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Rigidity properties for commuting automorphisms on tori and solenoids

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In memory of Anatole Katok

Abstract. Assuming positive entropy, we prove a measure rigidity theorem for higher rank actions on tori and solenoids by commuting automorphisms. We also apply this result to obtain a complete classification of disjointness and measurable factors for these actions.

Key words: entropy, invariant measures, invariant σ-algebras, measurable factors, joinings, toral automorphisms, solenoid automorphism
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Contents

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1. *Introduction and main results*

The map $T_p : x \mapsto px$ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has many closed invariant sets and many invariant measures. Furstenberg initiated the study of jointly invariant sets in his seminal paper [14]. A set $A \subseteq \mathbb{T}$ is called *jointly invariant* under T_p and T_q if $T_p(A) \subseteq A$ and $T_q(A) \subseteq A$. Furstenberg proved that if p and q are multiplicatively independent integers, then any closed jointly invariant set is either finite or all of $\mathbb{T}$.

Furstenberg also raised the question concerning what are the jointly invariant measures, that is, which probability measures μ on $\mathbb{T}$ satisfy $(T_p)_*\mu = (T_q)_*\mu = \mu$. The obvious ones are the Lebesgue measure, atomic measures supported on finite invariant sets, and (non-ergodic) convex combinations of these.

In the following, a *solenoid* X is a compact, connected, abelian group whose Pontryagin dual $\widehat{X}$ can be embedded into a finite-dimensional vector space over $\mathbb{Q}$. The simplest example is a finite-dimensional torus. A $\mathbb{Z}^d$-action α by automorphisms of a solenoid X is called *irreducible* if there is no proper infinite closed subgroup which is invariant under α, and *totally irreducible* if there is no finite index subgroup $\Lambda \subseteq \mathbb{Z}^d$ and no proper infinite closed subgroup $Y \subseteq X$ which is invariant under the induced action α_Λ. A $\mathbb{Z}^d$-action is *virtually cyclic* if there exists $\mathbf{n} \in \mathbb{Z}^d$ such that for every element $\mathbf{m} \in \Lambda$ of a finite index subgroup $\Lambda \subseteq \mathbb{Z}^d$, there exists some $k \in \mathbb{Z}$ with $\alpha^{\mathbf{m}} = \alpha^{k\mathbf{n}}$.

We briefly summarize the history of this problem. The topological generalization of Furstenberg's result to higher dimensions was given by Berend [1, 2]: An action on a torus or solenoid has no proper, infinite, closed, and invariant subsets if and only if it is totally irreducible, not virtually cyclic, and contains a hyperbolic element.

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The first partial result for the measure problem on $\mathbb{T}$ was given by Lyons [31] under a strong additional assumption. Rudolph [34] weakened this assumption considerably, and proved the following theorem.

THEOREM 1.1. [34, Theorem 4.9] *Let $p, q \geq 2$ be relatively prime positive integers, and let μ be a T_p, T_q-invariant, and ergodic measure on $\mathbb{T}$. Then either $\mu = m_{\mathbb{T}}$ is the Lebesgue measure on $\mathbb{T}$, or the entropy of T_p and T_q is zero.*

Johnson [17] lifted the relative primality assumption, by showing it is enough to assume that p and q are multiplicatively independent. Feldman [13], Parry [33], and Host [15] have found different proofs of this theorem, but positive entropy remains a crucial assumption.

Anatole Katok and Spatzier [22, 23] obtained the first analogous results for actions on higher dimensional tori and homogeneous spaces. However, their method required either an additional ergodicity assumption on the measure (satisfied for example if every one parameter subgroup of the suspension acts ergodically), or that the action is totally non-symplectic (TNS). A careful and readable account of these results has been written by Kalinin and Anatole Katok [18], which also fixed some minor inaccuracies. The following theorem (already proven in the announcement [7]) gives a full generalization of the result of Rudolph and Johnson to actions on higher-dimensional solenoids.

THEOREM 1.2. [7, Theorem 1.1] *Let α be a totally irreducible, not virtually cyclic $\mathbb{Z}^d$-action by automorphisms of a solenoid X. Let μ be an α-invariant and ergodic probability measure. Then either $\mu = m_X$ is the Haar measure of X, or the entropy $\mathrm{h}_\mu(\alpha^{\mathbf{n}}) = 0$ vanishes for all $\mathbf{n} \in \mathbb{Z}^d$.*

1.1. *The general positive entropy measure rigidity theorem.* Without total irreducibility, the Haar measure of the group is no longer the only measure with positive entropy. Thus our main theorem below is (necessarily) longer in its formulation than Theorem 1.2. It strengthens e.g. [18, Theorem 3.1] which has a similar conclusion but stronger assumptions.

THEOREM 1.3. (Positive entropy rigidity theorem) *Let α be a $\mathbb{Z}^d$-action ($d \geq 2$) by automorphisms of a solenoid X. Suppose α has no virtually cyclic factors, and let μ be an α-invariant and ergodic probability measure on X. Then there exists a subgroup $\Lambda \subseteq \mathbb{Z}^d$ of finite index and a decomposition $\mu = (1/J)(\mu_1 + \cdots + \mu_J)$ of μ into mutually singular measures with the following properties for every $j = 1, \ldots, J$.*

(1) *The measure μ_j is α_Λ-ergodic, where α_Λ is the restriction of α to Λ.*
(2) *There exists an α_Λ-invariant closed subgroup G_j such that μ_j is invariant under translation with elements in G_j, that is, $\mu_j(B) = \mu_j(B + g)$ for all $g \in G_j$ and every measurable set $B \subseteq X$.*
(3) *For $\mathbf{n} \in \mathbb{Z}^d$, $\alpha^{\mathbf{n}}_* \mu_j = \mu_k$ for some $k \in \{1, \ldots, J\}$ and $\alpha^{\mathbf{n}}(G_j) = G_k$.*
(4) *The measure μ_j induces a measure on the factor X/G_j with $\mathrm{h}_{\mu_j}(\alpha^{\mathbf{n}}_{X/G_j}) = 0$ for any $\mathbf{n} \in \Lambda$. (Here α_{X/G_j} denotes the action induced on X/G_j.)*

We remark that in the topological category, there is a big gap between our understanding of the totally irreducible case and the general case of $\mathbb{Z}^d$-actions by automorphisms on a

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solenoid. In the totally irreducible case, Berend [2] gave an if-and-only-if condition for a $\mathbb{Z}^d$-action to have the property that every orbit is either finite or dense, and the same methods could be pushed further to give a complete classification of closed invariant subsets for a totally irreducible $\mathbb{Z}^d$-action on the solenoid; for $\mathbb{Z}^d$-action on tori, this is due to Z. Wang [36, Theorem 1.10], and his proof certainly works also for solenoids though this does not seem to have been written (a special case, with a very nice application, can be found in Manner's paper [32]). In the non-irreducible case, orbit closures and closed invariant sets are much less understood. We refer to [30] by Z. Wang and the second named author for some results in this direction and additional details.

The proofs of Theorem 1.2 and Theorem 1.3 follow the outline of Rudolph's proof of Theorem 1.1. One of the main ingredients there was the observation that $\mathsf{h}_\mu(T_p)/\log p = \mathsf{h}_\mu(T_q)/\log q$ (and a relativized version of this equality). This follows from the particularly simple geometry of this system where both T_p and T_q expand the one-dimensional space $\mathbb{T}$ with fixed factors. There is no simple geometrical reason why such an equality should be true for more complicated $\mathbb{Z}^d$-actions on solenoids, and indeed is easily seen to fail in the reducible case. However, somewhat surprisingly, such an equality *is* true for irreducible $\mathbb{Z}^d$-actions, even though this is true for subtler reasons (see Theorem 7.1 below).

In the following two subsections we also apply Theorem 1.3 to obtain new information about the measurable structure, with respect to the Haar measure, of algebraic $\mathbb{Z}^d$-actions on tori and solenoids.

1.2. *Characterization of disjointness.* Let α_1 and α_2 be two measure-preserving $\mathbb{Z}^d$-actions on the probability spaces $(X_1, \mathcal{B}_{X_1}, \mu_1)$ and $(X_2, \mathcal{B}_{X_2}, \mu_2)$. A *joining* between α_1 and α_2 is an $\alpha_1 \times \alpha_2$-invariant probability measure ν on $X_1 \times X_2$, which projects to μ_1 and μ_2 under the projection maps π_1 and π_2. In other words we require $\nu(\alpha_1^{\mathbf{n}} \times \alpha_2^{\mathbf{n}}(C)) = \nu(C)$ for $\mathbf{n} \in \mathbb{Z}^d$ and $C \in \mathcal{B}_{X_1 \times X_2}$, $\nu(A \times X_2) = \mu_1(A)$ for $A \in \mathcal{B}_{X_1}$, and also $\nu(X_1 \times B) = \mu_2(B)$ for $B \in \mathcal{B}_{X_2}$. The product measure $\mu_1 \times \mu_2$ is always a joining, called the *trivial joining*. If the trivial joining is the only joining, the two actions are *disjoint*. This implies that the two actions are measurably non-isomorphic. In fact if they are disjoint, there is no non-trivial common factor of the two systems, see for instance §9 where we recall the construction of the relatively independent joining over a common factor.

Let now α_j be measure preserving $\mathbb{Z}^d$-actions on $(X_j, \mathcal{B}_{X_j}, \mu_j)$ for $j = 1, \dots, r$. A *joining between α_j for $j = 1, \dots, r$* is a measure ν on $\prod_{j=1}^r X_j$ which projects to μ_j under the coordinate projections π_j for $j = 1, \dots, r$, and is invariant under the $\mathbb{Z}^d$-action $\alpha_1 \times \cdots \times \alpha_r$. The product measure is the *trivial joining*, and the $\mathbb{Z}^d$-actions are *mutually disjoint* if the trivial joining is the only joining.

Suppose now α_1 and α_2 are actions by automorphisms on solenoids X_1 and X_2, respectively. We will classify disjointness with respect to the Haar measures m_{X_j} on the group X_j for $j = 1, 2$. If $\varphi : X_1 \to X_2$ is a continuous surjective homomorphism and satisfies $\alpha_2^{\mathbf{n}} \circ \varphi = \varphi \circ \alpha_1^{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^d$, we say φ is an *algebraic factor map*. If α_1 and α_2 are both finite-to-one factors of each other by algebraic factor maps, we say they are *algebraically weakly isomorphic*. Equivalently, α_1 and α_2 are algebraically weakly isomorphic if they have a common finite-to-one algebraic factor.

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The following generalizes a theorem of Kalinin and Anatole Katok [19, Theorem 3.1] and of Kalinin and Spatzier [20, Theorem 4.7], where the main difference is that we do not assume that the actions are totally non-symplectic or hyperbolic.

COROLLARY 1.4. (Classification of disjointness) *If α_1 and α_2 are totally irreducible and not virtually cyclic, then they are not disjoint (with respect to the Haar measures) if and only if there exists a finite index subgroup $\Lambda \subseteq \mathbb{Z}^d$ for which $\alpha_{1,\Lambda}$ and $\alpha_{2,\Lambda}$ are algebraically weakly isomorphic.*

More generally, let α_j be $\mathbb{Z}^d$-actions on solenoids (not necessarily irreducible) without virtually cyclic factors for $j = 1, \ldots, r$. Then they are not mutually disjoint if and only if there exist indices $i, j \in \{1, \ldots, r\}$ with $i \neq j$, a finite index subgroup $\Lambda \subseteq \mathbb{Z}^d$, and a non-trivial Λ-action β on a solenoid Y which is an algebraic factor of $\alpha_{i,\Lambda}$ and $\alpha_{j,\Lambda}$.

