)$, it is enough to show that $\tau(b) = 0$ for every limit trace $\tau = \lim_{n \to \omega} \tau_n \in T(B_\omega)$. This follows from the computation

$$|\tau(b)| = \lim_{n \to \omega} |\tau_n(b_n)| \le \lim_{n \to \omega} \tau_n(|b_n|) \le \lim_{n \to \omega} \|b_n\|_{2,T(B)} = \|\pi(b)\| = 0.$$

The first and second inequalities follow from [12, Theorem 2.7] and [12, Corollary 2.8], respectively.   ∎

The following statement holds somewhat more generally (see [57, Corollary 5.1]), but this is the setting which will be relevant for us in Theorem 4.5.

Theorem 2.9 [57, Corollary 5.1]   *Let A be a (nonzero) unital, simple, exact, $\mathcal{Z}$-stable C*-algebra. Then A is either purely infinite or finite.*

Remark 2.10   When A is unital, simple, and $\mathcal{Z}$-stable, the proof of the preceding theorem shows that the statement of [57, Theorem 6.7] can be slightly strengthened to say that the following three statements are equivalent:

(1) A is finite.

(2) *A* has stable rank one.

(3) *A* is stably finite.

2.4 The Cuntz category

If a and b are elements of an ordered set M, we will say that a is *(countably) compactly contained* in b, and we write $a \ll b$, if for any increasing sequence $(b_k)_{k=1}^{\infty}$ in M with $\sup_k b_k \geq b$ (or such that every upper bound of the sequence majorizes b), eventually $b_k \geq a$. An increasing sequence with each term compactly contained in the next is called *rapidly increasing*. An element which is compactly contained in itself is called *compact* (more precisely, *countably compact*).

Cuntz category semigroups are ordered abelian semigroups with an additive identity with the following four properties:

(O1) Every increasing sequence in S has a supremum in S.

(O2) Any element of S is the supremum of a rapidly increasing sequence.

(O3) If a_i and b_i are elements of S such that $a_i \ll b_i$ for $i = 1, 2$, then $a_1 + a_2 \ll b_1 + b_2$.

(O4) If (a_n) and (b_n) are increasing sequences in S, then $\sup_n(a_n + b_n) = \sup_n a_n + \sup_n b_n$.

Cuntz category morphisms $f : S \to T$ are ordered semigroup maps (i.e., preserving order, addition, and the additive identity) which preserve suprema of increasing sequences and compact containment. A *generalized Cuntz category morphism* is a Cuntz category morphism which does not necessarily preserve compact containment. The Cuntz category has as objects the ordered semigroups and ordered semigroup maps with the properties stipulated above (see ([18, 33, 54])). It is easily checked, and it will be used without mention that Cartesian products and direct sums exist in this category.

It was shown in [18] that there is a functor $\mathrm{Cu}(\cdot)$ from the category of C*-algebras to the Cuntz category. Recall that, for a C*-algebra A, $\mathrm{Cu}(A)$ is the ordered semigroup of Cuntz equivalence classes of positive elements in the stabilization of A and is a Cuntz category semigroup. We shall denote by $[a]$ the Cuntz equivalence class of a positive element a of $A \otimes \mathcal{K}$. We shall denote by $V(A)$ the semigroup of Murray–von Neumann equivalence classes of projections in the stabilization of A. For a compact convex subset K of a locally convex topological vector space, we shall denote by $\mathrm{LAff}_+(K)$ the collection of lower semicontinuous extended positive real-valued affine functions on K which are strictly positive, except for the zero function which we include in $\mathrm{LAff}_+(K)$, and are the pointwise supremum of an increasing sequence of continuous and finite-valued such functions. If K is metrizable, then the latter condition is automatic. It was observed in Section 2 of [31] (and in Section 3 of [62]) that $\mathrm{LAff}_+(K)$, equipped with pointwise order and addition, is a Cuntz category semigroup. The addition and order structure defined on the disjoint union decomposition appearing in the following important computation can be found in Section 6 of [33]. Related work on the unstabilized Cuntz semigroup that was done prior to the result stated here can be found in [9, 21, 49]. Recent developments that go well beyond what is needed in this paper include [3, Theorem 7.15] and [62, Theorem 8.11].

Theorem 2.11 [33, Corollary 6.8]   *Let A be a unital, simple, separable, exact, $\mathcal{Z}$-stable, and finite C^*-algebra. Then, in a natural way,*

$$\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}_+(TA)\backslash\{0\}.$$

Lemma 2.12   *We note that $\mathrm{LAff}_+(TA)$ is a subobject of $\mathrm{Cu}(A)$ in the setting of Theorem* 2.11, *according to the embedding there.*

Proof   Since the elements of $V(A)$ are compact in $\mathrm{Cu}(A)$ (by Corollary 5 of [18]; this statement does not use stable rank one), it is clear that the embedding of $\mathrm{LAff}_+(TA)$ into $\mathrm{Cu}(A)$ is a generalized Cuntz category morphism. To show that compact containment is preserved, let $f \ll g$ in $\mathrm{LAff}_+(TA)$ and let $(g_n)_{n=1}^\infty$ be an increasing sequence in $\mathrm{Cu}(A)$ such that $\sup_n g_n$ majorizes g. Passing to a subsequence, we may suppose that the sequence $(g_n)_{n=1}^\infty$ is composed entirely of projections or entirely of affine functions. In the latter case, since the embedding of $\mathrm{LAff}_+(TA)$ in $\mathrm{Cu}(A)$ preserves increasing sequential suprema, we have $\sup_n g_n \in \mathrm{LAff}_+(TA)$ and so there is, by assumption, some g_n which majorizes f. Let us now consider the former case. Denote by $\widehat{g_k}$ the rank of g_k—the element of $\mathrm{LAff}_+(TA)$ gotten by evaluation of traces in TA at the projection g_k.

By the characterization of compact containment in $\mathrm{LAff}_+(TA)$ in Section 2 of [31] (or [62, Lemma 3.6]), there exists an $h \in \mathrm{LAff}_+(TA)$ which is continuous and finite-valued and an $\varepsilon > 0$ such that $f \le h < (1+\varepsilon)h \le g$. Since $0 \notin \mathrm{LAff}_+(TA)$, h is strictly positive. It follows that there is, for each $\tau \in TA$, a neighbourhood of τ on which f is strictly majorized by some $\widehat{g_k}$. By compactness of TA and the fact that $(g_n)_{n=1}^\infty$ is increasing, f is majorized by some $\widehat{g_n}$ on all of TA. This shows $f \ll g$ in $\mathrm{Cu}(A)$.

To check that $\mathrm{LAff}_+(TA)$ is a subobject, it remains to note that $f \ll g$ holds in $\mathrm{LAff}_+(TA)$ if it holds in $\mathrm{Cu}(A)$. This follows immediately from the fact that increasing sequential suprema in $\mathrm{LAff}_+(TA)$ are the same in $\mathrm{Cu}(A)$ (see above). ∎

Examples of Cuntz category semigroups not explicitly involving a C^*-algebra include $\overline{\mathbb{N}} := \{0, 1, 2, \ldots, \infty\}$ and $[0, \infty]$ with the usual order and addition taken from $\mathbb{R}$. In fact, these Cuntz objects both arise from C^*-algebras. Let $\mathrm{Lsc}(X, M)$ denote the collection of lower semicontinuous functions from a space X to a Cuntz category semigroup M, equipped with pointwise order and addition. Then $\mathrm{Lsc}(X, M)$ is a Cuntz category semigroup whenever X is a second-countable compact Hausdorff space with finite covering dimension and M is countably based (see [4, Theorem 5.17]). It was also shown in [66, Corollary 4.22] that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a Cuntz category object when X is a compact metric space. It was proved in [14, Theorem 10.1] and [38, Theorem 6.11] that $\mathrm{Cu}(C_0(X)) \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$, via the rank map, if X is $[0, 1]$ or $(0, 1]$. More generally, if X is a locally compact Hausdorff space with covering dimension at most two and $\check{H}_2(K) = 0$ (Čech cohomology with integer coefficients) for every compact subset K of X, then again (via the rank map) $\mathrm{Cu}(C_0(X)) \cong \mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$, the ordered semigroup of lower semicontinuous extended positive integer-valued functions f such that $f^{-1}(k, \infty]$ is σ-compact for each $k \in \mathbb{N}$ (see [55, Theorem 1.1]). If $C_0(X)$ is separable, these are the exact conditions needed on X for $\mathrm{Cu}(C_0(X))$ to be isomorphic (via the rank map) to $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ (see [55, Theorem 1.3]). Note that X is a hereditarily Lindelöf locally compact Hausdorff space (as is the case when $C_0(X)$ is separable) if, and only if, $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}}) = \mathrm{Lsc}(X, \overline{\mathbb{N}})$. In general, $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ is a Cuntz

category object, and in fact (Theorem 3 of [26]) a subobject of $\mathrm{Cu}(C_0(X))$ (this is noted in [63, Corollary 4.8] in the case that X is metrizable and compact).

Proposition 2.13 *Let X be a locally compact Hausdorff space. Then $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ is a Cuntz category object.*

Proof Let us first recall a characterization of compact containment for σ-compact elements in the lattice of open subsets of X, ordered by inclusion. If there is a compact subset K of X sitting in between open sets U and V, then U is compactly contained in V. For the converse, suppose U is compactly contained in V and that V is σ-compact. Since X is locally compact Hausdorff, V is locally compact and by assumption, V is σ-compact so there exist an increasing sequence of open sets $(V_k)_{k=1}^\infty$ and an increasing sequence of compact sets $(K_k)_{k=1}^\infty$ such that $V_k \subseteq K_k \subseteq V_{k+1}$ for each k and $\sup_k K_k = V$. Since U is compactly contained in V, there exists a k such that $U \subseteq V_k \subseteq K_k \subseteq V$.

In the latter construction, we claim each V_k can be chosen to be σ-compact. Set $V_1 := \varnothing$. By Urysohn's lemma, there exists a continuous function f_{k+1} from X into $[0,1]$ which is equal to 1 on K_k and is equal to 0 on $X \setminus V_{k+1}$. The open support of f_{k+1}, with which we replace V_{k+1}, is an F_σ-set sitting in between K_k and K_{k+1}, and is therefore σ-compact.