1.3. *Algebraicity of factors.* Anatole Katok, Svetlana Katok, and Schmidt [21, Theorem 5.6] studied measurable factor maps between $\mathbb{Z}^d$-actions by automorphisms of tori. Our second application gives an extension of this by characterizing the structure of measurable factors (or equivalently invariant σ-algebras). We start by giving two algebraic constructions that give invariant σ-algebras.

- If $X' \subseteq X$ is a closed α-invariant subgroup and $\pi : X \to X/X'$ denotes the canonical projection map, then the preimage $\mathcal{A} = \pi^{-1}\mathcal{B}_{X/X'}$ of the Borel σ-algebra $\mathcal{B}_{X/X'}$ of X/X' is α-invariant.
- If Γ is a finite group of affine automorphisms that is normalized by α, then the σ-algebra $\mathcal{B}_X^\Gamma$ of Γ-invariant Borel subsets of X is α-invariant.

COROLLARY 1.5. (Algebraicity of measurable factors) *Let α be a $\mathbb{Z}^d$-action by automorphisms of the solenoid X without virtually cyclic factors, and let $\mathcal{A} \subseteq \mathcal{B}_X$ be an invariant σ-algebra. Then there exists a closed α-invariant subgroup $X' \subseteq X$ and a finite group Γ of affine automorphisms of X/X' that is normalized by the action $\alpha_{X/X'}$ induced by α on X/X' such that*

$$\mathcal{A} = \pi^{-1}(\mathcal{B}_{X/X'}^\Gamma) \text{ modulo } m_X.$$

In other words, the corollary states that every measurable factor of α arises by a combination of the two algebraic constructions given above.

In the irreducible case, the theorem gives that every non-trivial measurable factor of α is a quotient of X by the action of a finite affine group. The simplest examples of such groups are finite translation groups. However, more complicated examples are also possible; for example, let $w \in X$ be any α-fixed point. Then the action of $G = \{\mathrm{Id}, -\mathrm{Id} + w\}$ on X commutes with α.

The proof of Corollary 1.5 uses the relatively independent joining of the Haar measure with itself over the factor $\mathcal{A}$, which gives an invariant measure on $X \times X$ analyzable by Theorem 1.3. This is similar to the proof of isomorphism rigidity in [21], which followed a suggestion by Thouvenot.

We will discuss further corollaries towards factors in §9.

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1.4. *Remarks and acknowledgements.* The results of this paper were obtained in 2002 and announced in [7]; indeed this was the first result we worked on together. Since then, there was always another newer result that we wanted to write, and we never seemed to have the time to finally write down the general case of the results announced in [7]. One important ingredient in this work is the product structure for coarse Lyapunov foliations developed around that time by Anatole Katok and the first author.

The ideas behind the proof of Theorem 1.3 were used by Z. Wang to prove his strong measure classification result for invariant measures on nilmanifolds [37]. Actions by automorphisms on nilmanifolds generalize actions on tori which are covered by the results of this paper; solenoids are more general than tori, but more importantly, in that paper, Z. Wang does not allow for zero entropy factors, as we do here. Hence the results of this paper are (to the best of our knowledge) 'new' in the sense that they have not appeared in print before. We thank Z. Wang for encouraging us to write down the complete proof of [7] and for his willingness to help us do so. We also would like to thank the anonymous referee and Manuel Luethi for their comments.

2. *Actions on adelic solenoids*

2.1. *Adeles, local and global fields.* We review some basic facts and definitions regarding local fields, global fields, and the adeles. A general reference to these topics is Weil's classical book [38, Chs. I–IV]; note that Weil calls what is now commonly referred to as global fields $\mathbb{A}$-fields. Throughout this paper, the term *local field* will denote a locally compact field of characteristic zero; these include $\mathbb{R}$ and $\mathbb{C}$ as well as finite extensions of the field of p-adic numbers $\mathbb{Q}_p$. (The terminology of global and local fields was introduced to incorporate both the positive and zero characteristic cases on an equal footing, but dynamically there are rather fundamental differences (see e.g. [3, 24]) and we restrict ourselves in this paper to the zero characteristic case.) Let $\mathbb{K}$ be a local field and let $\lambda_{\mathbb{K}}$ be the Haar measure on $\mathbb{K}$. We define $\delta(\mathbb{K})$ as the degree of the field extension $\mathbb{K}$ over the closure of $\mathbb{Q}$ in $\mathbb{K}$, which can be isomorphic to either $\mathbb{R}$ if $\mathbb{K}$ is Archimedean or $\mathbb{Q}_p$ for some prime p otherwise (to make the notation more consistent, we will also write $\mathbb{Q}_\infty$ for $\mathbb{R}$). Local fields come equipped with an absolute value $|\cdot|_{\mathbb{K}}$, which we will always normalize to coincide with the usual absolute value on $\mathbb{R}$ or $\mathbb{Q}_p$. We note that in any of these cases we have

$$\lambda_{\mathbb{K}}(aC) = |a|_{\mathbb{K}}^{\delta(\mathbb{K})}\lambda_{\mathbb{K}}(C) \tag{2.1}$$

for any measurable set $C \subseteq \mathbb{K}$.

We recall that a *global field* $\mathbb{K}$ is a finite field extension of $\mathbb{Q}$. We will denote the completions of $\mathbb{K}$ by $\mathbb{K}_\sigma$, where σ stands for the (Archimedean or non-Archimedean) *place*—that is, an equivalence class of absolute values. We choose the representative to coincide with either $|\cdot|_\infty$ of $|\cdot|_p$ on $\mathbb{Q}$. We recall that $\mathbb{K}_\sigma$ is a local field and will use the abbreviation $|\cdot|_\sigma = |\cdot|_{\mathbb{K}_\sigma}$ for the norms satisfying (2.1) on $\mathbb{K}_\sigma$. If $|\cdot|_\sigma$ coincides with $|\cdot|_p$ on $\mathbb{Q}$, then we say that σ lies over p; if $|\cdot|_\sigma$ is Archimedean, we say that σ is an infinite place of $\mathbb{K}$.

For a global field $\mathbb{K}$, the ring of adeles $\mathbb{A}_{\mathbb{K}}$ over $\mathbb{K}$ is defined as the restricted direct product of all completions of $\mathbb{K}$ with respect to the maximal compact subrings for all

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non-Archimedean completions. In other words, $(t_\sigma)_\sigma \in \mathbb{A}_\mathbb{K}$ if $t_\sigma \in \mathbb{K}_\sigma$ for all places σ of $\mathbb{K}$ and, except for finitely many σ (an exceptional set that is assumed to include all infinite places), we have that in fact t_σ lies in the maximal compact subring $\mathcal{O}_{\mathbb{K},\sigma} < \mathbb{K}_\sigma$. In the special case $\mathbb{K} = \mathbb{Q}$, this takes the form

$$\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p = \mathbb{R} \times \bigcup_S (\prod_{p\in S} \mathbb{Q}_p \times \prod_{p\notin S} \mathbb{Z}_p),$$

where the union runs over all finite subsets S of the primes. The general case of the ring of adeles $\mathbb{A}_\mathbb{K}$ over a global field $\mathbb{K}$ is defined similarly, but can also be obtained via

$$\mathbb{A}_\mathbb{K} = \mathbb{A} \otimes_\mathbb{Q} \mathbb{K}. \tag{2.2}$$

We shall identify $\mathbb{K}_\sigma$ with the corresponding subring in $\mathbb{A}_\mathbb{K}$. Using a basis of $\mathbb{K}$ over $\mathbb{Q}$, we obtain an additive group isomorphism (indeed, an isomorphism of vector spaces over $\mathbb{Q}$)

$$\mathbb{A}_\mathbb{K} = \mathbb{A} \otimes_\mathbb{Q} \mathbb{K} \cong \mathbb{A}^{[\mathbb{K}:\mathbb{Q}]}. \tag{2.3}$$

We recall moreover that $\mathbb{Q}$ diagonally embedded into $\mathbb{A}$ is discrete and cocompact and that the Pontryagin dual $\widehat{\mathbb{A}}$ of $\mathbb{A}$ can be identified with $\mathbb{A}$ itself. Finally the isomorphism between $\widehat{\mathbb{A}}$ and $\mathbb{A}$ can be chosen so that the annihilator of $\mathbb{Q}$ is $\mathbb{Q}$ itself, which implies that the Pontryagin dual of $\mathbb{Q}$ can be identified with $\mathbb{A}/\mathbb{Q}$. This extends similarly to global fields, see e.g. [**38**, pp. 64–69].

2.2. *Adelic actions.* For us, the adelic setup gives a concrete language to discuss actions on general solenoids. We note however that for automorphisms on tori, it suffices to consider all Archimedean places of $\mathbb{K}$ and for irreducible actions, it would suffice to consider only finitely many places (see also [**7**] for the latter).

Indeed, let us fix a dimension $m \geq 1$, a rank $d \geq 1$, and d commuting matrices $A_1, \ldots, A_d \in \mathrm{GL}_m(\mathbb{Q})$. We use them to define a linear representation $\widetilde{\alpha}$ of $\mathbb{Z}^d$ on $\mathbb{Q}^m$. Using the matrices in the same way as within vector spaces, this extends to an action of $\mathbb{Z}^d$ by group automorphisms on $\mathbb{A}^m$, which we will also denote by $\widetilde{\alpha}$. Finally, we take the quotient by the discrete cocompact invariant subgroup $\mathbb{Q}^m$ and obtain an action α of $\mathbb{Z}^d$ by automorphisms on the solenoid

$$X_m = \mathbb{A}^m/\mathbb{Q}^m.$$

We will refer to X_m as an *adelic solenoid* and to this action as the *adelic action* on X_m defined by the matrices (or equivalently the linear maps) $A_1, \ldots, A_d$.

Since every group automorphism of $\mathbb{Q}^m$ is in fact $\mathbb{Q}$-linear and defined by an invertible matrix in $\mathrm{GL}_m(\mathbb{Q})$, it follows from Pontryagin duality that every action of $\mathbb{Z}^d$ by automorphisms on X_m can be defined this way. We will explain this step in a more general form in §4.1.

We say that a closed subgroup $Y < X_m$ of an adelic solenoid $X_m = \widehat{\mathbb{Q}}^m$ is *adelic* if it is a linear subspace over $\mathbb{Q}$ (that is, $\mathbb{Q}Y \subseteq Y$). Since this notion will be useful for us, we wish to study it briefly in the following lemma.

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LEMMA 2.1. *Let $m \geq 1$ and let $Y \leq X_m$ be a closed subgroup. Then the following conditions are equivalent:*

(1) *$Y \leq X_m$ is an adelic subgroup;*
(2) *the annihilator $Y^\perp \leq \mathbb{Q}^m$ is a $\mathbb{Q}$-linear subspace;*
(3) *there exists a $\mathbb{Q}$-linear subspace $V \leq \mathbb{Q}^m$ so that Y is the image of $\mathbb{A} \otimes_{\mathbb{Q}} V \leq \mathbb{A}^m$ modulo $\mathbb{Q}^m$.*

Proof. The equivalence of (1) and (2) follows from Pontryagin duality. Indeed $aY = Y$ for $a \in \mathbb{Z} \setminus \{0\}$ (and then also $a \in \mathbb{Q} \setminus \{0\}$) is equivalent to $a(Y^\perp) = Y^\perp$ (since $(aY)^\perp = a^{-1}Y^\perp$).

Suppose now $V < \mathbb{Q}^m$ is a linear subspace as in (3). Then $\mathbb{A} \otimes_{\mathbb{Q}} V$ is clearly invariant under $\mathbb{Q}$ and hence defines modulo $\mathbb{Q}^m$, an adelic subgroup.

Finally assume that Y is adelic as in (1) (and equivalently (2)). Let $W = Y^\perp < \mathbb{Q}^m$ so that W is a linear subspace and $Y = W^\perp$ by Pontryagin duality. By [38, Ch. IV], there exists a character $\chi_0 \in \widehat{\mathbb{A}}$ so that the isomorphism $\widehat{\mathbb{A}} \cong \mathbb{A}$ is induced by the definition $\langle a, b\rangle = \chi_0(ab)$ for all $a, b \in \mathbb{A}$ and with this isomorphism, we have $\mathbb{Q}^\perp = \mathbb{Q}$. Moreover, this also gives $\widehat{\mathbb{A}^m} \cong \mathbb{A}^m$ using the pairing

$$\langle (a_1, \dots, a_m), (b_1, \dots, b_m)\rangle = \chi_0(a_1b_1 + \cdots + a_mb_m)$$

for all $(a_1, \dots, a_m), (b_1, \dots, b_m) \in \mathbb{A}^m$. Since $W < \mathbb{Q}^m$ is a linear subspace, we may apply a linear isomorphism $A \in \mathrm{GL}_m(\mathbb{Q})$ so that $W_1 = A(W)$ is precisely the span of the first k standard basis vectors. Applying the inverse of the dual (transpose) linear automorphism to Y, this shows that $Y_1 = (A^t)^{-1}(Y)$ satisfies that $Y_1^\perp = W_1$. Now let $(a_1, \dots, a_m) \in Y_1$. Hence we have $\chi_0(a_jb) = 1$ for all $b \in \mathbb{Q}$ and $j = 1, \dots, k$. However, this gives by the properties of χ_0 that $a_j \in \mathbb{Q}$ for $j = 1, \dots, k$. It follows that $Y_1 = \mathbb{Q}^m + \mathbb{A} \otimes_{\mathbb{Q}} V_1$, where V_1 is the linear hull of the last $m - k$ basis vectors. Applying A^t to this claim gives the description of Y as in (3). □

2.3. *Irreducible adelic actions.* We say that an adelic action on X_m is $\mathbb{A}$ *-irreducible* if the associated linear representation of $\mathbb{Z}^d$ on $\mathbb{Q}^m$ is irreducible over $\mathbb{Q}$, that is, if there does not exist a rational nontrivial proper invariant subspace. Note however that $\mathbb{A}$-irreduciblity does not coincide with the notion of irreducibility defined on p. 5. In fact, an adelic action is never irreducible but it will be convenient to study $\mathbb{A}$-irreducible adelic actions as basic building blocks of other adelic actions.

We note that given a global field $\mathbb{K}$ and $d \geq 1$ elements $\zeta_1, \dots, \zeta_d \in \mathbb{K}$, we may consider multiplication by these elements as a $\mathbb{Q}$-linear map on the vector space $\mathbb{K}$ over $\mathbb{Q}$ to define an adelic action of $\mathbb{Z}^d$ on $\mathbb{A}_{\mathbb{K}}/\mathbb{K}$. Using a fixed basis of $\mathbb{K}$ over $\mathbb{Q}$, we may identify $\mathbb{K}$ with $\mathbb{Q}^m$ and multiplication by $\zeta_1, \dots, \zeta_d$ with certain matrices $A_1, \dots, A_d$. In this way, our discussions of §2.2 also apply to the multiplication maps by $\zeta_1, \dots, \zeta_d$ on $\mathbb{K}$. The point of the following proposition is that every $\mathbb{A}$-irreducible action of $\mathbb{Z}^d$ arises in this way from a global number field and d of its elements.

PROPOSITION 2.2. (Diagonalization of $\mathbb{A}$-irreducible action) *Let $m, d \geq 1$ and let α be an $\mathbb{A}$-irreducible adelic action of $\mathbb{Z}^d$ on $X_m = (\mathbb{A}/\mathbb{Q})^m$. Then there exists a global field*

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$\mathbb{K}$ of degree m over $\mathbb{Q}$ and d non-zero elements $\zeta_1, \dots, \zeta_d \in \mathbb{K}^\times$ so that α is isomorphic to the action on $\mathbb{A}_\mathbb{K}/\mathbb{K}$ generated by the maps $a \in \mathbb{K} \mapsto \zeta_j a \in \mathbb{K}$ for $j = 1, \dots, d$. More explicitly, this action on $\mathbb{A}_\mathbb{K}/\mathbb{K}$ (which as an additive topological group is isomorphic to X_m) can be given as follows:

$$\widetilde{\alpha}^{\mathbf{n}} : (v_\sigma)_\sigma \in \mathbb{A}_\mathbb{K} \mapsto (\underbrace{\zeta_{1,\sigma}^{n_1} \dots \zeta_{d,\sigma}^{n_d}}_{=\zeta_{\mathbf{n},\sigma}} v_\sigma)_\sigma,$$

where $\zeta_{1,\sigma}, \dots, \zeta_{d,\sigma} \in \mathbb{K}_\sigma$ and $\zeta_{\mathbf{n},\sigma}$ denote the image of $\zeta_1, \dots, \zeta_d$ respectively of $\zeta_\mathbf{n} = \zeta_1^{n_1} \dots \zeta_d^{n_d}$ in the completion $\mathbb{K}_\sigma$.

Proof. Let $\zeta_j = \widetilde{\alpha}^{\mathbf{e}_j} \in \mathrm{GL}_m(\mathbb{Q})$ for $j = 1, \dots, d$ be the matrices that define the action $\widetilde{\alpha}$ on $\mathbb{Q}^m$ and $\mathbb{A}^m$ associated to α.

We define $\mathbb{K} = \mathbb{Q}[\zeta_1, \dots, \zeta_d] \subseteq \mathrm{GL}_m(\mathbb{Q})$ to be the ring of polynomial expressions f in the matrices $\zeta_1, \dots, \zeta_d$ and with rational coefficients. We note that Lemma 2.1 implies that $\mathbb{Q}^m$ has no proper rational subspaces invariant under $\mathbb{K}$. Since $\zeta_1, \dots, \zeta_d$ commute, it follows that any such polynomial expression $f \in \mathbb{K}$ is either zero or is invertible (as an element of $\mathrm{GL}_m(\mathbb{Q})$). In particular, we have that $\mathbb{K}$ is an integral domain. As it is also a finite dimensional algebra over $\mathbb{Q}$, it follows that $\mathbb{K}$ is field extension of $\mathbb{Q}$. Once more because $\mathbb{Q}^m$ has no proper invariant subspaces, it also follows that $\varphi : a \in \mathbb{K} \mapsto a(\mathbf{e}_1) \in \mathbb{Q}^m$ must be surjective. By definition the kernel $\ker(\varphi)$ is an ideal, which implies that φ is injective since $\mathbb{K}$ is a field. It follows that φ is a linear isomorphism.

To summarize, we have found a global field $\mathbb{K}$ and elements $\zeta_1, \dots, \zeta_d \in \mathbb{K}^\times$ so that up to a linear isomorphism our linear representation $\widetilde{\alpha}^{\mathbf{n}}$ on $\mathbb{Q}^m$ is defined for every $\mathbf{n} \in \mathbb{Z}^d$ by multiplication by $\zeta_\mathbf{n} = \zeta_1^{n_1} \dots \zeta_d^{n_d}$ on the vector space $\mathbb{K}$.

To obtain the adelic action, we tensorize with $\mathbb{A}$. On one hand, for the action on $\mathbb{Q}^m$ this gives the action of $\mathbb{Z}^d$ on $\mathbb{A}^m$ we started with. On the other hand, we may tensorize the linear isomorphism between $\mathbb{Q}^m$ and $\mathbb{K}$ with $\mathbb{A}$ to obtain the group isomorphism

$$\mathbb{A}^m = \mathbb{Q}^m \otimes_\mathbb{Q} \mathbb{A} \cong \mathbb{K} \otimes_\mathbb{Q} \mathbb{A} \cong (\mathbb{K} \otimes_\mathbb{Q} \mathbb{R}) \times \textstyle\prod_p'(\mathbb{K} \otimes_\mathbb{Q} \mathbb{Q}_p).$$

Now notice that we can identify $\mathbb{K}$ with the quotient $\mathbb{Q}[x]/(p(x))$ for some irreducible polynomial $p(x) \in \mathbb{Q}[x]$, which implies that $\mathbb{K} \otimes \mathbb{R}$ is isomorphic $\mathbb{R}[x]/(p(x))$. Since the irreducible factors of $p(x) \in \mathbb{R}[x]$ correspond precisely to the roots of $p(x)$ (all appearing with multiplicity one) and hence also to the Galois embeddings of $\mathbb{K}$ into $\mathbb{C}$, it follows that $\mathbb{K} \otimes \mathbb{R}$ is as a ring isomorphic to $\prod_{\sigma|\infty} \mathbb{K}_\sigma$, where the product runs over all places of $\mathbb{K}$ lying above ∞, that is, over all Archimedean completions of $\mathbb{K}$.

This argument applies similarly for the tensor product with $\mathbb{Q}_p$ so that $\mathbb{K} \otimes_\mathbb{Q} \mathbb{Q}_p$ is isomorphic as a ring to the product $\prod_{\sigma|p} \mathbb{K}_\sigma$ and σ denotes here all places of $\mathbb{K}$ above p, see also [**38**, p. 56]. Applying this argument at all places of $\mathbb{Q}$, we obtain that $\mathbb{A}^m$ is isomorphic to $\mathbb{A}_\mathbb{K}$.

Application of $\widetilde{\alpha}^{\mathbf{e}_j}$ corresponds under this isomorphism from $\mathbb{A}^m$ to $\mathbb{A}_\mathbb{K} = \prod_\sigma' \mathbb{K}_\sigma$ to multiplication by the image of ζ_j in the factor $\mathbb{K}_\sigma$ for every place σ of $\mathbb{K}$. This gives the proposition. □

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Let us write $\delta(\sigma) = \delta(\mathbb{K}_\sigma) \in \mathbb{N}$ for any place σ of $\mathbb{K}$. The following product formula is a crucial ingredient in our proof.

PROPOSITION 2.3. (Product formula) *Let α be an $\mathbb{A}$-irreducible adelic $\mathbb{Z}^d$-action as in Proposition 2.2. Then we have*

$$\prod_\sigma |a|_\sigma^{\delta(\sigma)} = 1 \quad \text{for every } a \in \mathbb{K} \setminus \{0\} \tag{2.4}$$

and this applies in particular to $a = \zeta_{\mathbf{n}}$ for every $\mathbf{n} \in \mathbb{Z}^d$.

We note that one way to obtain this result is precisely to interpret the product as the modular character for the automorphism defined by multiplication by a on the compact group $\widehat{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}$ (cf. (2.1)). We refer to [38, p. 75] for a proof along these lines.

2.4. *A filtration by $\mathbb{A}$-irreducible adelic actions.* The following lemma reveals an advantage of adelic actions by connecting structural questions concerning α to linear algebra on the dual.

LEMMA 2.4. (Decomposition into $\mathbb{A}$-irreducible factors) *Let $m, d \geq 1$ and let α be an adelic $\mathbb{Z}^d$-action on X_m. Then there exist closed α-invariant adelic subgroups*

$$Y_0 = \{0\} < Y_1 < \cdots < Y_r = X_m,$$

such that the action induced by α on Y_j/Y_{j-1} is an $\mathbb{A}$-irreducible adelic $\mathbb{Z}^d$-action for all $j = 1, \ldots, r$.

We will refer to the $\mathbb{A}$-irreducible adelic actions appearing in Lemma 2.4 as the $\mathbb{A}$-irreducible factors associated to α.

Proof. By Pontryagin duality, we may consider instead of α the linear representation $\widehat{\alpha}$ on $\mathbb{Q}^m$. Let $V_1 \subseteq \mathbb{Q}^m$ be a non-trivial subspace that is invariant under $\widehat{\alpha}$ and of minimal dimension. Note that this implies that the restriction of $\widehat{\alpha}$ to V_1 is irreducible over $\mathbb{Q}$. If $V_1 \neq \mathbb{Q}^m$, we let V_2 be a subspace that is invariant under $\widehat{\alpha}$, strictly contains V_1 and is among these of minimal dimension. Once more this implies that V_2/V_1 is irreducible over $\mathbb{Q}$ (for the representation induced by $\widehat{\alpha}$).