(O1) Let $(f_n)_{n=1}^\infty$ be an increasing sequence in $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ and denote by $f := \sup_n f_n$. Then for each $k \in \mathbb{N}$, $\{f > k\} = \bigcup_{n=1}^\infty \{f_n > k\}$. (Abbreviated notation.) Since f_n is lower semicontinuous, each $\{f_n > k\}$ is open and so f is lower semicontinuous. Since each $\{f_n > k\}$ is σ-compact, a diagonalization argument shows $\{f > k\}$ is σ-compact. This shows $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ is closed under suprema of increasing sequences.

(O4) is easily verified. It is also easy to see that if χ_U and χ_V are characteristic functions in $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$, then $\chi_U \ll \chi_V$ if, and only if, $U \ll V$. We will now show that if $f, g \in \mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$, then $f \ll g$ if, and only if, f is bounded and $\{f > k\} \ll \{g > k\}$ for each k. Suppose f is compactly contained in g. Each $\{g > k\}$ is the supremum of a rapidly increasing sequence of σ-compact open sets $(U_{k,i})_{i=0}^\infty$ and so there exist compact sets $K_{k,i}$ such that $U_{k,i} \subseteq K_{k,i} \subseteq U_{k+1,i}$ for each i. Moreover, by taking finite unions we may choose $U_{k,i}$ and $K_{k,i}$ so that $U_{k,i} \supseteq U_{k+1,i}$ and $K_{k,i} \supseteq K_{k+1,i}$ whenever $k \le i$. This gives the following inclusions:

$$
\begin{array}{ccccccccc}
U_{0,0} & \hookrightarrow & K_{0,0} & \hookrightarrow & U_{0,1} & \hookrightarrow & K_{0,1} & \hookrightarrow & U_{0,2} & \hookrightarrow & K_{0,2} \\
& & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & & & U_{1,1} & \hookrightarrow & K_{1,1} & \hookrightarrow & U_{1,2} & \hookrightarrow & K_{1,2} \qquad \cdots \\
& & & & & & & & \uparrow & & \uparrow \\
& & & & & & & & U_{2,2} & \hookrightarrow & K_{2,2}
\end{array}
$$

The supremum of the increasing sequence of functions $(\sum_{k=0}^i \chi_{U_{k,i}})_{i=0}^\infty$ is g. Since f is compactly contained in g, f is majorized by $\sum_{k=0}^n \chi_{U_{k,n}}$ for some n and so f is bounded. By construction, $\{f > k\} \subseteq U_{k,n} \subseteq K_{k,n} \subseteq \{g > k\}$ for $k \le n$ (and if $k > n$, then $\{f > k\} = \varnothing \ll \{g > k\}$). This shows $\{f > k\}$ is compactly contained in $\{g > k\}$ for each k.

Conversely, suppose f is bounded and that $\{f > k\} \ll \{g > k\}$ for each k. Let an increasing sequence $(g_i)_{i=1}^{\infty}$ whose supremum majorizes g be given. Since f is bounded, $f = \sum_{k=0}^{n} \chi_{\{f>k\}}$ for some n. Since $\{f > k\} \ll \{g > k\}$ for each k, there exists a compact set K_k which sits in between $\{f > k\}$ and $\{g > k\}$. Using that $(g_i)_{i=1}^{\infty}$ is increasing, and that g_i is extended positive integer-valued and lower semicontinuous, by compactness of each K_k, g_i eventually majorizes $\sum_{k=0}^{n} \chi_{K_k}$ and therefore g_i eventually majorizes f. This shows f is compactly contained in g.

(O3) Suppose $f_i \ll g_i$ for $i = 1, 2$. Then $f_1 + f_2$ is bounded (since f_1 and f_2 are bounded). There is, for each k, a compact set $K_{i,k}$ sitting in between $\{f_i > k\}$ and $\{g_i > k\}$ for $i = 1, 2$. Taking the convention that $K_{i,-1} = X$, we have

$$\{f_1 + f_2 > k\} = \bigcup_{i=0}^{k+1} \{f_1 > k - i\} \cap \{f_2 > i - 1\}$$
$$\subseteq \bigcup_{i=0}^{k+1} (K_{1,k-i} \cap K_{2,i-1})$$
$$\subseteq \bigcup_{i=0}^{k+1} \{g_1 > k - i\} \cap \{g_2 > i - 1\} = \{g_1 + g_2 > k\}.$$

This shows that $\{f_1 + f_2 > k\} \ll \{g_1 + g_2 > k\}$ for each k, so (see the proof of (O4) above) $f_1 + f_2$ is compactly contained in $g_1 + g_2$.

(O2) Let $g \in \mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ be given. Each $\{g > k\}$ is the supremum of a rapidly increasing sequence of open sets $(U_{k,i})_{i=0}^{\infty}$ and so g is the supremum of the increasing sequence $(\sum_{k=0}^{n} \chi_{U_{k,n}})_{n=0}^{\infty}$. By (O3), this sequence is rapidly increasing. ∎

Theorem 2.14 [55, Theorem 1.1]   *Let X be a locally compact Hausdorff space of covering dimension at most two with $\check{H}_2(K) = 0$ for every compact subset K of X. Then $\mathrm{Cu}(C_0(X)) \cong Lsc_\sigma(X, \overline{\mathbb{N}})$, the isomorphism being given by the rank map.*

In order to lift Cuntz category morphisms to ∗-homomorphisms, we will need the following classification theorem of [14]. Further developments in this direction include [15, Theorem 1], [54, Theorem 1.0.1], and [56, Theorem 2].

Theorem 2.15 [14, Theorem 4.1 and remark page 29]   *Let B be a stable rank one C^*-algebra, and let s_B be a strictly positive element of B. If $\sigma : \mathrm{Cu}(C_0(0,1]) \to \mathrm{Cu}(B)$ is a Cuntz category morphism taking $[t]$ (where t denotes the identity map on $(0,1]$) into $[s_B]$, or into $g \le [s_B]$, then there is a ∗-homomorphism $\pi : C_0(0,1] \to B$, which is unique up to approximate unitary equivalence, such that $\mathrm{Cu}(\pi) = \sigma$.*

2.5 Embedding theorems

Theorem 2.16 [40, Theorem 2.8]   *If A is a unital, separable, and exact C^*-algebra, then there exists a unital embedding of A into $\mathcal{O}_2$.*

Theorem 2.17   *If A is a unital, finite, $\mathcal{Z}$-stable, simple, separable amenable C^*-algebra satisfying the UCT), then there is a unital embedding of A into a unital, simple, separable AF algebra D giving rise to an isomorphism of tracial simplices.*

Proof Let $\mathcal{Q}$ denote the universal UHF algebra. Since there is an embedding of A into $A \otimes \mathcal{Q}$ which induces an isomorphism of tracial simplices, we may suppose, without loss of generality, that A is $\mathcal{Q}$-stable. (This will be used in applying [37] below.)

Let $\rho: K_0(A) \to \text{Aff}(T(A))$ denote the map associated with the canonical pairing of $K_0(A)$ with $T(A)$. Choose a countable dense subgroup G of $\text{Aff}(T(A))$ containing the image of ρ. By [2, 44], $\text{Aff}(T(A))$ with the strict pointwise order has the Riesz interpolation property, and therefore G does also. By [23], there exists a unital, simple, separable AF algebra D with $K_0(D) = G$. Since $K_0(D)$ is dense in $\text{Aff}(T(A))$, the map $T(A) \to S(K_0(D))$ is an isomorphism. Since D is an AF algebra, $T(D) = S(K_0(D))$, and so we have an isomorphism $\Phi: T(D) \to T(A)$. We now have compatible maps $\rho: K_0(A) \to K_0(D)$ and $\Phi: T(D) \to T(A)$. Furthermore, A and we may suppose also D are $\mathcal{Q}$-stable. By [11, Theorem A], [29, Theorem 1.1], and [64, Theorem A], $A \otimes \mathcal{Q}$ has generalized tracial rank at most one—and therefore, A does too as it is $\mathcal{Q}$-stable. The only additional constituent of the invariant (see [37, Definition 2.4]) used in the homomorphism theorem [37, Corollary 21.11] is a K_1-map which we can take to be zero. The corresponding algebra map is the desired embedding. ∎

2.6 Uniqueness theorems

A couple of the key technical ingredients used in the proof of Theorem 4.5 are the following uniqueness theorems for maps from unital, separable amenable C*-algebras into unital, simple, separable, $\mathcal{Z}$-stable C*-algebras. These results were developed in [6] with certain restrictions on the trace space. These tracial assumptions were removed in [11]. For the statement involving a Kirchberg algebra (i.e., a unital, purely infinite, simple, separable amenable C*-algebra) as the codomain, we specialize to the case that the maps are injective $*$-homomorphisms. Such maps φ induce injective c.p.c. order zero maps $(\varphi - t)_+$ for each $t \in [0, 1)$ and so [6, Corollary 9.11] is applicable to them. (To see this, it suffices to show that $(h_\varphi - t)_+$ is a nonzero scalar multiple of the projection $\pi(1_A)$, where notation is as in Theorem 2.1 (note that, here, $\pi_\varphi = \varphi$). Since h_φ is a projection (namely, $\pi_\varphi(1_A)$), $(h_\varphi - t)_+ = (1 - t)h_\varphi$.)

Theorem 2.18 [6, Corollary 9.11] *Let A be a unital, separable, amenable C*-algebra, and let B be a Kirchberg algebra. Let $\phi_1, \phi_2: A \to B_\omega$ be a pair of injective $*$-homomorphisms. Then, there exist contractions $v_i, w_i \in B_\omega$ for $i = 1, 2$, such that*

$$\phi_1(a) = w_1 \phi_2(a) w_1^* + w_2 \phi_2(a) w_2^*,$$
$$\phi_2(a) = v_1 \phi_1(a) v_1^* + v_2 \phi_1(a) v_2^*,$$

for all $a \in A$, and

$$v_1^* v_1 + v_2^* v_2 = w_1^* w_1 + w_2^* w_2 = 1_{B_\omega}$$

with $w_i^* w_i \in \phi_2(A)'$ and $v_i^* v_i \in \phi_1(A)'$ for $i = 1, 2$.

Definition 2.19 Let A and B be unital C*-algebras. Any two order zero maps $\phi_1, \phi_2: A \to B_\omega$ satisfying the conclusion of Theorem 2.18 will be said to be *colored equivalent* (see [6, Section 6]).

For the statement involving a finite algebra as the codomain, only one map is assumed to be a homomorphism.