Continuing like this, we obtain a partial flag

$$V_0 = \{0\} < V_1 < V_2 < \cdots < V_r = \mathbb{Q}^m \tag{2.5}$$

consisting of $\widehat{\alpha}$-invariant subspaces so that V_j/V_{j-1} is irreducible over $\mathbb{Q}$. Applying Pontryagin duality (and reversing the indexing), this gives the lemma. □

3. *Leafwise measures, invariant foliations, and entropy*

We briefly recall the main properties of leafwise measures. These have been introduced in the context of higher rank rigidity theorems (under the name of conditional measures for foliations) by Anatole Katok and Spatzier in [22] and have since become an essential tool for all of the theorems in the area. Implicitly leafwise measures appear already in the proof of Rudolph's Theorem in [34]. A general reference for this section is [10, §§6 and 7].

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3.1. *Leafwise measures.* Given a quotient $X = G/\Gamma$ of a locally compact abelian group G by a lattice $\Gamma < G$ and a closed subgroup $V < G$ with $V \cap \Gamma = \{0\}$, we consider the foliation of X into V-orbits. Let π_X denote the natural projection $G \to X$. As we will reduce our main theorem to the adelic case (Theorem 4.1), we consider the case that G is $\mathbb{A}^m$ (though everything we say below is equally valid for $G = \mathbb{R}^m$, or a finite product of local fields). We note that the metric on $\mathbb{A}^m$ is chosen so that the balls $B_r^V = B_r(0) \cap V$ have compact closures for all $r > 0$.

For our purposes, it will be important to work with an extension of X—a product $\widetilde{X} = X \times \Omega$ of X with an arbitrary compact metric space Ω. We let V act on $\widetilde{X}$ by translation on the first coordinate and trivially on the second coordinate, obtaining in this way a foliation of $\widetilde{X}$ into V orbits. This foliation of $\widetilde{X}$ into V-orbits does not admit in general a Borel cross-section, and typically one cannot find a countably generated σ-algebra on $\widetilde{X}$ whose atoms coincide with almost every (a.e.) V-orbits. Given a probability measure μ on $\widetilde{X}$, the foliation into V-orbits gives rise to a system of *leafwise measures* on $\widetilde{X}$: a Borel measurable map $x \mapsto \mu_x^V$ from a subset of full measure $\widetilde{X}' \subset \widetilde{X}$ to locally finite (possibly infinite) measures on V. We say that a leafwise measure μ_x^V is *trivial* if it is a multiple of the Dirac measure at the identity; we say that the system of leafwise measures is trivial if it is trivial at a.e. point. We also note that almost surely 0 belongs to the support of μ_x^V.

The system of leafwise measure satisfies the following *compatibility condition*: for any $v \in V$ and $x \in \widetilde{X}'$ so that $x + v$ is also in $\widetilde{X}'$

$$(\mu_{x+v}^V + v) \propto \mu_x^V. \tag{3.1}$$

Here and in the following, we write $\nu \propto \nu'$ for two measures ν, ν' if there exists $c > 0$ with $\nu = c\nu'$.

One way to characterize the leafwise measures is through the notion of subordinate σ-algebras:

Definition 3.1. (Subordinate σ-algebras) A σ-algebra $\mathcal{A}$ of Borel subsets of $\widetilde{X}$ is *subordinate to* V if $\mathcal{A}$ is countably generated, for every $x \in \widetilde{X}$ the atom $[x]_{\mathcal{A}}$ of x with respect to $\mathcal{A}$ is contained in the leaf $x + V$, and for a.e. x

$$x + B_\epsilon^V \subseteq [x]_{\mathcal{A}} \subseteq x + B_\rho^V \quad \text{for some } \epsilon > 0 \text{ and } \rho > 0.$$

For these x, we define the *shape* S_x of the atom $[x]_{\mathcal{A}}$ (from the point of view of x) as the subset of V satisfying $x + S_x = [x]_{\mathcal{A}}$.

Let $\mathcal{A}$ be a countably generated σ-algebra on $\widetilde{X}$ that is subordinate to V. Then the leafwise measures for μ with respect to V and the conditional measures of μ with respect to $\mathcal{A}$ satisfy that for a.e. $x \in \widetilde{X}$, the conditional measure $\mu_x^{\mathcal{A}}$ at x with respect to $\mathcal{A}$ equals the normalized push forward $x + (\mu_x^V|_{S_x})$ of the restriction $\mu_x^V|_{S_x}$ under the addition map $v \mapsto x + v$ from $V \to \widetilde{X}$.

A slightly subtler but important feature of the system of leafwise measures is that while they may be (and typically are) infinite measures, they have certain *a priori* restrictions on how fast they grow: There exists a concrete function f_V on V (only depending on V) that is integrable with respect to μ_x^V for every x in a set of full measure (that we may as

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well assume already contains the conull set $\widetilde{X}'$), see [10, Theorem 6.30]. In fact, f_V can be chosen with very mild (polynomial-like) decay properties.

In particular, if $x \in \widetilde{X}'$ and $v \in V$ satisfies

$$(\mu_x^V + v) \propto \mu_x^V, \tag{3.2}$$

this in fact implies the formally stronger conclusion that μ_x^V is translation invariant by the same v, that is,

$$(\mu_x^V + v) = \mu_x^V, \tag{3.3}$$

for otherwise μ_x^V would have exponential growth, which would contradict the polynomial-like growth condition. Finally we recall the following proposition.

PROPOSITION 3.2. (Leafwise measures supported on subgroups [9, Lemma 3.2], [28, §3]) *Let $X = G/\Gamma$ and let $W < V < G$ be closed subgroups. Let μ be a probability measure on X. Suppose that for μ-a.e. x, the measure μ_x^V is supported on W. Then, identifying locally finite measures on W with locally finite measures on V supported on W in the obvious way, we have that for μ-a.e. x, $\mu_x^W \propto \mu_x^V$.*

3.2. *Entropy and leafwise measures.* Suppose now $T : G \to G$ is a group automorphism preserving Γ and V. Recall that we have restricted ourselves without loss of generality to the case where X is the Pontryagin dual to an m-dimensional vector space L over $\mathbb{Q}$. Fixing a choice of basis in L, we can restrict our attention to the case of $G = \mathbb{A}^m$, $\Gamma = \mathbb{Q}^m$, and hence the automorphism T is defined by a rational matrix $\mathbf{T} \in \mathrm{GL}_m(\mathbb{Q})$. We note however that some of the definitions below, e.g. the 'sufficiently fine' condition in (3.7), depend on the choice of basis used to give the isomorphism $L \cong \mathbb{Q}^m$ (which induces an isomorphism $\Gamma \cong \mathbb{Q}^m$).

We denote the resulting automorphism of $X = G/\Gamma$ also by T, and consider an extension $\widetilde{T} = T \times T_\Omega : \widetilde{X} \to \widetilde{X}$ with $T_\Omega : \Omega \to \Omega$ measurable. Furthermore, suppose the probability measure μ on $\widetilde{X}$ is invariant under $\widetilde{T}$. Then the characterizing properties of the leafwise measures μ_x^V imply the *equivariance formula*

$$\mu_{\widetilde{T}x}^V \propto T_*(\mu_x^V) \tag{3.4}$$

for a.e. $x \in \widetilde{X}'$, see e.g. [10, Lemma 7.16].

We say that a closed subgroup $V < \mathbb{A}^m$ is *S-linear* where S is a finite set of places of $\mathbb{Q}$ (that is, a set of prime numbers or infinity) if for each $\sigma \in S$ there is a subspace $V_\sigma < \mathbb{Q}_\sigma^m$ so that V is the direct product of the V_σ for $\sigma \in S$. Below we will frequently use the *stable horospherical subgroup*

$$U_T^- = \{a \in \mathbb{A}^m : T^n a \to 0 \text{ as } n \to \infty\}$$

for T and the *unstable horospherical subgroup* $U_T^+ = U_{T^{-1}}^-$. If

$$S = \{p : \mathbf{T} \notin \mathrm{GL}(m, \mathbb{Z}_p)\} \cup \{\infty\}, \tag{3.5}$$

then the horospherical subgroups U_T^- and U_T^+ are S-linear for this S.

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3.3. *Increasing subordinate σ-algebras and entropy.* We continue with the notation of §3.2. We say that a countably generated σ-algebra $\mathcal{A}$ on $\widetilde{X}$ is *increasing* with respect to $\widetilde{T}$ if $\mathcal{A} \subseteq \widetilde{T}\mathcal{A}$ (modulo μ), that is, the atom $[\widetilde{T}x]_{\mathcal{A}}$ almost surely (a.s.) contains $\widetilde{T}([x]_{\mathcal{A}})$.

Let $\mathcal{P}$ be a finite partition of $\widetilde{X}$, which we identify with the corresponding finite algebra of sets. For any $\epsilon > 0$, let

$$\partial_\epsilon^V \mathcal{P} = \{\widetilde{x} \in \widetilde{X} : \widetilde{x} + B_\epsilon^V \not\subseteq [\widetilde{x}]_{\mathcal{P}}\}.$$

The following lemma follows quickly from monotonicity of the function $r \in [0, \infty) \mapsto \mu(B_r(x))$ for $x \in \widetilde{X}$ (and its almost sure differentiability) together with compactness of $\widetilde{X}$.

LEMMA 3.3. [**10**, Lemma 7.27] *For any probability measure $\widetilde{\mu}$ on $\widetilde{X}$, there exists a finite partition $\mathcal{P}$ of $\widetilde{X}$ into arbitrarily small sets such that for some fixed C and for every $\epsilon > 0$*

$$\mu(\partial_\epsilon^V \mathcal{P}) < C\epsilon. \tag{3.6}$$

For more details, see [**10**, §7]. A partition $\mathcal{P}$ satisfying the conclusion of the above lemma will be said to have *thin boundaries*. We will assume throughout that any finite partition $\mathcal{P}$ of X we will consider below is *sufficiently fine* in the sense that

$$P - P \subset \pi_X\left(\prod_{v\in S} B_{\mathbb{Q}_v^m}(r_v) \times \prod_{v\notin S} \mathbb{Z}_v^m\right) \quad \text{for every } P \in \mathcal{P}, \tag{3.7}$$

with $r_v = 0.1 \max(\|\mathbf{T}\|_v, \|\mathbf{T}^{-1}\|_v)^{-1}$ (with respect to the operator norm on $\mathrm{GL}(\mathbb{Q}_v)$).

For any σ-algebra $\mathcal{A}$ and $-\infty \le k_0 < k_1 \le \infty$ set

$$\mathcal{A}^{k_0} = \widetilde{T}^{-k_0}\mathcal{A} \text{ and } \mathcal{A}^{(k_0,k_1)} = \bigvee_{k_0\le i\le k_1} \widetilde{T}^{-i}\mathcal{A}$$

(for k_0 or $k_1 = \pm\infty$ strict inequality instead of $\le$ should be used).

An easy Borel–Cantelli argument gives that if $\mathcal{P}$ is a sufficiently fine finite partition of X with small boundaries in the sense of (3.6) and (3.7), then

$$\mathcal{C}_{\mathcal{P}} = \mathcal{P}^{(0,\infty)}$$

is a countably generated σ-algebra satisfying one of the two conditions required by Definition 3.1 for $V = U_T^-$, namely for a.e. x it holds that there is an $\epsilon > 0$ so that $x + B_{U_T^-}(\epsilon) \subset [x]_{\mathcal{P}}$. (With a bit more care, using a countable partition $\mathcal{P}$ with finite entropy, one can get that $\mathcal{P}^{(0,\infty)}$ is actually U_T^--subordinate; we achieve a similar goal by a cruder approach below.)

A modified version of this increasing σ-algebra $\mathcal{P}^{(0,\infty)}$ can be used to construct for any S-linear T-normalized subgroup $V < U_T^-$ of $\mathbb{A}^m$ an increasing V subordinate σ-algebra $\mathcal{C}_V$ on $\widetilde{X} = X \times \Omega$ so that moreover $\mathcal{C}_V = \tilde{T}^{-1}\mathcal{C}_V \vee \mathcal{P}$. Indeed, first we construct starting from a (sufficiently fine, with small boundaries) finite partition $\mathcal{P}$ on X a σ-algebra $\mathcal{P}_V$ on $\widetilde{X}$ as follows: for each $P \in \mathcal{P}$, lift it to a subset $\widetilde{P} \subset \mathbb{A}^m$ contained in a translate of $\prod_{v\in S} B_{\mathbb{Q}_v^m}(r_v) \times \prod_{v\notin S} \mathbb{Z}_v^m$ (up to translation by an element of $\mathbb{Q}^m$, this lift is uniquely defined). Now take the countably generated σ-ring $\widetilde{\mathcal{Q}}_P$ of subsets C of $\widetilde{P}$ with the property that if $x \in C$ then $(x + V) \cap \widetilde{P} \subseteq C$. Since π_X is a bijection from $\widetilde{P}$ to P, the image $\mathcal{Q}_P$ of $\widetilde{\mathcal{Q}}_P$ in X is a countably generated σ-ring, and now we define $\mathcal{P}_V$ to be the σ-algebra

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of subsets of X generated by the σ-rings $\mathcal{Q}_P \times \mathcal{B}_\Omega$ for all $P \in \mathcal{P}$. Then $\mathcal{C}_V = \mathcal{P}_V^{(0,\infty)}$ is a T-increasing, countably generated σ-algebra subordinate to V satisfying $\mathcal{C}_V = \widetilde{T}^{-1}\mathcal{C}_V \vee \mathcal{P}$. Note also that by the way $\mathcal{C}_V$ is defined, there is a fixed $\rho > 0$ so that $[x]_{\mathcal{C}_V} \subseteq x + B_\rho^V$ for *all* $x \in \widetilde{X}$. For more details, the reader is referred again to [10, §7].

For $V \leq U_T^-$ as above and $\mathcal{C}_V$ as above, we define the *entropy contribution of* V to be

$$\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V) = \mathsf{H}_{\widetilde{\mu}}(\mathcal{C}_V \mid \widetilde{T}^{-1}\mathcal{C}_V). \tag{3.8}$$

We also need the conditional form of this definition: if $\mathcal{Y}$ is a $\widetilde{T}$-invariant σ-algebra, then the *entropy contribution of* V *conditional on* $\mathcal{Y}$ is defined to be

$$h_{\widetilde{\mu}}(\widetilde{T}, V \mid \mathcal{Y}) = H_{\widetilde{\mu}}(\mathcal{C}_V \mid \widetilde{T}^{-1}\mathcal{C}_V \vee \mathcal{Y}), \tag{3.8$'$}$$

where, as usual, we will identify a factor Y of $\widetilde{X}$ with the corresponding $\widetilde{T}$-invariant σ-algebra $\mathcal{Y}$ of subsets of $\widetilde{X}$. Formally the conditional entropy contribution is included in the previous case, replacing Ω by $\Omega \times Y$, but notationally it will be useful to allow additional explicit conditioning. The following propositions shows that—as implied by the notation— $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V)$ does not depend on the choice of $\mathcal{P}$ and $\mathcal{C}_V$.

PROPOSITION 3.4. *Let $V \leq U_T^-$ be an S-linear subgroup normalized by T, and let $\mathcal{C}_V$ be as above. Then*

$$\operatorname{vol}(\widetilde{T}, V, \widetilde{x}) = \lim_{|N|\to\infty} \frac{1}{N} \log \widetilde{\mu}_{\widetilde{x}}^V(T^{-N}(B_1^V(0))) \tag{3.9}$$

exists almost everywhere and $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V) = \int \operatorname{vol}(\widetilde{T}, V, \widetilde{x})\, d\widetilde{\mu}(\widetilde{x})$. *Moreover, outside a set of $\widetilde{\mu}$-measure zero,* $\operatorname{vol}(\widetilde{T}, V, x) = 0$ *if and only if $\widetilde{\mu}_x^V$ is trivial. Furthermore,* $\mathsf{h}_{\widetilde{\mu}}(\widetilde{T}, V) \leq h_{\widetilde{\mu}}(\widetilde{T} \mid \Omega)$, *with equality holding for $V = U_T^-$. In particular, $h_\mu(\widetilde{T} \mid \Omega) > 0$ if and only if $\widetilde{\mu}_x^{U_T^-}$ is not almost everywhere trivial.*

By the remark above, this proposition also covers the case of entropy contributions conditional on a factor. This proposition is essentially well known, and is e.g. heavily used by Ledrappier and Young in [25, 26] (though we are using a version of these results relative to the factor Ω of $\widetilde{X}$). For proof, we refer the reader to [10, §7] where an exposition in the spirit of this paper can be found. (In [10, §7]. It is assumed that the acting group acts in a semisimple way on the leaves, which does not necessarily hold in our case as we are explicitly allowing actions with non-trivial Jordan form. However, the arguments of [10, §7] can be easily modified to handle this situation; we leave the details to the readers.)

Assuming that μ is invariant and ergodic under a $\mathbb{Z}^d$-action $\widetilde{\alpha}$ with $\widetilde{T} = \widetilde{\alpha}^{\mathbf{n}}$ for some $\mathbf{n} \in \mathbb{Z}^d$, the value of the limit in (3.9) defines an invariant function for $\widetilde{\alpha}$, hence is almost everywhere constant, and so equals $h_\mu(\widetilde{T}, V \mid \Omega)$ for a.e. $x \in \widetilde{X}$. In particular, assuming $\widetilde{\alpha}$ is ergodic, the following three statements are equivalent: (i) $h_\mu(\widetilde{T} \mid \Omega) > 0$; (ii) $\mu_x^{U_T^-}$ is non-trivial almost everywhere; (iii) $\mu_x^{U_T^-}$ is non-trivial on a set of positive measure.

The entropy contributions for $V < U_T^+$ (also denoted $h_{\widetilde{\mu}}(\widetilde{T}, V)$) are defined similarly, and satisfy that

$$h_{\widetilde{\mu}}(\widetilde{T}, V) = h_{\widetilde{\mu}}(\widetilde{T}^{-1}, V).$$

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Let $0 \to L_1 \to L \to K \to 0$ be an exact sequence of finite dimensional vector space over $\mathbb{Q}$ and let $0 \to Y \to X \to X_1 \to 0$ be the corresponding dual exact sequence of adelic solenoids. Let $T : X \to X$ be the dual map to a linear map in $\mathrm{GL}(L)$ fixing L_1. Then T also induces a map $T_1 : X_1 \to X_1$. Let $V_{Y,\mathbb{A}} < \mathbb{A}^m$ be the rational subspace projecting modulo $\mathbb{Q}^m$ to Y. Let $T_\Omega : \Omega \to \Omega$ be a continuous map on the compact metric space Ω, and denote $\widetilde{X} = X \times \Omega$, $\widetilde{T} = T \times T_\Omega$, $\widetilde{X}_1 = X_1 \times \Omega$, etc. We let π denote both the projection from $\mathbb{A}^m \to \mathbb{A}^m / V_{Y,\mathbb{A}}$ as well as the corresponding projection $X \to X_1$.

PROPOSITION 3.5. (Entropy contribution of factor and fiber, cf. [**11**, Proposition 6.4] or [**8**, Proposition 3.1]) *With the notation above, let $\widetilde{\mu}$ be a $\widetilde{T}$-invariant measure on X, let V be a S-linear subgroup of $U_T^- < \mathbb{A}^m$, and let V_1 be a S-linear subgroup of $U_{T_1}^- < \mathbb{A}^m / V_{Y,\mathbb{A}}$ so that $V \leq \pi^{-1}(V_1)$. Let $\widetilde{\mu}_1 = \pi_* \widetilde{\mu}$. Then*

$$h_{\widetilde{\mu}}(\widetilde{T}, V) \leq h_{\widetilde{\mu}_1}(\widetilde{T}_1, V_1) + h_{\widetilde{\mu}}(\widetilde{T}, V \cap V_{Y,\mathbb{A}}), \tag{3.10}$$

with equality holding for $V = U_T^-$ and $V_1 = U_{T_1}^-$.

Proof. See [**11**, Proposition 6.4] (while the setting is a bit different, the proof there works verbatim also in our setting). □

Note that by definition, $h_{\widetilde{\mu}}(\widetilde{T}, V) = h_{\widetilde{\mu}}(\widetilde{T}, V \mid \Omega)$ and similarly for the other terms in (3.10).

4. *An adelic version of the positive entropy theorem*

We show in this section that it suffices to prove the following more special version of Theorem 1.3.

THEOREM 4.1. (Adelic theorem) *Let $m \geq 1$, $d \geq 2$, and let α be an adelic $\mathbb{Z}^d$-action on X_m without virtually cyclic factors. We suppose furthermore that the action satisfies that every adelic subgroup of X_m that is invariant under the restriction of α to a finite index subgroup of $\mathbb{Z}^d$ is actually invariant under $\mathbb{Z}^d$. Let μ be an α-invariant and ergodic probability measure on X_m. Then there exists an adelic subgroup $G < X_m$ so that μ is invariant under translation by elements of G and $h_\mu(\alpha^{\mathbf{n}}_{X/G}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^d$.*

We note that the reader interested in the heart of the argument may skip most of this section, which is dedicated to the reduction of Theorem 1.3 to Theorem 4.1, and should instead continue with §4.4.

4.1. *Extension to adelic action.* Suppose that α is a $\mathbb{Z}^d$-action by automorphisms on a solenoid X as in Theorem 1.3. By definition, this means that the Pontryagin dual $\widehat{X}$ is isomorphic to a subgroup $V \subseteq \mathbb{Q}^m$ for some $m \in \mathbb{N}$. We may assume that m is minimal, that $\widehat{X} = V$, and by applying some linear automorphism if necessary, we may also assume that the standard basis vectors of $\mathbb{Q}^m$ belong to V. By Pontryagin duality, we also have that X is isomorphic to the quotient of $X_m = \widehat{\mathbb{Q}^m} = \mathbb{A}^m / \mathbb{Q}^m$ modulo the annihilator $K = V^\perp < X_m$ of V.

Moreover, for every $\mathbf{n} \in \mathbb{Z}^d$ the dual $\widehat{\alpha}^{\mathbf{n}}$ of the automorphism $\alpha^{\mathbf{n}}$ is an automorphism of $V \subseteq \mathbb{Q}^m$. Using the standard basis of $\mathbb{Q}^m$ (contained in V by the above assumption), we can

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represent this dual automorphism by a rational matrix and extend it to linear automorphism of $\mathbb{Q}^m$ (since $V \otimes_{\mathbb{Q}} \mathbb{Q}$ can be identified with $\mathbb{Q}^m$). This shows that the dual action extends to a linear representation of $\mathbb{Z}^d$ on $\mathbb{Q}^m$.

We take the transpose of this representation (that is, of each of the matrices defining $\widehat{\alpha}^{\mathbf{e}_j}$ for $j = 1, \dots, d$) to define an action $\widetilde{\alpha}$ of $\mathbb{Z}^d$ by automorphisms on $\mathbb{Q}^m$ and $\mathbb{A}^m$. As discussed in §2.2, this defines an adelic action of $\mathbb{Z}^d$ by automorphisms of $X_m = \mathbb{A}^m/\mathbb{Q}^m$. Moreover, since $V \subseteq \mathbb{Q}^m$ is invariant under $\widehat{\alpha}$, the annihilator $K = V^\perp$ is a closed invariant subgroup for $\widetilde{\alpha}$ and the induced action on X_m/K is isomorphic to the original action on X.