Theorem 2.20 [6, Theorem 6.6], [11]   *Let A be a unital, separable amenable C^*-algebra, and let B be a unital, finite, $\mathcal{Z}$-stable, simple, separable amenable C^*-algebra. Let $\phi_1 : A \to B_\omega$ be a totally full $*$-homomorphism (i.e., $\phi_1(a)$ generates B_ω as a closed two-sided ideal for every nonzero $a \in A$), and let $\phi_2 : A \to B_\omega$ be a c.p.c. order zero map with*

$$\tau \circ \phi_1 = \tau \circ \phi_2^m$$

for all $\tau \in T(B_\omega)$ and all $m \in \mathbb{N}$. Then, ϕ_1 and ϕ_2 are colored equivalent.

Proof   By [11, Theorem I], B has complemented partitions of unity. Now the proof is exactly as in [6, Theorem 6.6] using [11, Lemma 4.8] in place of [6, Theorem 5.5]. The same lemma also permits relaxing the tracial extreme boundary hypothesis in [6, Theorem 6.2]. ∎

In practice, the $*$-homomorphism typically used in Theorem 2.20 is essentially the canonical embedding $\iota_A : A \to A_\omega$ (e.g., [6, Corollary 6.5 and Theorem 7.5]), which is totally full if A is simple. To see this, let a nonzero element $a \in A$ be given. If A is unital, $\iota_A(a)$ can be cut down to any coordinate with a projection. In the nonunital case, the coordinate projection can be replaced with an approximate unit in a coordinate. Since A is simple, a copy of A is generated in each coordinate of A_ω. This shows ι_A is totally full.

2.7 Colored isomorphism

We introduce two notions of colored isomorphism. The first one (Definition 2.21) is symmetrically formulated, but it is a little long. A weaker notion (Definition 2.23) is all that's needed to establish Theorem 4.1. Together with Theorem 4.5, it follows that these two notions coincide in the classifiable setting.

Definition 2.21   Unital C^*-algebras A and B will be said to be *colored isomorphic* if there exist c.p.c. order zero maps $\varphi : A_\omega \to B_\omega$ and $\psi : B_\omega \to A_\omega$ such that $\psi\varphi\iota_A$ is colored equivalent to ι_A (see Definition 2.19) and $\varphi\psi\iota_B$ is colored equivalent to ι_B; and φ^* and ψ^* preserve constant limit traces.

Remark 2.22   Colored isomorphism is reflexive and symmetric. It follows from Corollary 4.6 that transitivity holds for classifiable C^*-algebras (cf. [6, Section 6]), but transitivity is not clear in general.

Definition 2.23   Unital C^*-algebras A and B will be said to be *minimalist colored isomorphic* if there exist constant limit trace preserving c.p.c. order zero maps $\varphi : A_\omega \to B_\omega$ and $\psi : B_\omega \to A_\omega$, and contractions $u_1, \ldots, u_m \in A_\omega$ and $v_1, \ldots, v_n \in B_\omega$, such that

$$(2.7) \qquad \sum_{i=1}^m u_i \psi\varphi(a) u_i^* = a \quad \text{and} \quad \sum_{j=1}^n v_j \varphi\psi(b) v_j^* = b$$

for all $a \in A$ and for all $b \in B$, and, in addition,

$$(2.8) \qquad \sum_{i=1}^m u_i^* u_i = 1_{A_\omega}, \quad \sum_{j=1}^n v_j^* v_j = 1_{B_\omega}.$$

We note that (2.7) is weaker than requiring that $\psi\varphi\iota_A$ and ι_A are colored equivalent (and requiring that $\psi\varphi\iota_B$ and ι_B are colored equivalent), as required by Definition 2.21. It follows from Theorems 4.1 and 4.5 that φ and ψ can be chosen so that the maps $\psi\varphi\iota_A$ and ι_A are colored equivalent (and likewise for $\psi\varphi\iota_B$ and ι_B) in the classifiable setting.

The identity (2.7) implies that φ and ψ are injective; furthermore, the commutation relations coming from the colored equivalences of maps—the symmetrized form of colored isomorphism holding automatically in the classifiable case—guarantee that $u_i\psi\varphi(\cdot)u_i^*$ and $v_j\varphi\psi(\cdot)v_j^*$ are order zero for $i = 1,\dots,m$ and for $j = 1,\dots,n$.

3 Functionals preserving compact containment

The main result of this section characterizes when a functional on the Cuntz semigroup $\mathrm{Cu}(A)$ arising from a faithful densely defined lower semicontinuous traces on a commutative C*-algebra, A, with the property that the rank map gives rise to an isomorphism $\mathrm{Cu}(A) \cong \mathrm{Lsc}_\sigma(\widehat{A}, \overline{\mathbb{N}})$, preserves compact containment. Let us first establish some necessary conditions for a functional to preserve compact containment in a more general context. Suppose λ is a functional (i.e., a generalized Cuntz category morphism into $[0,\infty]$) on a positive Cuntz category object M. In order for λ to preserve compact containment, λ must be finite on any element that is compactly contained in some element of M. Moreover, λ must be finite on any element that is majorized by a finite sum of such elements. It is also necessary that λ vanish on compact elements. Now recall, for instance from [33, Proposition 4.2], that a lower semicontinuous trace τ on a C*-algebra induces a functional d_τ on $\mathrm{Cu}(A)$ (the trace of the range projection of a positive element). (In later sections, we will not make a distinction between τ and d_τ.) The finiteness condition requires d_τ to be finite-valued on the Pedersen ideal of $A \otimes \mathcal{K}$. If τ is faithful, the latter condition requires $\mathrm{Cu}(A)$ to have no nonzero compact elements. If A is stably finite, then, by [8, Corollary 3.6], this is equivalent to A being stably projectionless. If A is, moreover, commutative, this is equivalent to the spectrum of A not containing any nonempty compact open subsets. To see this, suppose K is a nonempty compact open subset of the spectrum of A. Then $[\chi_K \otimes p]$ is a nonzero compact element of $\mathrm{Cu}(A)$ for any nonzero projection $p \in \mathcal{K}$. Conversely, suppose $A \otimes \mathcal{K}$ contains a nonzero projection p. Then the map $\eta : \widehat{A} \ni x \mapsto \|p(x)\| \in \mathbb{C}$ is continuous. Since p vanishes at infinity and is a nonzero projection, the same is true of η. This shows A contains a nonzero projection.

Theorem 3.1 *Suppose that μ is a faithful densely defined lower semicontinuous trace on a commutative C*-algebra, $C_0(X)$, with the property that $\mathrm{Cu}(C_0(X)) \cong \mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ (the isomorphism being given by the rank map) (for example, X is as in Theorem 2.14). Then the functional $d_\mu : \mathrm{Cu}(C_0(X)) \to [0,\infty]$ preserves compact containment if, and only if, X does not contain any nonempty compact open sets.*

Proof Denote by ι the natural set-theoretical mapping of $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ into $\mathrm{Cu}(C_0(X))$:

$$f = \sum_{k=0}^{\infty} \chi_{\{f>k\}} \mapsto \sum_{k=0}^{\infty} [f_k \otimes p_k],$$

where f_k is a positive function in $C_0(X)$ with open support equal to $\{f > k\}$ and $(p_k)_{k=0}^\infty$ is a family of mutually orthogonal rank-one projections in $\mathcal{K}$. By hypothesis, rank: $\mathrm{Cu}(C_0(X)) \to \mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ is an isomorphism. It follows, since as easily checked, $\mathrm{rank} \circ \iota = \mathrm{id}_{\mathrm{Lsc}_\sigma(X,\overline{\mathbb{N}})}$, that ι is an isomorphism. Denote by μ: $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}}) \to [0, \infty]$ the functional $d_\mu \circ \iota$ and denote also by μ the locally finite (as the Pedersen ideal of $C_0(X)$ is $C_c(X)$ and by [48, Theorem 1.3]) extended Radon measure induced by the densely defined lower semicontinuous trace μ. Since $d_\mu = \mu \circ \mathrm{rank}$, to show that d_μ preserves compact containment, it suffices to show that μ preserves compact containment. Suppose f and g are elements of $\mathrm{Lsc}_\sigma(X, \overline{\mathbb{N}})$ and that $f \ll g$. If $\mu(f) = 0$, then $\mu(f) \ll \mu(g)$ because 0 is compactly contained in every element of $[0, \infty]$. Let us now consider the case where $\mu(f) > 0$. Since f is compactly contained in g, by the proof of (O4) in Proposition 2.13, $f = \sum_{k=0}^n \chi_{\{f>k\}}$ for some n and there exists a compact set $K \subseteq X$ such that (with abbreviated notation) $\{f > 0\} \subseteq K \subseteq \{g > 0\}$, and so $\mu(f) \le (n+1)\mu(K) < \infty$. Now suppose $\mu(f) = \mu(g)$, i.e., $\sum_{k=0}^\infty \mu(\{f > k\}) = \sum_{k=0}^\infty \mu(\{g > k\})$. Since $f \le g$ and since $\mu(f)$ is finite, we have $\mu(\{f > k\}) = \mu(\{g > k\})$ for each k. If $K = \{g > 0\}$, then since X contains no nonempty compact open sets, $\{g > 0\}$ is empty. This implies g is zero and hence f is zero, which is a contradiction. In other words, K is properly contained in $\{g > 0\}$, and so there is a nonempty open subset of $\{g > 0\}$ which is disjoint from K and therefore disjoint from $\{f > 0\}$ as well. Since μ is faithful, $\mu(\{f > 0\}) < \mu(\{g > 0\})$ and so (since $f \le g$ and $\mu(f)$ is finite) $\mu(f) < \mu(g)$.   ∎

Remark 3.2 A different proof of Theorem 3.1, in the case of the half-open interval (the case pertinent to this paper), using the characterization of compact containment given in [4] is possible.