Extending the above discussion, the following lemma allows us to switch our attention to the setting of an adelic $\mathbb{Z}^d$-action α on X_m for some $m \in \mathbb{N}$.

LEMMA 4.2. *Suppose Theorem 1.3 holds for adelic actions, then it also holds for all actions of $\mathbb{Z}^d$ by automorphisms on solenoids.*

Proof. Let α be a $\mathbb{Z}^d$-action by automorphisms on a solenoid X and let μ be an α-invariant and ergodic probability measure on X. Applying the above discussion, we can construct an adelic action (again denoted by α) on X_m for some $m \geq 1$ and an invariant compact subgroup K such that the action on X_m/K is isomorphic to the original action.

Next, we can define a probability measure μ_K on X_m that is invariant under K and modulo K equals μ. This describes μ_K uniquely. By invariance of K under α and uniqueness, this measure is also α-invariant. If it is not ergodic with respect to α, we may consider an ergodic component $\tilde{\mu}$ of μ_K. Due to ergodicity of μ, almost surely the ergodic components will project to μ. Let $\tilde{\mu}$ be one such ergodic component.

By our assumption (in Lemma 4.2), we know that Theorem 1.3 already holds for $\tilde{\mu}$. In other words, there exists a finite index subgroup $\Lambda < \mathbb{Z}^d$ and a decomposition $\tilde{\mu} = (1/J)(\tilde{\mu}_1 + \cdots + \tilde{\mu}_J)$ of $\tilde{\mu}$ into α_Λ-invariant and ergodic probability measures, and there exist closed α_Λ-invariant subgroup $G_j < X_m$ so that $\tilde{\mu}_j$ is invariant under translation by elements of G_j for $j = 1, \dots, J$. Moreover, the entropy of $\alpha^{\mathbf{n}}_{X_m/\tilde{G}_j}$ with $\mathbf{n} \in \Lambda$ with respect to $\tilde{\mu}_j$ vanishes for all $j = 1, \dots, J$. Taking the quotient of X_m by the invariant subgroup K, all of these statements become the corresponding statements for the push forwards μ_j of $\tilde{\mu}_j$ under the quotient map $X_m \to X_m/K \cong X$. We also note that the α_Λ-ergodic components of μ are either equal or singular to each other. Hence, if μ_j equals μ_k for some $j \neq k$, we may simply collect equal terms and would again obtain a decomposition into mutually singular measures (of necessary equal weight due to ergodicity with respect to α). Together, we obtain the conclusions of Theorem 1.3 for the original measure μ. This gives the lemma. □

4.2. *Choosing a good finite index subgroup $\Lambda < \mathbb{Z}^d$.*

LEMMA 4.3. *Let $m, d \geq 1$ and let α be an adelic $\mathbb{Z}^d$-action on X_m. Then there exists a finite index subgroup $\Lambda < \mathbb{Z}^d$ with the following property: applying Lemma 2.4 to the restriction α_Λ of α to Λ, we obtain finitely many $\mathbb{A}$-irreducible adelic actions. Then each one of them remains $\mathbb{A}$-irreducible if we restrict α_Λ further to a finite index subgroup $\Lambda' < \Lambda$. Moreover, Λ may be chosen so that an adelic subgroup $Y < X$ is invariant under α_Λ if and only if it is invariant under $\alpha_{\Lambda'}$ for a finite index subgroup $\Lambda' < \Lambda$.*

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For the proof of the lemma and some of the following arguments, we first recall the Jordan decomposition. Given a matrix $A \in \mathrm{GL}_m(\mathbb{Q})$, there exist matrices $D, U \in \mathrm{GL}_m(\mathbb{Q})$ so that $A = DU = UD$, D is semisimple (that is, is diagonalizable over $\overline{\mathbb{Q}}$), and U is unipotent (that is, has only 1 as eigenvalue). This decomposition is unique and if A commutes with a matrix B, then D and U as above also commute with B. Moreover, a subspace $V < \mathbb{Q}$ is invariant under A if and only if V is invariant under both D and U.

By applying this to each of the matrices $\alpha^{\mathbf{e}_i}$ for $i = 1, \ldots, d$, we obtain two representations of $\mathbb{Z}^d$ on $\mathbb{Q}^m$: the first representation α_{diag} is semisimple, and the second α_{uni} is by unipotent matrices, the two representations commute, and we have $\alpha^{\mathbf{n}} = \alpha^{\mathbf{n}}_{\mathrm{diag}}\alpha^{\mathbf{n}}_{\mathrm{uni}}$ for all $\mathbf{n} \in \mathbb{Z}^d$.

Proof of Lemma 4.3. We first consider an $\mathbb{A}$-irreducible action α. As the proof of Lemma 2.2 shows, α corresponds in this case to a global field $\mathbb{K}$ generated by d elements $\zeta_1, \ldots, \zeta_d$ (obtained directly from the matrix representations of $\widetilde{\alpha}^{\mathbf{e}_j}$). Restricting the action to a finite index subgroup results in replacing $\zeta_1, \ldots, \zeta_d$ by d monomial expressions $\xi_1, \ldots, \xi_d$ (corresponding to a basis of $\Lambda < \mathbb{Z}^d$) in the numbers $\zeta_1, \ldots, \zeta_d$. This in turn may result in $\xi_1, \ldots, \xi_d$ generating instead of $\mathbb{K}$ a subfield $\mathbb{L}$ of $\mathbb{K}$. In this case, the $\mathbb{A}$-irreducible representation of $\mathbb{Z}^d$ on $\mathbb{K}$ obtained in the proof of Lemma 2.2 becomes, when restricted to Λ, isomorphic to a direct sum of $[\mathbb{K} : \mathbb{L}]$ many copies of the $\mathbb{A}$-irreducible representation defined by multiplication by $\xi_1, \ldots, \xi_d$ on $\mathbb{L}$. If this indeed happens, we may choose Λ so that $\mathbb{L}$ is minimal in dimension. Hence for any finite index subgroup $\Lambda' < \Lambda$, the monomial expressions in the variables $\xi_1, \ldots, \xi_d$ corresponding to a basis of Λ' will still generate the same field $\mathbb{L}$ and so the $[\mathbb{K} : \mathbb{L}]$ many $\mathbb{A}$-irreducible representations for the restriction to Λ will remain irreducible for the restriction to Λ'. This proves the first part of the lemma in the $\mathbb{A}$-irreducible case.

Let now α be a general adelic action. Let

$$V_0 = \{0\} < V_1 < V_2 < \cdots < V_r = \mathbb{Q}^m$$

be as in Lemma 2.4, equation (2.5), with the action induced by $\widehat{\alpha}$ on V_i / V_{i-1} irreducible over $\mathbb{Q}$, and let $\mathbb{K}_i$ be the corresponding finite extension of $\mathbb{Q}$ as in Proposition 2.2 for V_i / V_{i-1} (or more precisely, for the dual adelic solenoid) for $i = 1, \ldots, r$. Applying the above discussion on each irreducible quotient by passing to a finite index subgroup $\Lambda < \mathbb{Z}^d$, we may assume that these quotients remain irreducible even if we pass to a further finite index subgroup of Λ, establishing the first part of the lemma.

Now let M be the least common multiple of the orders of the (finitely many) roots of unity in some finite degree Galois extension of $\mathbb{Q}$ containing the fields $\mathbb{K}_1, \ldots, \mathbb{K}_r$. We claim that if $\Lambda_1 = M\Lambda$ and if $V < \mathbb{Q}^m$ is invariant under $\widehat{\alpha}(\Lambda')$ for some $\Lambda' < \Lambda_1$, then V is invariant under $\widehat{\alpha}(\Lambda_1)$.

Indeed, as discussed above, for any $\Lambda' \leq \mathbb{Z}^d$, the space $V = Y^{\perp}$ is $\widehat{\alpha}(\Lambda')$-invariant if and only if it is invariant under both $\widehat{\alpha}_{\mathrm{diag}}(\Lambda')$ and $\widehat{\alpha}_{\mathrm{uni}}(\Lambda')$.

Since the map $\mathbf{n} \mapsto \widehat{\alpha}_{\mathrm{uni}}(\mathbf{n})$ is polynomial, and since any finite index subgroup of $\mathbb{Z}^d$ is Zariski dense in affine d-dimensional space, a space V is $\widehat{\alpha}_{\mathrm{uni}}(\Lambda')$-invariant if and only if it is $\widehat{\alpha}_{\mathrm{uni}}(\mathbb{Z}^d)$-invariant (hence in particular $\widehat{\alpha}_{\mathrm{uni}}(\Lambda_1)$-invariant).

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Suppose V is $\widehat{\alpha}_{\mathrm{diag}}(\Lambda')$-invariant but not $\widehat{\alpha}_{\mathrm{diag}}(\Lambda_1)$-invariant. Let $\mathbf{n} = M\mathbf{n}' \in \Lambda_1$ so that $\widehat{\alpha}_{\mathrm{diag}}(\mathbf{n})$ does not fix V. Since $\widehat{\alpha}_{\mathrm{diag}}(\Lambda')$ leaves V invariant, it follows that there is some $k \in \mathbb{N}$ so that $\widehat{\alpha}_{\mathrm{diag}}(k\mathbf{n})$ leaves V invariant.

By Proposition 2.2, for every i, the action induced by $\alpha(\mathbf{n}')$ on each V_i/V_{i-1} can be identified with multiplication by some $\xi \in \mathbb{K}_i$ on $\mathbb{K}_i$. If $\widehat{\alpha}_{\mathrm{diag}}(\mathbf{n}) = \widehat{\alpha}_{\mathrm{diag}}(\mathbf{n}')^M$ does not fix V but $\widehat{\alpha}_{\mathrm{diag}}(k\mathbf{n})$ does, this implies that there is a j and $\xi' \in \mathbb{K}_j$ as well as embeddings σ, σ' of $\mathbb{K}_i$ and $\mathbb{K}_j$ into $\mathbb{C}$ so that $\sigma(\xi)^M \neq \sigma'(\xi')^M$ but $\sigma(\xi)^{kM} = \sigma(\xi')^{kM}$. Then $\sigma(\xi)\sigma'(\xi')^{-1}$ is an element of the compositum of $\sigma(\mathbb{K}_i)$, $\sigma'(\mathbb{K}_j)$ that is a root of unity of order not dividing M—a contradiction. □

In particular, Lemma 4.3 shows that it is possible to restrict any adelic action to a finite index subgroup so that each of the $\mathbb{A}$-irreducible adelic actions associated to its restriction are in fact totally $\mathbb{A}$-irreducible.

4.3. *Reduction to Theorem 4.1.* Using the above preparations, we are now ready to explain the following reduction step.

Proof of Theorem 1.3 assuming Theorem 4.1. By Lemma 4.2 it suffices to consider adelic actions for the proof of Theorem 1.3. So let $d \geq 2$, let α be an adelic $\mathbb{Z}^d$-action without virtually cyclic factors, and let μ be an α-invariant and ergodic probability measure.

By Lemma 4.3 there exists a finite index subgroup $\Lambda < \mathbb{Z}^d$ so that the restriction of α to Λ satisfies the assumptions to Theorem 4.1. Note however, that the measure μ might not be ergodic with respect to α_Λ. Hence we may have to apply the ergodic decomposition. Since Λ has finite index in $\mathbb{Z}^d$, this ergodic decomposition simply takes the form

$$\mu = \frac{1}{J}(\mu_1 + \cdots + \mu_J),$$

where the probability measures μ_j are α_Λ-invariant and ergodic for $j = 1, \ldots, J$. Since μ is invariant under the full action α, we also have that for every $\mathbf{n} \in \mathbb{Z}^d$ and every index $j \in \{1, \ldots, J\}$ there exists an index k with $\alpha^{\mathbf{n}}_* \mu_j = \mu_k$. Since ergodic measure are either equal or singular to each other, we may also assume that the measure $\mu_1, \ldots, \mu_J$ are mutually singular to each other. By ergodicity of μ with respect to α, we also have that for every pair of indices j, k there exists some $\mathbf{n} \in \mathbb{Z}^d$ so that $\alpha^{\mathbf{n}}\mu_j = \mu_k$.