Proof Suppose $f, g \in \mathrm{Lsc}((0,1], \overline{\mathbb{N}})$ are such that $f \ll g$. Let μ be as in the proof of Theorem 3.1 and suppose f is nonzero. Extend both f and g to be zero at zero and call these extensions $\widetilde{f}$ and $\widetilde{g}$. It is easily checked that these extensions are lower semicontinuous and that $\widetilde{f}$ remains compactly contained in $\widetilde{g}$ in $\mathrm{Lsc}([0,1], \overline{\mathbb{N}})$ (one could also use [14, Theorem 10.1], [16, Theorem 5], and [38, Theorem 6.11]). An application of [4, Proposition 5.5] at the point zero shows that $\widetilde{f}$ is zero on a neighbourhood of zero because $\widetilde{g}(0) = 0$. This shows f is compactly supported, and since f is bounded, $\mu(f)$ is finite. Since f is nonzero and extended positive integer-valued, its extension $\widetilde{f}$ is necessarily discontinuous. Its first point of discontinuity, x_0, must occur within $(0,1)$. By [4, Proposition 5.5] applied to the point x_0, there exists a neighbourhood U of x_0 which is contained in $(0,1)$ and a nonzero constant $c \in \overline{\mathbb{N}}$ such that $\widetilde{f} \le c \ll \widetilde{g}$ on U. Since $\widetilde{f}$ is zero to the left of x_0, and since $\widetilde{f} \le \widetilde{g}$, the integral of f on U is strictly less than the integral of c on U, and hence also of g. This implies $\mu(f) < \mu(g)$.   ∎

Remark 3.3 The characterization of compact containment in [18] can be used to show that if τ is a faithful densely defined lower semicontinuous trace on a C^*-algebra A, then d_τ preserves compact containment if, and only if, $\mathrm{Cu}(A)$ does not contain nonzero compact elements. Since the only case of interest in this paper is covered by Theorem 3.1, we will not prove this.

Remark 3.4 A simple reduction to the faithful case shows that if τ is a (not necessarily faithful) densely defined lower semicontinuous trace on A and $\mathrm{Cu}(A/N_\tau)$ does not contain nonzero compact elements, where N_τ denotes the kernel of τ, then d_τ preserves compact containment.

Proof Denote by $\pi_{N_\tau} : A \to A/N_\tau$ the canonical quotient map and let $\eta : (A/N_\tau)_+ \to [0,\infty]$ be the faithful densely defined lower semicontinuous trace induced by τ. Suppose $\mathrm{Cu}(A/N_\tau)$ contains no nonzero compact elements. Then, by Remark 3.3, the functional $d_\eta : \mathrm{Cu}(A/N_\tau) \to [0,\infty]$ preserves compact containment and by [18, Theorem 2], π_{N_τ} induces a Cuntz category morphism $\mathrm{Cu}(\pi_{N_\tau}) : \mathrm{Cu}(A) \to \mathrm{Cu}(A/N_\tau)$. Since the functional d_τ factors through $\mathrm{Cu}(A/N_\tau)$, via the commutative diagram

$$
\begin{array}{ccc}
\mathrm{Cu}(A) & \xrightarrow{\quad d_\tau \quad} & [0,\infty] \\
\mathrm{Cu}(\pi_{N_\tau}) \searrow & & \nearrow d_\eta \\
& \mathrm{Cu}(A/N_\tau) &
\end{array}
\quad ,
$$

it preserves compact containment. ∎

Remark 3.5 It is possible for the functional d_τ induced by a lower semicontinuous trace τ on a C*-algebra A to fail to preserve compact containment even if d_τ is assumed to vanish on every compact element of $\mathrm{Cu}(A)$. For example, let μ be the trace induced by the measure on $(0,1]$ which is the Lebesgue measure on $(1/2,1]$ and zero on $(0,1/2]$. The condition that d_μ vanishes on every compact element of $\mathrm{Cu}(C_0(0,1])$ is automatic since $C_0(0,1]$ is stably projectionless.

To show that d_μ fails to preserve compact containment, it suffices, by the proof of Theorem 3.1, to show that μ fails to preserve compact containment. By a characterization of compact containment in the proof of Proposition 2.13, $\chi_{(1/2,1]}$ is compactly contained in $\chi_{(0,1]}$, but $\mu((1/2,1]) = \mu((0,1])$.

4 Colored classification of C*-algebras

Theorem 4.1 *Any two unital C*-algebras which are colored isomorphic have isomorphic tracial simplices. Moreover, the order zero maps implementing a colored isomorphism induce mutually inverse isomorphisms of tracial simplices (as described in Section 2.3).*

Proof The same conclusion holds with the weaker hypothesis that A and B are unital C*-algebras which are minimalist colored isomorphic, so this is what we prove. Let the notation be as in Definition 2.23. Since $\varphi : A_\omega \to B_\omega$ and $\psi : B_\omega \to A_\omega$ are c.p.c. order zero maps, they induce mappings of bounded traces $\varphi^* : \mathbb{R}_+ T(B_\omega) \to \mathbb{R}_+ T(A_\omega)$ and $\psi^* : \mathbb{R}_+ T(A_\omega) \to \mathbb{R}_+ T(B_\omega)$ (see [68, Corollary 3.4]). In fact, φ^* and ψ^* are mutually inverse affine isomorphisms of the cones of bounded traces. To see this, let $\tau \in T(A_\omega)$ and $a \in A_\omega$ be given. Then

$$\tau(a) \stackrel{(2.7)}{=} \tau\left(\sum_{i=1}^m u_i \psi\varphi(a) u_i^*\right) = \sum_{i=1}^m \tau(u_i \psi\varphi(a) u_i^*)$$

$$= \sum_{i=1}^m \tau(\psi\varphi(a) u_i^* u_i) \stackrel{(2.8)}{=} \tau(\psi\varphi(a)) = \varphi^* \psi^*(\tau)(a).$$

A symmetric calculation shows $\psi^* \varphi^*(\tau) = \tau$ for each $\tau \in T(B_\omega)$.

Let us now show that the cones of bounded traces on A and B are affinely isomorphic. Denote by $c_B : \mathbb{R}_+ T(B) \to \mathbb{R}_+ T(B_\omega)$ the embedding of bounded traces on B as bounded limit traces on B_ω and by $\iota_A : A \to A_\omega$ the canonical embedding of A into A_ω. Denote by ι_A^* the dual map induced on bounded traces and define $\Phi : \mathbb{R}_+ T(B) \to \mathbb{R}_+ T(A)$ to be the composition

$$\mathbb{R}_+ T(B) \xrightarrow{c_B} \mathbb{R}_+ T(B_\omega) \xrightarrow{\varphi^*} \mathbb{R}_+ T(A_\omega) \xrightarrow{\iota_A^*} \mathbb{R}_+ T(A),$$

and define $c_A : \mathbb{R}_+ T(A) \to \mathbb{R}_+ T(A_\omega)$, $\iota_B : B \to B_\omega$, and $\Psi : \mathbb{R}_+ T(A) \to \mathbb{R}_+ T(B)$ similarly. Then Φ and Ψ are weak* continuous and affine, and for $\tau \in \mathbb{R}_+ T(B)$,

$$\Psi\Phi(\tau) = \iota_B^* \psi^* c_A \iota_A^* \varphi^* c_B(\tau)$$
$$= \iota_B^* \psi^* \varphi^* c_B(\tau) = \iota_B^* c_B(\tau) = \tau.$$

The second equality follows from the assumption that φ^* takes constant limit traces to constant limit traces so that $c_A \iota_A^* \varphi^* c_B(\tau) = \varphi^* c_B(\tau)$. The third equality follows from the previous paragraph. A symmetric calculation shows that $\Phi\Psi(\tau) = \tau$ for $\tau \in \mathbb{R}_+ T(A)$. This shows Φ and Ψ are mutually inverse isomorphisms of the topological convex cones $\mathbb{R}_+ T(A)$ and $\mathbb{R}_+ T(B)$.

To see that Φ and Ψ preserve the tracial simplices, i.e., are isometries, note that they are contractions since they are the compositions of contractions. By the preceding paragraph, $\Phi\Psi = \mathrm{id}_{T(A)}$ and so we have $1 = \|\Phi\Psi\| \le \|\Phi\|\|\Psi\| \le 1$ which implies that the norms of Φ and Ψ are both one so these maps are isometries and therefore constitute an isomorphism of the tracial simplices. ∎

Remark 4.2   Colored isomorphism (as defined in Section 2.7), extended in a natural way to the nonunital case, also preserves, up to isometric isomorphism, the topological convex cone of lower semicontinuous traces of [33]. To see this, we will need to know that c.p.c. order zero maps induce mappings of lower semicontinuous traces. This follows from [68, Corollary 3.4]. The result now follows by extending the maps considered in the proof of Theorem 4.1 to the cone of lower semicontinuous traces rather than just the cone of bounded traces.

From this one can deduce that colored isomorphism, in the present sense, preserves ideal lattices, up to isomorphism (cf. [10, Theorem 5.4.9]). Let the notation be as in the proof of Theorem 4.1 with Ψ assumed to be an isomorphism of cones of lower semicontinuous traces. Recall, from [33], that there is an order-reversing bijection α_A between closed two-sided ideals I of A and lower semicontinuous traces on A taking only the values 0 and ∞ given by

$$I \mapsto \tau_I(x) := \begin{cases} 0, & \text{if } x \in I_+, \\ \infty, & \text{if } x \notin I_+. \end{cases}$$

Such traces are characterized by the property $\tau + \tau = \tau$, and so Ψ restricts to a mapping between these traces. Therefore, the composition $\alpha_B^{-1} \circ \Psi \circ \alpha_A$ gives an order-preserving bijection of ideal lattices. If the prime ideals of A and B are primitive, as is the case with separable or postliminary C*-algebras (see [22, Theorems 4.3.5 and 4.4.5] and [52, Theorem A.50]), then the primitive ideal spaces of A and B are homeomorphic. By Gelfand duality, it follows that colored isomorphism (in the present sense) is a rigid notion for commutative C*-algebras (i.e., if A is a commutative C*-algebra which is colored isomorphic, in the present sense, to a C*-algebra B, then A is isomorphic to B) (cf. [10, Proposition 5.4.13]). It is easily seen that colored isomorphism, in the present sense, is also a rigid notion for finite-dimensional C*-algebras (cf. [10, Proposition 5.4.12]).

Theorem 4.3 *Let B be a unital, simple, separable, exact, $\mathcal{Z}$-stable C*-algebra with stable rank one, and let D be a simple, unital AF algebra. Let μ be a faithful trace on $C_0(0,1]$ with norm at most one, and let $\Phi \colon T(B) \to T(D)$ be a continuous affine map of simplices. There exists a c.p.c. order zero map $\varphi \colon D \to B$ which satisfies the identity*

$$(4.1) \qquad\qquad \tau\varphi^n = \mu(t^n)\Phi(\tau)$$

for all $n \in \mathbb{N}$ and for all $\tau \in T(B)$, where t denotes the identity map on $(0,1]$.