We now apply Theorem 4.1 to μ_1 and the restriction α_Λ. Therefore there exists a closed subgroup $G_1 < X_m$ so that μ_1 is invariant under translation by elements in G_1 and for any $\mathbf{n} \in \Lambda$, we have $h_{\mu_1}(\alpha^{\mathbf{n}}_{X_m/G_1}) = 0$. Applying the above transitivity claim we obtain the theorem. □

4.4. *Standing assumptions for the proof of Theorem 4.1.* Since we have shown that Theorem 4.1 implies Theorem 1.3, our aim for the the next sections is to show the former. Hence we will assume that α is an adelic $\mathbb{Z}^d$-action on X_m satisfying the assumptions of Theorem 4.1. Furthermore we assume that μ is an α-invariant and ergodic probability measure.

Suppose that μ is translation invariant under an adelic subgroup $Y < X_m$, then by invariance of μ under α, it is also translation invariant under $\alpha^{\mathbf{n}}(Y)$. Taking the closed

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subgroup generated by these, it follows that there exists a maximal adelic subgroup $Y < X_m$ so that μ is invariant under translation by elements of Y and, moreover, that Y is α-invariant. We may replace X_m by X_m/Y for this maximal α-invariant adelic subgroup Y and consider the push forward of μ under the canonical projection map $X_m \to X_m/Y$. If this new measure has zero entropy, Theorem 4.1 already holds for this action. Hence we may and will assume for the proof of Theorem 4.1 that:

(i) μ is not invariant under any adelic subgroup; and
(ii) there exists some $\mathbf{n} \in \mathbb{Z}^d$ so that $h_\mu(\alpha^{\mathbf{n}}) > 0$.

Our aim is to derive a contradiction from these assumptions.

To be able to apply the method introduced in [7] (relying on arithmetic properties of $\mathbb{A}$-irreducible actions), we start by applying Lemma 2.4 to the adelic action α on X_m. We now think of the chain of invariant subgroups as defining a chain of factor maps,

$$\begin{aligned} X = X_{(0)} = X_m \to X_{(1)} = X/Y_1 \to \cdots \\ \to X_{(j)} = X/Y_j \to X_{(j+1)} = X/Y_{j+1} \to \cdots \to X_{(r)} = \{0\}. \end{aligned}$$

Applying the Rokhlin entropy addition formula inductively to these factors, we have

$$h_\mu(\alpha^{\mathbf{n}}) = \sum_{j=0}^{r-1} h_\mu(\alpha^{\mathbf{n}}_{X_{(j)}} \mid X_{(j+1)})$$

for all $\mathbf{n} \in \mathbb{Z}^d$, where we write μ for the invariant measure on all of the factors, write $\alpha^{\mathbf{n}}_{X_{(j)}}$ for the induced action on the factor $X_{(j)}$, and write $h_\mu(\alpha^{\mathbf{n}}_{X_{(j)}} \mid X_{(j+1)})$ for the conditional entropy of $\alpha^{\mathbf{n}}_{X_{(j)}}$ conditioned on the next factor $X_{(j+1)}$.

By assumption (ii) above, there exists some $\mathbf{n} \in \mathbb{Z}^d$ so that $h_\mu(\alpha^{\mathbf{n}}) > 0$. Hence, we may choose the minimal $s \in \{0, \ldots, r-1\}$ so that $h_\mu(\alpha^{\mathbf{n}}_{X_{(s)}} \mid X_{(s+1)}) > 0$ for some $\mathbf{n} \in \mathbb{Z}^d$. We define $Y_{\text{base}} = Y_{s+1}$, $X_{\text{base}} = X_{(s+1)}$, and will always consider conditional entropy over the factor X_{base}. We also define $Y_{\text{pos}} = Y_s$ (contained in Y_{base}) and $X_{\text{pos}} = X/Y_{\text{pos}}$ (which factors on X_{base}). In this sense, the factor X_{pos} will be important for us as it is a 'positive entropy extension' of X_{base} and at the same time, an 'adelic extension with $\mathbb{A}$-irreducible fibers'. In fact by choice of s, there exists some $\mathbf{n} \in \mathbb{Z}^d$ so that $h_\mu(\alpha^{\mathbf{n}}_{X_{\text{pos}}} \mid X_{\text{base}}) > 0$, we have $X_{\text{pos}} = X/Y_{\text{pos}}$, $X_{\text{base}} = X/Y_{\text{base}}$, and that the action induced on the fibers $Y_{\text{irred}} = Y_{\text{base}}/Y_{\text{pos}}$ is (totally) $\mathbb{A}$-irreducible. Finally, as we have chosen s minimally, the original system X is a zero entropy extension of X_{pos} in the sense that $\mathsf{h}_\mu(\alpha^{\mathbf{n}} \mid X_{\text{pos}}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^d$.

To summarize the factor maps,

$$X \to X_{\text{pos}} \to X_{\text{base}} = X_{\text{pos}}/Y_{\text{irred}} \tag{4.1}$$

describe the original action as a zero-entropy extension of $X_{\text{pos}} = X/Y_{\text{pos}}$, and $X_{\text{base}} = X/Y_{\text{base}}$ as the quotient of X_{pos} by an α-invariant $\mathbb{A}$-irreducible adelic subgroup $Y_{\text{irred}} < X_{\text{pos}}$ so that X_{pos} is a positive entropy extension over X_{base}. More precisely,

$$h_\mu(\alpha^{\mathbf{n}} \mid X_{\text{pos}}) = 0 \quad \text{for all } \mathbf{n} \in \mathbb{Z}^d$$

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but

$$h_\mu(\alpha^{\mathbf{n}}_{X_{\text{pos}}} \mid X_{\text{base}}) > 0 \quad \text{for some } \mathbf{n} \in \mathbb{Z}^d.$$

To the exact sequence $0 \to Y_{\text{irred}} \to X_{\text{pos}} \to X_{\text{base}} \to 0$, there is attached an exact sequence of $\mathbb{Q}$-vector spaces $0 \to L_{\text{base}} \to L_{\text{pos}} \to L_{\text{irred}} \to 0$ where $L_{\text{base}} = \widehat{X_{\text{base}}}$, $L_{\text{pos}} = \widehat{X_{\text{pos}}}$, and $L_{\text{irred}} = \widehat{Y_{\text{irred}}}$ can be identified as $\mathbb{Q}$-vector space with a number field $\mathbb{K}$. Viewing X_{pos} as a quotient of $\mathbb{A}^{m_1}$ amounts to choosing a $\mathbb{Q}$-basis $v_1, \dots, v_{m_1}$ for the vector space L^*_{pos}.

Identifying $\mathbb{K}$ as a $\mathbb{Q}$-vector space with its dual using the trace form, we have an embedding $\mathbb{K} \to L^*_{\text{pos}}$, and we will always choose our $\mathbb{Q}$-basis so that $v_1, \dots, v_{[\mathbb{K}:\mathbb{Q}]}$ will be a basis for $\mathbb{K} < L^*_{\text{pos}}$.

5. *A bound on the entropy contribution*

In this section, we prove the following theorem.

THEOREM 5.1. (Cf. [7, Theorem 4.1]) *Let $m, d \geq 1$, and let α be a $\mathbb{Z}^d$-action on an adelic solenoid $X = \mathbb{A}^m/\mathbb{Q}^m$. We also let α denote the corresponding action on $\mathbb{A}^m$. Let $Y_{\text{irred}} \leq X$ be an α-invariant $\mathbb{A}$-irreducible adelic subspace, and set $X_{\text{base}} = X/Y_{\text{irred}}$. Let α_Ω be a $\mathbb{Z}^d$-action on a compact metric space Ω, $\widetilde{\alpha} = \alpha \times \alpha_\Omega$, and let $\widetilde{\mu}$ be a $\widetilde{\alpha}$-invariant measure on $\widetilde{X} = X \times \Omega$. Fix $\mathbf{n} \in \mathbb{Z}^d$, and let $V < U^-_{\mathbf{n}} < \mathbb{A}^m$ be a closed α-invariant subspace. Let $V_{\text{irred},\mathbb{A}} < \mathbb{A}^m$ be the rational subspace projecting modulo $\mathbb{Q}^m$ to Y_{irred}. Then*

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V \mid X_{\text{base}} \times \Omega) \leq \frac{h_\lambda(\alpha^{\mathbf{n}}_{\text{irred}}, V \cap V_{\text{irred},\mathbb{A}})}{h_\lambda(\alpha^{\mathbf{n}}_{\text{irred}})} \cdot h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}} \mid X_{\text{base}} \times \Omega). \tag{5.1}$$

Notice that this estimate is sharp for a product measure $\widetilde{\mu} = \lambda \times \nu$ with λ being the Haar measure on X. Our treatment here follows closely in content (if not in notation) that of [7, §4]. A special case of this theorem appeared in [27, Theorem 2.4].

Proof. We first note that

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V \cap V_{\text{irred},\mathbb{A}} \mid X_{\text{base}} \times \Omega) = h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V \mid X_{\text{base}} \times \Omega).$$

Indeed, if $\pi : X \to X_{\text{base}}$ is the natural projection, then for any $x \in X$, we have that

$$(x + V) \cap \pi^{-1} \circ \pi(x) = x + (V \cap V_{\text{irred},\mathbb{A}}),$$

hence if $\mathcal{C}_V$ is a decreasing V-subordinate σ-algebra of subsets of X, then $\mathcal{C}_V \vee \mathcal{B}_{\text{base}}$ is a decreasing σ-algebra subordinate to $V \cap V_{\text{irred},\mathbb{A}}$ (with $\mathcal{B}_{\text{base}}$ denoting the σ algebra of Borel subsets of X_{base}, or more precisely the image under π^{-1} of this σ-algebra in X). Therefore applying (3.8′) twice, once for the V-subordinate σ-algebra $\mathcal{C}_V$ and once for the $V \cap V_{\text{irred},\mathbb{A}}$-subordinate σ-algebra $\mathcal{C}_V \vee \mathcal{B}_{\text{base}}$, we get

$$\begin{aligned} h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V \mid X_{\text{base}} \times \Omega) &= H_{\widetilde{\mu}}(\mathcal{C}_V \mid \widetilde{\alpha}^{-\mathbf{n}}\mathcal{C}_V \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega) \\ &= H_{\widetilde{\mu}}(\mathcal{C}_V \vee \mathcal{B}_{\text{base}} \mid \widetilde{\alpha}^{-\mathbf{n}}\mathcal{C}_V \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega) \\ &= h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V \cap V_{\text{irred},\mathbb{A}} \mid X_{\text{base}} \times \Omega). \end{aligned}$$

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Since the right-hand side of (5.1) depends only on $V \cap V_{\text{irred},\mathbb{A}}$, we may (and will, for the remainder of the proof) assume that $V \leq U_{\mathbf{n}}^- \cap V_{\text{irred},\mathbb{A}}$. We may also assume V is a proper subgroup of $U_{\mathbf{n}}^- \cap V_{\text{irred},\mathbb{A}}$ since for $V = U_{\mathbf{n}}^- \cap V_{\text{irred},\mathbb{A}}$ by first applying the above discussion and then using the second part of Proposition 3.4 twice,

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, U_{\mathbf{n}}^- \cap V_{\text{irred},\mathbb{A}} \mid X_{\text{base}} \times \Omega) = h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, U_{\mathbf{n}}^- \mid X_{\text{base}} \times \Omega) = h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}} \mid X_{\text{base}} \times \Omega),$$
$$h_\lambda(\alpha_{\text{irred}}^{\mathbf{n}}, U_{\mathbf{n}}^- \cap V_{\text{irred},\mathbb{A}}) = h_\lambda(\alpha_{\text{irred}}^{\mathbf{n}}),$$

establishing (5.1) in this case.