Proof Let $(D_i)_{i=1}^{\infty}$ be an increasing sequence of finite-dimensional C*-subalgebras of D such that $\bigcup_{i=1}^{\infty} D_i$ is dense in D. Denote by ι_i the embedding of D_i into D. We claim that the map

$$\sigma_i : \mathrm{Cu}(C_0(0,1] \otimes D_i) \to \mathrm{LAff}_+(TB)$$

determined by the rule

$$[d] \mapsto (\mu \otimes (\iota_i^* \circ \Phi(\cdot)))[d]$$

is a Cuntz category morphism.

We first show that σ_i in fact maps into $\mathrm{LAff}_+(TB)$. The image of $[d] \in \mathrm{Cu}(C_0(0,1] \otimes D_i)$ is clearly a positive real-valued affine function on TB. By Section 5 of [33], $\sigma_i([d])$ is lower semicontinuous. By assumption, μ is faithful and $\iota_i^* \circ \Phi(\tau)$, being the restriction of a faithful trace on D (as D is simple), is also faithful for each $\tau \in TD_i$. Therefore, $(\mu \otimes (\iota_i^* \circ \Phi(\cdot)))[d]$ is (pointwise) strictly positive whenever $[d]$ is nonzero. Since B is unital and separable, TB is metrizable. Therefore, by [1, Corollary I.1.4] and [62, Lemma 3.6], $\sigma_i([d])$ is the pointwise supremum of an increasing sequence of continuous finite-valued functions in $\mathrm{LAff}_+(TB)$. This shows that $\sigma_i([d])$ is an element of $\mathrm{LAff}_+(TB)$.

That σ_i is a generalized Cuntz category morphism follows from Section 4 of [33], so all that remains is to show σ_i preserves compact containment. Since D_i is a finite-dimensional C*-algebra, we may identify it with a finite direct sum of matrix algebras, $\bigoplus_{j=1}^{k} M_{n_j}$. Using that $C_0(0,1] \otimes (\bigoplus_{j=1}^{k} M_{n_j})$ is isomorphic to $\bigoplus_{j=1}^{k}(C_0(0,1] \otimes M_{n_j})$, we make the identification $\mathrm{Cu}(C_0(0,1] \otimes D_i) = \bigoplus_{j=1}^{k} \mathrm{Cu}(C_0(0,1] \otimes M_{n_j})$. By [18, Appendix 6], the embedding of $C_0(0,1]$ into the upper-left corner of $C_0(0,1] \otimes M_{n_j}$ induces an isomorphism at the level of the Cuntz category. For each j, we denote by e_j the nonzero minimal projection in the upper-left corner of M_{n_j}, and we denote by

ρ_j the trace on M_{n_j}. These maps induce an isomorphism $\bigoplus_{j=1}^k \mathrm{Cu}(C_0(0,1] \otimes M_{n_j}) \cong$ $\bigoplus_{j=1}^k \mathrm{Cu}(C_0(0,1])$. By Theorem 2.14, $\mathrm{Cu}(C_0(0,1]) \cong \mathrm{Lsc}((0,1],\overline{\mathbb{N}})$ and so under these isomorphisms, an arbitrary element $[d] \in \mathrm{Cu}(C_0(0,1] \otimes D_i)$ is a direct sum of elements $d_1,\dots,d_k$ of $\mathrm{Lsc}((0,1],\overline{\mathbb{N}})$. Pick $d_j^i \in C_0(0,1]$ to have open support exactly $\{d_j > i\}$ ($j = 1,\dots,k$ and $i \in \mathbb{N}$). Let $(p_i)_{i=0}^\infty$ be a sequence of mutually orthogonal rank-one projections and let $\iota : \mathrm{Lsc}((0,1],\overline{\mathbb{N}}) \to \mathrm{Cu}(C_0(0,1])$ and $\mu : \mathrm{Lsc}((0,1],\overline{\mathbb{N}}) \to [0,\infty]$ be as in the proof of Theorem 3.1. The direct sum of copies of ι gives an isomorphism $\bigoplus_{j=1}^k \mathrm{Lsc}((0,1],\overline{\mathbb{N}}) \to \bigoplus_{j=1}^k \mathrm{Cu}(C_0(0,1])$. Under the aforementioned isomorphisms, we have

$$[d] = \sum_{j=1}^k \sum_{i=0}^\infty [d_j^i \otimes e_j \otimes p_i].$$

Therefore,

$$\sigma_i([d]) = \sum_{j=1}^k \sum_{i=0}^\infty \mu(d_j^i)(\iota_i^* \circ \Phi(\cdot))(e_j)\mathrm{Tr}(p_i)$$
$$= \sum_{j=1}^k \sum_{i=0}^\infty \mu(d_j^i)(\iota_i^* \circ \Phi(\cdot))(e_j)$$
$$= \sum_{j=1}^k \mu(d_j)(\iota_i^* \circ \Phi(\cdot))(e_j).$$

Let us use the above calculation to show that $\sigma_i([d])$ is continuous. If $\sigma_i([d])$ is bounded, then $\sigma_i([d])$ is an affine extension of the continuous function taking on the real values $\mu(d_j)$ at the extreme points ρ_j of TD_i. Since D_i is finite-dimensional, TD_i is a Bauer simplex and so $\sigma_i([d])$ is continuous. If $\sigma_i([d])$ is not bounded, then $\mu(d_j)$ must be equal to ∞ for some j. Since $\iota_i^* \circ \Phi(\cdot)$ maps into the faithful part of TD_i, $\sigma_i([d])$ is the continuous function which is constant and equal to ∞.

Suppose $[f] \ll [g]$ in $\mathrm{Cu}(C_0(0,1] \otimes D_i)$. Let the notation be as in the preceding two paragraphs for both $[f]$ and $[g]$. If $[f]$ is equal to zero, then $\sigma_i([f])$ is also equal to zero and it is compactly contained in every element of $\mathrm{LAff}_+(TB)$ (and in particular, $\sigma_i([g])$). If $[f]$ is nonzero, then $f_j \ll g_j$ for each j and f_j must be nonzero for some j. For this particular j, $\mu(f_j) < \mu(g_j)$, by faithfulness of μ, and for each j, $\mu(f_j)$ is finite and $\mu(f_j) \leq \mu(g_j)$ (these facts are established in the proof of Theorem 3.1). It follows from the above computation and the fact that $\iota_i^* \circ \Phi(\tau) > 0$ ($\tau \in TB$) that $\sigma_i([f]) < \sigma_i([g])$. By the previous paragraph, this also shows that $\sigma_i([f])$ is continuous and finite-valued. Since lower semicontinuous functions attain their infima on compact sets, it follows from the characterization of compact containment in Section 2 of [33] (or [62, Lemma 3.6]) that $\sigma_i([f])$ is compactly contained in $\sigma_i([g])$. This shows σ_i is a Cuntz category morphism.

Since $\mathrm{LAff}_+(TB)$ is (in a natural way) a subobject of $\mathrm{Cu}(B)$ (Lemma 2.12), σ_i extends to a Cuntz category morphism into $\mathrm{Cu}(B)$ which we denote again by σ_i. In order to lift σ_i to a $*$-homomorphism, let us check that σ_i takes a strictly positive

element into an element majorized by a strictly positive element:

$$\sigma_i([t \otimes 1_{D_i}]) = \sum_{j=1}^{k} \mu(\chi_{(0,1]})(\iota_i^* \circ \Phi(\cdot))[1_{M_{n_j}}] \le \sum_{j=1}^{k} (\iota_i^* \circ \Phi(\cdot))[1_{M_{n_j}}]$$

$$= \sum_{j=1}^{k} \Phi(\cdot)[\iota_i(1_{M_{n_j}})] = \Phi(\cdot)[\iota_i(1_{D_i})] \le \widehat{[1_B]}.$$

In the first inequality, we have used that μ is of norm at most one. By Theorem 2.15, i.e., by [14, Theorem 4.1], as modified in the remark on page 29 of [14] (to replace $[0,1]$ with $(0,1]$), for each i there exists a $*$-homomorphism $\pi_i : C_0(0,1] \otimes D_i \to B$, which is unique up to approximate unitary equivalence, such that $\mathrm{Cu}(\pi_i) = \sigma_i$. Let us now denote by ι_i the embedding of $C_0(0,1] \otimes D_i$ into $C_0(0,1] \otimes D_{i+1}$. Then with the embeddings $\mathrm{Cu}(\iota_i)$ on the top row and the identity map $\mathrm{Cu}(\mathrm{id}_B)$ on the bottom row, we have the following one-sided intertwining at the level of the Cuntz category:

$$
\begin{array}{ccccc}
\mathrm{Cu}(C_0(0,1] \otimes D_1)) & \longrightarrow & \mathrm{Cu}(C_0(0,1] \otimes D_2)) & \longrightarrow & \cdots \\
\sigma_1 \downarrow & & \sigma_2 \downarrow & & \\
\mathrm{Cu}(B) & \longrightarrow & \mathrm{Cu}(B) & \longrightarrow & \cdots
\end{array}
$$

The $*$-homomorphisms $\pi_i : C_0(0,1] \otimes D_i \to B$ may be corrected by inner automorphisms to obtain the following one-sided approximate intertwining (in the sense of [25, Section 2]):

$$
\begin{array}{ccccccc}
C_0(0,1] \otimes D_1) & \longrightarrow & C_0(0,1] \otimes D_2) & \longrightarrow & \cdots & \longrightarrow & C_0(0,1] \otimes D \\
\pi_1 \downarrow & & \pi_2 \downarrow & & & & \vdots\,\pi \\
B & \longrightarrow & B & \longrightarrow & \cdots & \longrightarrow & B
\end{array}
$$

By [25, Remark 2.3], there exists a $*$-homomorphism $\pi : C_0(0,1] \otimes D \to B$ as in the above diagram and the map $\varphi : D \to B$ determined by the rule $\varphi(d) := \pi(t \otimes d)$ is c.p.c. order zero, by Corollary 2.5. Let us now verify that φ satisfies the required tracial identity. By continuity, it suffices to check that the the tracial identity holds at the finite stages. Let $d \in D_i$, and let $\tau \in TB$ be given. Then

$$\tau \varphi^n(d) \overset{(2.6)}{=} \tau \pi(t^n \otimes d) = \tau \lim_{i \to \infty} \pi_i(t^n \otimes d)$$

$$= \lim_{i \to \infty} (\mu \otimes (\iota_i^* \circ \Phi)(\tau))(t^n \otimes d)$$

$$= \mu(t^n)\Phi(\tau)(d).$$

The second equality follows from the one-sided approximate intertwining, and the third equality follows from continuity of τ and by definition of σ_i. (Note that $\mathrm{Cu}(\pi) = \lim_{i \to \infty} \sigma_i$, but we do not actually use this.) ∎

Lemma 4.4 *There exists a sequence of fully supported Radon measures μ_k on $(0,1]$ with total mass one such that*

(4.2) $$\lim_{k \to \infty} \mu_k(t) = 1,$$

where t denotes the identity map on $(0, 1]$. *Necessarily, this holds also for* t^n, *rather than just t, for each* $n \in \mathbb{N}$.

Proof   Pick measures which are increasingly weighted to the right.   ∎

Theorem 4.5   *Any two unital, simple, $\mathcal{Z}$-stable, separable amenable C^*-algebras satisfying the UCT with isomorphic tracial simplices are colored isomorphic.*

Proof   We first consider the purely infinite case. This part of the theorem holds without assuming the UCT. Let A and B be Kirchberg algebras. By [13, Theorem 3.1], [17, Corollary 2], and [67, Proposition 7], separable amenable C^*-algebras are exact. Hence by Theorem 2.16, there is a unital embedding of A into $\mathcal{O}_2$. Since B is purely infinite, $K_0(B)$ consists of the Murray–von Neumann equivalence classes of nonzero properly infinite projections in B (see [19, Theorem 1.4]). In particular, there is a nonzero properly infinite projection $p \in B$ whose K_0-class is equal to zero. It follows from [58, Proposition 4.2.3] that there exists a unital embedding of $\mathcal{O}_2$ into pBp and therefore a (possibly) nonunital embedding into B. Denote by $\varphi: A \to B$ the composition of these embeddings and construct $\psi: B \to A$ symmetrically. Then since φ and ψ are injective and therefore isometric $*$-homomorphisms, the ultrapower maps $\psi\varphi: A_\omega \to B_\omega$ and $\varphi\psi: B_\omega \to A_\omega$ induced by their compositions are also isometric and in particular injective. By two applications of Theorem 2.18, the pairs $\psi\varphi\iota_A$ and ι_A; and $\varphi\psi\iota_B$ and ι_B are each colored equivalent, where ι_A (resp. ι_B) denote the canonical embedding of A (resp. B) into its ultrapower.

Now (as we may, in view of Theorem 2.9), suppose that A and B are two finite C^*-algebras satisfying the hypotheses. By Theorem 2.17, there exists a unital embedding α_A of A into a separable, unital, simple AF algebra D such that the induced map $\alpha_A^*: T(D) \to T(A)$ is an isomorphism of tracial simplices. This, combined with an isomorphism $T(B) \to T(A)$ (assumed to exist), yields an isomorphism $\Phi: T(B) \to T(D)$. Pairing this with the faithful tracial states μ_k on $C_0(0, 1]$ of Lemma 4.4, one obtains by Theorem 4.3 a sequence of c.p.c. order zero maps $\varphi_k: D \to B$ satisfying the identity

$$(4.3) \qquad \tau\varphi_k^n = \mu_k(t^n)\Phi(\tau)$$

for every $k, n \in \mathbb{N}$ and each $\tau \in T(B)$. Denote by $\varphi: A_\omega \to B_\omega$ the c.p.c. order zero map induced between the ultrapowers by the maps $\varphi_k\alpha_A: A \to B$. As above, by [68, Corollary 3.4], φ induces a mapping φ^* of bounded traces on B_ω into bounded traces on A_ω. Then we have

$$(4.4) \qquad \begin{aligned} \tau\varphi^n &\overset{(2.3)}{=} \lim_{k\to\omega} \tau_k(\varphi_k^n\alpha_A) \\ &\overset{(4.3)}{=} \alpha_A^*\left(\lim_{k\to\omega}(\mu_k(t^n)\Phi(\tau_k))\right) \overset{(4.2)}{=} \lim_{k\to\omega}(\alpha_A^*\Phi)(\tau_k) \end{aligned}$$

for each $n \in \mathbb{N}$ and limit trace $\tau = \lim_{k\to\omega} \tau_k \in T_\omega(B_\omega)$.

Symmetrically, there is, again by Theorem 2.17, a unital embedding α_B of B into a unital, simple, separable AF algebra E such that the induced map $\alpha_B^*: T(E) \to T(B)$ is an isomorphism of tracial simplices. Using as above the isomorphism $\Psi := (\alpha_A^*\Phi\alpha_B^*)^{-1}: T(A) \to T(E)$ and the faithful tracial states μ_k, we obtain a c.p.c. order

zero map $\psi \colon B_\omega \to A_\omega$ such that

$$(4.5) \qquad \tau\psi^n = \lim_{k\to\omega}(\alpha_B^*\Psi)(\tau_k)$$

for each $n \in \mathbb{N}$ and limit trace $\tau = \lim_{k\to\omega}\tau_k \in T_\omega(A_\omega)$. So for all limit traces $\tau = \lim_{k\to\omega}\tau_k \in T_\omega(A_\omega)$, we have

$$\begin{aligned}
(4.6) \qquad \tau(\psi\varphi)^n &\overset{(2.3)}{=} \tau\psi^n\varphi^n \overset{(4.5)}{=} (\varphi^n)^*\big(\lim_{k\to\omega}(\alpha_B^*\Psi)(\tau_k)\big) \\
&\overset{(4.4)}{=} \alpha_A^*\Phi\big(\lim_{k\to\omega}(\alpha_B^*\Psi)(\tau_k)\big) = \lim_{k\to\omega}\alpha_A^*\Phi\alpha_B^*\Psi(\tau_k) = \tau,
\end{aligned}$$

where the last equality follows from the definition of Ψ.

Weak* density of $T_\omega(A)$ in $T(A_\omega)$ (Theorem 2.7) extends the identity (4.6) to all tracial states on A_ω. Since A is simple, the canonical embedding of A into A_ω is totally full. Hence by Theorem 2.20, $\psi\varphi\iota_A$ and ι_A are colored equivalent. By a symmetric argument, $\varphi\psi\iota_B$ and ι_B are colored equivalent. It follows immediately from (4.4) and (4.5) that φ and ψ preserve constant limit traces. This shows A is colored isomorphic to B (with m and n of (2.7) equal to two). ∎

Corollary 4.6 *Let A and B be classifiable C*-algebras (i.e., ones satisfying the hypotheses of Theorem 4.5). Then the following statements are equivalent:*

(1) *There exist constant limit trace preserving c.p.c. order zero maps $\varphi : A_\omega \to B_\omega$ and $\psi \colon B_\omega \to A_\omega$ such that φ^n and ψ^n induce mutually inverse isomorphisms of $T(A_\omega)$ and $T(B_\omega)$ for each $n \in \mathbb{N}$.*

(2) *There exist constant limit trace preserving c.p.c. order zero maps $\varphi : A_\omega \to B_\omega$ and $\psi \colon B_\omega \to A_\omega$ such that φ^n and ψ^n induce mutually inverse isomorphisms of $T(A_\omega)$ and $T(B_\omega)$ for some $n \in \mathbb{N}$.*

(3) *A and B are colored isomorphic (Definition 2.21).*

(4) *A and B are minimalist colored isomorphic (Definition 2.23).*

(5) *$T(A)$ is isomorphic to $T(B)$.*

Every isomorphism of $T(A)$ with $T(B)$ arises from a colored isomorphism of A and B.

Proof (1) $\implies$ (2) is clear. (2) $\implies$ (1) Let $\pi \colon A_\omega \to A^\omega$ denote the canonical quotient map. By assumption, there exists an $n \in \mathbb{N}$ such that $\tau\psi^n\varphi^n = \tau$ for all $\tau \in T(A_\omega)$. By Lemma 2.8, this implies $\pi\psi^n\varphi^n = \pi\mathrm{id}_{A_\omega}$. In particular, we will use that $\pi\psi^n\varphi^n(1_{A_\omega}) = \pi(1_{A_\omega})$ below. Since

$$\pi\psi\varphi(1_{A_\omega}) = \pi\psi\varphi(1_{A_\omega}^n) = (\pi\psi\varphi)^n(1_{A_\omega}) = \pi\psi^n\varphi^n(1_{A_\omega}) = \pi(1_{A_\omega})$$

is a projection, it then follows from Corollary 2.2 that $\pi\psi\varphi$ is a $*$-homomorphism. We have used Corollaries 2.3 and 2.4 for the second and third equalities. Therefore, $\pi\psi^k\varphi^k = \pi(\psi\varphi)^k = (\pi\psi\varphi)^k = \pi\psi\varphi$ for each $k \in \mathbb{N}$. By Lemma 2.8 again, this implies $\tau\psi^k\varphi^k = \tau$ for each $\tau \in T(A_\omega)$ and each $k \in \mathbb{N}$. A symmetric argument shows that φ^k and ψ^k induce mutually inverse isomorphisms of $T(A_\omega)$ and $T(B_\omega)$.

(1) $\implies$ (3) follows (without the UCT assumption) from two applications of Theorem 2.20: once with $\psi\varphi\iota_A$ and ι_A where ι_A is the constant sequence embedding of A into A_ω and once more with $\varphi\psi\iota_B$ and ι_B. (3) $\implies$ (4) is immediate. (4) $\implies$ (5)

as mentioned in the proof of Theorem 4.1, the same conclusion holds with the weaker hypothesis that the two C*-algebras are minimalist colored isomorphic. (5) $\implies$ (1) is given by (4.6) in the proof of Theorem 4.5 (along with the analogous statement for $(\varphi\psi)^n$) and Corollary 2.4.

The last statement follows from the proof of Theorem 4.5, which constructs a colored isomorphism using a given isomorphism of $T(A)$ with $T(B)$, and the proof of Theorem 4.1, which recovers the given isomorphism of $T(A)$ with $T(B)$ from the constructed colored isomorphism. ∎

We are not sure to what extent the c.p.c. order zero maps involved in a colored isomorphism are unique.

Corollary 4.7 *Let A be a classifiable C*-algebra. If $T(A) = \varnothing$, then A is colored isomorphic to $\mathcal{O}_2$. If $T(A) \neq \varnothing$, then A is colored isomorphic to a unital simple AF algebra.*

Proof If $T(A) = \varnothing$, then by Theorem 4.5, A is colored isomorphic to $\mathcal{O}_2$. If $T(A) \neq \varnothing$, by [5, Theorem 3.10], there exists a unital AF algebra B whose tracial simplex isomorphic to $T(A)$. The conclusion now follows from Theorem 4.5. ∎

The existence step in establishing finite nuclear dimension from $\mathcal{Z}$-stability (see [6, Lemma 7.4] and [11, Lemma 5.2]) is the construction of a sequence of c.p.c. maps $\phi_i : A \to A$, where A is a finite, unital, simple, $\mathcal{Z}$-stable, separable amenable C*-algebra, which factorize through finite-dimensional C*-algebras F_i as

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ \phi_i\ \ } & A \\
& \theta_i \searrow \quad \nearrow \eta_i & \\
& F_i &
\end{array}
$$

with θ_i c.p.c. and η_i c.p.c. order zero, such that the induced maps $(\theta_i)_{i=1}^\infty : A \to \prod_\omega F_i$ and $\Phi = (\phi_i)_{i=1}^\infty : A \to A_\omega$ are order zero and
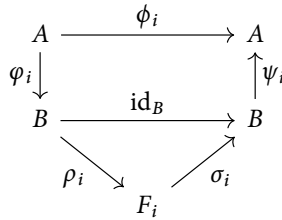
$$
\tau\Phi(a) = \tau(a)
$$

for each $a \in A$ and each $\tau \in T(A_\omega)$.

More precisely, the conclusion in [11, Lemma 5.2] is that Φ agrees with the canonical embedding ι_A of A into A_ω in the uniform tracial ultrapower A^ω, whereas the conclusion in [6, Lemma 7.4] is that $\tau\Phi = \tau\iota_A$ for each $\tau \in T(A_\omega)$. These conclusions are equivalent by Lemma 2.8 and Theorem 2.7.

In the presence of the UCT, a one-sided formulation of Theorem 4.5 (Corollary 4.8) can be viewed as a generalization of the existence step mentioned earlier since the conclusion follows from the special case that B is an AF algebra. Since the nuclear dimension of B is zero (see [69, Remark 2.2(iii)]), there exist finite-dimensional C*-algebras F_i and c.p.c. maps $\rho_i : B \to F_i$ and c.p.c. order zero maps $\sigma_i : F_i \to B$ such that

the triangle below commutes approximately.

$$
\begin{array}{ccc}
A & \xrightarrow{\ \phi_i\ } & A \\
{\scriptstyle\varphi_i}\downarrow & & \uparrow{\scriptstyle\psi_i} \\
B & \xrightarrow{\ \mathrm{id}_B\ } & B
\end{array}
$$

with ρ_i going from B down to F_i and σ_i going from F_i up to B.

The maps $\theta_i : A \to F_i$ on the left side of the diagram and the maps $\eta_i : F_i \to A$ on the right side of the diagram satisfy the conclusion of the aforementioned existence step.

Corollary 4.8 *Let A and B be finite classifiable C^*-algebras with isomorphic tracial simplices. Then there exist sequences of c.p.c. order zero maps $\varphi_i : A \to B$ and $\psi_i : B \to A$ for $i \in \mathbb{N}$, such that the induced c.p.c. order zero maps $\varphi : A_\omega \to B_\omega$ and $\psi : B_\omega \to A_\omega$ induce mutually inverse isomorphisms of $T(A_\omega)$ and $T(B_\omega)$. In particular,*

$$
\tau\psi\varphi(a) = \tau(a)
$$

for each $a \in A$ and each $\tau \in T(A_\omega)$.

Proof This is a special case of Corollary 4.6. ∎

Corollary 4.8 gives rise to the completely positive approximation property below. The nuclear dimension calculation in Corollary 4.9 is not new—in fact, it holds without the UCT assumption and without unitality (see [11, 14]), and our proof of it using the UCT still relies on the main technical results of [6] and [11] that were used in establishing finite nuclear dimension from $\mathcal{Z}$-stability in the context of the Toms–Winter conjecture (see [11, 14]).

Corollary 4.9 *Let A and B be finite classifiable C^*-algebras with isomorphic tracial simplices. Then there exist a sequence $\varphi_i : A \to B$ of c.p.c. order zero maps and a sequence $\xi_i : B \to A$ such that ξ_i is a sum of two c.p.c. order zero maps from B to A for $i \in \mathbb{N}$, and such that the following diagram approximately commutes.*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \mathrm{id}_A\ } & A \\
& {\scriptstyle\varphi_i}\searrow \quad \nearrow{\scriptstyle\xi_i} & \\
& B &
\end{array}
$$
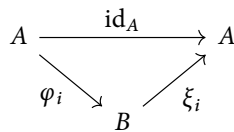
Since a possible choice for B is an AF algebra, it follows that the nuclear dimension of A is at most one. If A is not an AF algebra, then the nuclear dimension of A is exactly one.

Proof Let the notation for φ_i, ψ_i, φ, and ψ be as in Corollary 4.8, and let h be a positive element with full spectrum in $\mathcal{Z}$. Then since ι_A is totally full and since ι_A and $\psi\varphi\iota_A$ agree on traces, $\iota_A \otimes h$ and $\psi\varphi\iota_A \otimes h$ are approximately unitarily equivalent, by [11, Lemma 4.8]. Since $1_{\mathcal{Z}} - h$ is also a positive element of $\mathcal{Z}$ with full spectrum, $\iota_A \otimes (1_{\mathcal{Z}} - h)$ and $\psi\varphi\iota \otimes (1_{\mathcal{Z}} - h)$ are also approximately unitarily equivalent. Therefore, there exist unitaries u_1 and $u_2 \in (A \otimes \mathcal{Z})_\omega$ such that $a \otimes h = u_1(\psi\varphi(a) \otimes h)u_1^*$ and $a \otimes (1_{\mathcal{Z}} - h) = u_2(\psi\varphi(a) \otimes (1_{\mathcal{Z}} - h))u_2^*$. Since $\mathcal{Z}$ is strongly self-absorbing (see [39,

Theorem 8.7]) and therefore K_1-injective ((see [53, Theorem 10.12] and [57, Theorem 6.7]), [57, Theorem 6.7]), there is a $*$-homomorphism $\theta : A \otimes \mathcal{Z} \to A_\omega$ such that $\theta(a \otimes 1_{\mathcal{Z}}) = a$ for each $a \in A$, by [65, Theorem 2.3]. Let $h_1 := h$ and $h_2 := 1_{\mathcal{Z}} - h$. Taking representative sequences of $*$-homomorphisms $\theta_i : A \otimes \mathcal{Z} \to A$ (which exist by the proof of [65, Theorem 2.3]) and of unitaries $(u_1^{(i)})_{i=1}^\infty$ and $(u_2^{(i)})_{i=1}^\infty$ in $A \otimes \mathcal{Z}$ corresponding to u_1 and u_2, we have the following approximately commutative diagram.

$$
\begin{array}{ccccc}
A & \xrightarrow{\ \mathrm{id}_A\ } & A & \xleftarrow{\ \theta_i\ } & A \otimes \mathcal{Z} \\
 & \searrow{\varphi_i} & & & \big\uparrow{\sum_{k=1}^2 u_k^{(i)}(\cdot \otimes h_k)u_k^{(i)*}} \\
 & & B & \xrightarrow[\ \psi_i\ ]{} & A
\end{array}
$$

The maps $\varphi_i : A \to B$ and the maps $\xi_i : B \to A$ which factor though A and $A \otimes \mathcal{Z}$ give the desired completely positive approximation property.

Let us now specialize to the case that B is an AF algebra with $T(B) \cong T(A)$, which exists by [5, Theorem 3.10]. Since the nuclear dimension of B is zero (see [69, Remark 2.2(iii)]), there exist c.p.c. maps $\rho_i : B \to F_i$ into finite-dimensional C*-algebras F_i and c.p.c. order zero maps $\sigma_i : F_i \to B$ which make the following diagram approximately commute

$$
\begin{array}{ccc}
A & \xrightarrow{\ \mathrm{id}_A\ } & A \\
\varphi_i \big\downarrow & & \big\uparrow \xi_i \\
B & \xrightarrow[\ \mathrm{id}_B\ ]{} & B \\
 \ \ \searrow{\rho_i} & & \nearrow{\sigma_i}\ \ \\
 & F_i &
\end{array}\ .
$$

Since the maps $\rho_i \varphi_i : A \to F_i$ are and the maps $\xi_i \sigma_i : F_i \to A$ are c.p. and since $\xi_i \sigma_i$ is a sum of two c.p.c. order zero maps, A has nuclear dimension at most one (see [11, Section 1.1]). If A is not an AF algebra, then the nuclear dimension of A is exactly 1 (see [69, Remark 2.2(iii)]).   ∎

Question   All that is used about the (multiplicative) AF embeddings in Theorem 4.5 is that they are c.p.c. order zero maps which induce isomorphisms of simplices. In fact, it would be enough for each AF embedding to be replaced by a sequence of order zero maps which induce an isomorphism of tracial simplices in the sense described in Section 2.3. If this could be done without the UCT, then one would have Theorem 4.5 for classifiable C*-algebras not necessarily satisfying the UCT, as the UCT assumption is only used to produce AF embeddings. If A and B are unital, simple, exact, $\mathcal{Z}$-stable, separable C*-algebras with stable rank one, is there an order zero map $\varphi : A \to B$ realizing prescribed tracial data as in Theorem 4.3?

Chris Phillips for detailed comments on an earlier draft of this paper; and the anonymous reviewer for helpful comments.

References

[1] E. M. Alfsen, Compact convex sets and boundary integrals, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 57, Springer, New York, 1971.

[2] T. Andô, *On fundamental properties of a Banach space with a cone*. Pac. J. Math. **12**(1962), 1163–1169.

[3] R. Antoine, F. Perera, L. Robert, and H. Thiel, *C*-algebras of stable rank one and their Cuntz semigroups*. Duke Math. J. **171**(2022), no. 1, 33–99.

[4] R. Antoine, F. Perera, and L. Santiago, *Pullbacks, C(X)-algebras, and their Cuntz semigroup*. J. Funct. Anal. **260**(2011), no. 10, 2844–2880.

[5] B. E. Blackadar, *Traces on simple AF C*-algebras*, J. Funct. Anal. **38**(1980), no. 2, 156–168.

[6] J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White, and W. Winter, *Covering dimension of C*-algebras and 2-coloured classification*. Mem. Amer. Math. Soc. **257**(2019), no. 1233, 1–97.

[7] O. Bratteli, *Inductive limits of finite dimensional C*-algebras*. Trans. Amer. Math. Soc. **171**(1972), 195–234.

[8] N. P. Brown and A. Ciuperca, *Isomorphism of Hilbert modules over stably finite C*-algebras*. J. Funct. Anal. **257**(2009), no. 1, 332–339.

[9] N. P. Brown, F. Perera, and A. S. Toms, *The Cuntz semigroup, the Elliott conjecture, and dimension functions on C*-algebras*. J. Reine Angew. Math. **621**(2008), 191–211.

[10] J. Castillejos, *Decomposable approximations and coloured isomorphisms for C*-algebras*. Ph.D. thesis, University of Glasgow, 2016.

[11] J. Castillejos, S. Evington, A. Tikuisis, S. White, and W. Winter, *Nuclear dimension of simple C*-algebras*. Invent. Math. **224**(2021), no. 1, 245–290.

[12] M. D. Choi, *A Schwarz inequality for positive linear maps on C*-algebras*. Ill. J. Math. **18**(1974), 565–574.

[13] M. D. Choi and E. G. Effros, *Nuclear C*-algebras and the approximation property*. Amer. J. Math. **100**(1978), no. 1, 61–79.

[14] A. Ciuperca and G. A. Elliott, *A remark on invariants for C*-algebras of stable rank one*. Int. Math. Res. Not. IMRN **2008**(2008), no. 5, Article no. 158.

[15] A. Ciuperca, G. A. Elliott, and L. Santiago, *On inductive limits of type-I C*-algebras with one-dimensional spectrum*. Int. Math. Res. Not. IMRN **2011**(2011), no. 11, 2577–2615.

[16] A. Ciuperca, L. Robert, and L. Santiago, *The Cuntz semigroup of ideals and quotients and a generalized Kasparov stabilization theorem*. J. Operator Theory **64**(2010), no. 1, 155–169.

[17] A. Connes, *On the cohomology of operator algebras*. J. Funct. Anal. **28**(1978), no. 2, 248–253.

[18] K. T. Coward, G. A. Elliott, and C. Ivanescu, *The Cuntz semigroup as an invariant for C*-algebras*. J. Reine Angew. Math. **623**(2008), 161–193.

[19] J. Cuntz, *K-theory for certain C*-algebras. II*. J. Operator Theory **5**(1981), no. 1, 101–108.

[20] M. Dadarlat and S. Eilers, *On the classification of nuclear C*-algebras*. Proc. Lond. Math. Soc. **85**(2002), no. 1, 168–210.

[21] M. Dadarlat and A. S. Toms, *Ranks of operators in simple C*-algebras*. J. Funct. Anal. **259**(2010), no. 5, 1209–1229.

[22] J. Dixmier, *C*-algebras*, North-Holland, Amsterdam—New York—Oxford, 1977. Translated from the French by Francis Jellett. North-Holland Mathematical Library, Vol. 15.

[23] E. G. Effros, D. E. Handelman, and C. L. Shen, *Dimension groups and their affine representations*. Amer. J. Math. **102**(1980), no. 2, 385–407.

[24] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*. J. Algebra. **38**(1976), no. 1, 29–44.

[25] G. A. Elliott, *On the classification of C*-algebras of real rank zero*. J. Reine Angew. Math. **443**(1993), 179–219.

[26] G. A. Elliott, *On the Thiel diagonal semigroup of a C*-algebra of stable rank one*. In preparation, 2022.

[27] G. A. Elliott and D. E. Evans, *The structure of the irrational rotation C*-algebra*. Ann. Math. **138**(1993), no. 3, 477–501.

[28] G. A. Elliott, G. Gong, and L. Li, *On the classification of simple inductive limit C*-algebras. II. The isomorphism theorem*. Invent. Math. **168**(2007), no. 2, 249–320.

[29] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, *On the classification of simple amenable C*-algebras with finite decomposition rank, II*. Preprint, 2015. arXiv:1507.03437

[30] G. A. Elliott, G. Gong, H. Lin, and Z. Niu, *The classification of simple separable KK-contractible C*-algebras with finite nuclear dimension*. J. Geom. Phys. **158**(2020), Article no. 103861.

[31] G. A. Elliott, C. Ivanescu, and D. Kucerovsky, *The Cuntz semigroup of the tensor product of C*-algebras*. C. R. Math. Acad. Sci. Soc. R. Can. **41**(2019), no. 2, 32–44.

[32] G. A. Elliott and T. A. Loring, *AF embeddings of $C\left(\mathbf{T}^2\right)$ with a prescribed K-theory*. J. Funct. Anal. **103**(1992), no. 1, 1–25.

[33] G. A. Elliott, L. Robert, and L. Santiago, *The cone of lower semicontinuous traces on a C*-algebra*. Amer. J. Math. **133**(2011), no. 4, 969–1005.

[34] G. Gong, *On the classification of simple inductive limit C*-algebras. I. The reduction theorem*. Doc. Math. **7**(2002), 255–461.

[35] G. Gong and H. Lin, *On classification of non-unital amenable simple C*-algebras, II*. J. Geom. Phys. **158**(2020), Article no. 103865.

[36] G. Gong, H. Lin, and Z. Niu, *A classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras, I: C*-algebras with generalized tracial rank one*. C. R. Math. Acad. Sci. Soc. R. Can. **42**(2020), no. 3, 63–450.

[37] G. Gong, H. Lin, and Z. Niu, *A classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras, II: C*-algebras with rational generalized tracial rank one*. C. R. Math. Acad. Sci. Soc. R. Can. **42**(2020), no. 4, 451–539.

[38] C. Ivanescu, *On the classification of continuous trace C*-algebras with spectrum homeomorphic to the closed interval [0,1]*. In: Advances in operator algebras and mathematical physics, Theta Ser. Adv. Math., 5, Theta, Bucharest, 2005, pp. 109–135.

[39] X. Jiang and H. Su, *On a simple unital projectionless C*-algebra*. Amer. J. Math. **121**(1999), no. 2, 359–413.

[40] E. Kirchberg and N. C. Phillips, *Embedding of exact C*-algebras in the Cuntz algebra $\mathcal{O}_2$*. J. Reine Angew. Math. **525**(2000), 17–53.

[41] E. Kirchberg and M. Rørdam, *Central sequence C*-algebras and tensorial absorption of the Jiang–Su algebra*. J. Reine Angew. Math. **695**(2014), 175–214.

[42] H. Lin, *Classification of simple C*-algebras with unique traces*. Amer. J. Math. **120**(1998), no. 6, 1289–1315.

[43] H. Lin, *Classification of simple C*-algebras and higher dimensional noncommutative tori*. Ann. Math. **157**(2003), no. 2, 521–544.

[44] J. Lindenstrauss, *Extension of compact operators*. Mem. Amer. Math. Soc. **48**(1964), 112.

[45] H. Matui and Y. Sato, *Decomposition rank of UHF-absorbing C*-algebras*. Duke Math. J. **163**(2014), no. 14, 2687–2708.

[46] P. W. Ng and L. Robert, *Sums of commutators in pure C*-algebras*. Münster J. Math. **9**(2016), no. 1, 121–154.

[47] N. Ozawa, *Dixmier approximation and symmetric amenability for C*-algebras*. J. Math. Sci. Univ. Tokyo **20**(2013), no. 3, 349–374.

[48] G. K. Pedersen, *Measure theory for C* algebras*. Math. Scand. **19**(1966), 131–145.

[49] F. Perera and A. S. Toms, *Recasting the Elliott conjecture*. Math. Ann. **338**(2007), no. 3, 669–702.

[50] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C*-algebras*. Doc. Math. **5**(2000), 49–114.

[51] M. Pimsner and D. Voiculescu, *Imbedding the irrational rotation C*-algebra into an AF-algebra*. J. Operator Theory **4**(1980), no. 2, 201–210.

[52] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C*-algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998.

[53] M. A. Rieffel, *Dimension and stable rank in the K-theory of C*-algebras*. Proc. Lond. Math. Soc. **46**(1983), no. 2, 301–333.

[54] L. Robert, *Classification of inductive limits of 1-dimensional NCCW complexes*. Adv. Math. **231**(2012), no. 5, 2802–2836.

[55] L. Robert, *The cone of functionals on the Cuntz semigroup*. Math. Scand. **113**(2013), no. 2, 161–186.

[56] L. Robert and L. Santiago, *Classification of C*-homomorphisms from C_0 $(0,1)$ to a C*-algebra*. J. Funct. Anal. **258**(2010), no. 3, 869–892.

[57] M. Rørdam, *The stable and the real rank of $\mathcal{Z}$-absorbing C*-algebras*. Int. J. Math. **15**(2004), no. 10, 1065–1084.

[58] M. Rørdam and E. Størmer, *Classification of nuclear C*-algebras. Entropy in operator algebras*. Encyclopaedia of Mathematical Sciences, vol. 126, Springer, Berlin, 2002.

[59] Y. Sato, S. White, and W. Winter, *Nuclear dimension and $\mathcal{Z}$-stability*. Invent. Math. **202**(2015), no. 2, 893–921.

[60] C. Schafhauser, *A new proof of the Tikuisis–White–Winter theorem*. J. Reine Angew. Math. **759**(2020), 291–304.

[61] C. Schafhauser, *Subalgebras of simple AF-algebras*. Ann. Math. **192**(2020), no. 2, 309–352.

[62] H. Thiel, *Ranks of operators in simple C^*-algebras with stable rank one*. Commun. Math. Phys. **377**(2020), no. 1, 37–76.

[63] H. Thiel and E. Vilalta, *Covering dimension of Cuntz semigroups*. Adv. Math. **394**(2022), Article no. 108016.

[64] A. Tikuisis, S. White, and W. Winter, *Quasidiagonality of nuclear C^*-algebras*. Ann. Math. **185**(2017), no. 1, 229–284.

[65] A. S. Toms and W. Winter, *Strongly self-absorbing C^*-algebras*. Trans. Amer. Math. Soc. **359**(2007), no. 8, 3999–4029.

[66] E. Vilalta, *The Cuntz semigroup of unital commutative AI-algebras*. Preprint, 2021.
arXiv:2104.08165

[67] S. Wassermann, *Tensor products of free-group C^*-algebras*. Bull. Lond. Math. Soc. **22**(1990), no. 4, 375–380.

[68] W. Winter and J. Zacharias, *Completely positive maps of order zero*. Münster J. Math. **2**(2009), 311–324.

[69] W. Winter and J. Zacharias, *The nuclear dimension of C^*-algebras*. Adv. Math. **224**(2010), no. 2, 461–498.

Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada
e-mail: elliott@math.toronto.edu   jim@math.toronto.edu