Let the rank of $V_{\text{irred},\mathbb{A}}$ as a free $\mathbb{A}$-module be k. Since Y_{irred} is α-invariant and $\mathbb{A}$-irreducible, by Proposition 2.2, there is a global field $\mathbb{K}$ with $[\mathbb{K} : \mathbb{Q}] = k$, an injective homomorphism of $\mathbb{Q}$-vector spaces $\phi : \mathbb{K} \to \mathbb{Q}^m$, and d non-zero elements $\zeta_1, \ldots, \zeta_d \in \mathbb{K}^\times$ so that $\mathbb{A} \otimes \phi(\mathbb{K}) = V_{\text{irred},\mathbb{A}}$ and so that for any $\mathbf{n} = (n_1, \ldots, n_d)$ and $\xi \in \mathbb{A}_{\mathbb{K}} = \mathbb{A} \otimes \mathbb{K}$

$$\alpha^{\mathbf{n}}.\phi_{\mathbb{A}}(\xi) = \phi_{\mathbb{A}}(\zeta_{\mathbf{n}}\xi) \quad \zeta_{\mathbf{n}} := \zeta_1^{n_1} \ldots \zeta_d^{n_d}$$

with $\phi_{\mathbb{A}}$ the isomorphism of $\mathbb{A}$-modules $\mathbb{A}_{\mathbb{K}} \to V_{\text{irred},\mathbb{A}}$ induced from ϕ.

Fix $\mathbf{n} \in \mathbb{Z}^d$. Then $U_{\mathbf{n}}^-$ is S-linear for a finite set S of places of $\mathbb{Q}$ (including ∞) as in (3.5). Let $S_{\mathbb{K}}$ be the (finite) set of places of $\mathbb{K}$ lying over the places S of $\mathbb{Q}$. The $\mathbb{A}$-irreducibility of the action of α on $Y_{\text{irred}} \cong \mathbb{A}_{\mathbb{K}}/\mathbb{K}$ implies that every α-invariant subspace $V \leq V_{\text{irred},\mathbb{A}} \cap U_{\mathbf{n}}^-$ has the form

$$V = \phi_{\mathbb{A}}\Big(\prod_{\sigma \in S'_{\mathbb{K}}} \mathbb{K}_\sigma\Big), \tag{5.2}$$

with $S'_{\mathbb{K}} \subseteq S_{\mathbb{K}}$. Similarly,

$$V_{\text{irred},\mathbb{A}} \cap U_{\mathbf{n}}^- = \phi_{\mathbb{A}}\Big(\prod_{\sigma \in S_{\mathbb{K}}^-} \mathbb{K}_\sigma\Big),$$

with $S_{\mathbb{K}}^- \subseteq S_{\mathbb{K}}$ a finite set of places of $\mathbb{K}$ with $S'_{\mathbb{K}} \subset S_{\mathbb{K}}^-$. It follows from the relation between entropy contribution and leafwise measures in Proposition 3.4 that

$$h_\lambda(\alpha_{\text{irred}}^{\mathbf{n}}, V) = \sum_{\sigma \in S'_{\mathbb{K}}} \delta_\sigma \log 1/|\zeta_{\mathbf{n}}|_\sigma,$$
$$h_\lambda(\alpha_{\text{irred}}^{\mathbf{n}}) = h_\lambda(\alpha_{\text{irred}}^{\mathbf{n}}, U_{\mathbf{n}}^- \cap V_{\text{irred},\mathbb{A}}) = \sum_{\sigma \in S_{\mathbb{K}}^-} \delta_\sigma \log 1/|\zeta_{\mathbf{n}}|_\sigma;$$

note that by definition of $U_{\mathbf{n}}^-$ and $\zeta_{\mathbf{n}}$, we have that $|\zeta_{\mathbf{n}}|_\sigma < 1$ for every $\sigma \in S_{\mathbb{K}}^-$. Let

$$\kappa = \frac{\sum_{\sigma \in S'_{\mathbb{K}}} \delta_\sigma \log |\zeta_{\mathbf{n}}|_\sigma}{\sum_{\sigma \in S_{\mathbb{K}}^-} \delta_\sigma \log |\zeta_{\mathbf{n}}|_\sigma} = \frac{h_\lambda(\alpha_{\text{irred}}^{\mathbf{n}}, V)}{h_\lambda(\alpha_{\text{irred}}^{\mathbf{n}})} < 1.$$

Let $\mathcal{P}$ be a sufficiently fine finite partition with small boundaries of X, as in (3.6) and (3.7) (indeed, we will take it to be even finer), let $\mathcal{P}_V$ be a corresponding σ-algebra of subsets of $X \times \Omega$ as in p. 50, and let $\mathcal{C}_V = \bigcup_{i \geq 0} \widetilde{\alpha}^{-i\mathbf{n}} \mathcal{P}_V$ be as in §3.3. Then $\mathcal{C}_V$ and $\mathcal{P}_V$ are both

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subordinate to V, and $\mathcal{C}_V$ is in addition decreasing with respect to $\widetilde{\alpha}^{\mathbf{n}}$. Moreover,

$$\mathcal{C}_V = \widetilde{\alpha}^{-\mathbf{n}}\mathcal{C}_V \vee \mathcal{P} = \widetilde{\alpha}^{-\mathbf{n}}\mathcal{C}_V \vee \mathcal{P}_V. \tag{5.3}$$

Setting $\widetilde{T} = \widetilde{\alpha}^{\mathbf{n}}$, it follows from (5.3) that for any $j \in \mathbb{N}$,

$$\mathcal{C}_V = \mathcal{P}^{(0,j-1)} \vee \mathcal{C}_V^j,$$

where we recall that $\mathcal{C}_V^j = \widetilde{T}^{-j}\mathcal{C}_V$ and $\mathcal{P}^{(0,j-1)} = \bigvee_{0\le i\le j-1} \widetilde{T}^{-i}\mathcal{P}$. The key point in the proof of Theorem 5.1 is that *for $\ell = \lceil \kappa j \rceil$, the atoms of $\mathcal{P}^{(0,\ell)} \vee \mathcal{C}_V^j$ are already very close to being equal to the atoms of $\mathcal{C}_V = \mathcal{P}^{(0,j-1)} \vee \mathcal{C}_V^j$* (recall that $\kappa < 1$). In some simple cases (e.g. that considered in [27]), these σ-algebras literally coincide, though in general there may be a small disparity. What we now proceed to show (cf. [7, Lemma 4.2]) is that there is a set X_j with $\mu(X \setminus X_j) \le \exp(-cj)$ for appropriate $c > 0$ so that

$$[x]_{\mathcal{P}^{(0,\ell)}} \cap [x]_{\mathcal{C}_V^j} = [x]_{\mathcal{C}_V} \quad \text{for any } x \in X_j. \tag{5.4}$$

Since $\mathcal{C}_V^j$ is V-subordinate for V as in (5.2), there is some $B \subset \prod_{\sigma\in S'_{\mathbb{K}}} \mathbb{K}_\sigma$ (depending on x and j) so that

$$[x]_{\mathcal{C}_V^j} = x + \phi_{\mathbb{A}}(B);$$

moreover for any $\eta > 0$, by choosing $\mathcal{P}$ sufficiently fine depending on η, one can ensure that

$$B \subset \prod_{\sigma\in S'_{\mathbb{K}}} \{t \in \mathbb{K}_\sigma : |t|_\sigma \le \eta |\zeta_{\mathbf{n}}|_\sigma^{-j}\}.$$

Notice that $[x]_{\mathcal{P}^{(0,\ell)}\vee\mathcal{B}_{\text{base}}} \subset x + V_{\text{irred},\mathbb{A}}$, hence there is some $D \subset \mathbb{A}_{\mathbb{K}}$ so that

$$[x]_{\mathcal{P}^{(0,\ell)}\vee\mathcal{B}_{\text{base}}} = x + \phi_{\mathbb{A}}(D).$$

If the partition $\mathcal{P}$ was chosen to be sufficiently fine (again, depending on the parameter $\eta > 0$ introduced above), we may assume that

$$D \subset \prod_{\sigma\in S_{\mathbb{K}}} \{t \in \mathbb{K}_\sigma : |t|_\sigma \le \eta \min(1, |\zeta_{\mathbf{n}}|_\sigma^{-\ell})\} \times \prod_{\sigma\notin S_{\mathbb{K}}} \mathcal{O}_{\mathbb{K},\sigma}.$$

Note that $|\zeta_{\mathbf{n}}|_\sigma < 1$ for every $\sigma \in S'_{\mathbb{K}}$ but may be < 1, $= 1$, or > 1 for $\sigma \in S_{\mathbb{K}}$; however by choice of S (hence of $S_{\mathbb{K}}$), $|\zeta_{\mathbf{n}}|_\sigma = 1$ for $\sigma \notin S_{\mathbb{K}}$. Since $\mathbb{Q}^m \cap V_{\text{irred},\mathbb{A}} = \phi_{\mathbb{A}}(\mathbb{K})$, and using the fact that $\mathcal{B}_{\text{base}} \subseteq \mathcal{C}_V^j$ (modulo $\widetilde{\mu}$), we see that

$$\begin{aligned}[x]_{\mathcal{P}^{(0,\ell)}} \cap [x]_{\mathcal{C}_V^j} &= [x]_{\mathcal{P}^{(0,\ell)}\vee\mathcal{B}_{\text{base}}} \cap [x]_{\mathcal{C}_V^j} \\ &= x + \phi_{\mathbb{A}}\Big(\bigcup_{\xi\in\mathbb{K}} (B \cap (D + \xi))\Big).\end{aligned}$$

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However, if $B \cap (D + \xi) \neq \emptyset$, that is $\xi \in B - D$, then

$$|\xi|_\sigma \leq \begin{cases} 2\eta|\zeta_{\mathbf{n}}|_\sigma^{-j} & \text{if } \sigma \in S'_{\mathbb{K}}, \\ 2\eta & \text{if } \sigma \in S^-_{\mathbb{K}} \setminus S'_{\mathbb{K}}, \\ 2\eta|\zeta_{\mathbf{n}}|_\sigma^{-\ell} & \text{if } \sigma \in S_{\mathbb{K}} \setminus S^-_{\mathbb{K}}, \\ 1 & \text{otherwise.} \end{cases}$$

By Proposition 2.3, if moreover $\xi \in \mathbb{K}^\times$ and η was chosen small enough ($< 1/2$),

$$1 = \prod_{\text{places } \sigma \text{ of } \mathbb{K}} |\xi|_\sigma^{\delta(\sigma)} < \prod_{\sigma \in S'_{\mathbb{K}}} |\zeta_{\mathbf{n}}|_\sigma^{-\delta(\sigma)j} \times \prod_{\sigma \in S_{\mathbb{K}} \setminus S^-_{\mathbb{K}}} |\zeta_{\mathbf{n}}|_\sigma^{-\delta(\sigma)\ell}. \tag{5.5}$$

Applying Proposition 2.3 to $\zeta_{\mathbf{n}}$,

$$1 = \prod_{\text{places } \sigma \text{ of } \mathbb{K}} |\zeta_{\mathbf{n}}|_\sigma^{\delta(\sigma)} = \prod_{\sigma \in S^-_{\mathbb{K}}} |\zeta_{\mathbf{n}}|_\sigma^{\delta(\sigma)} \times \prod_{\sigma \in S_{\mathbb{K}} \setminus S^-_{\mathbb{K}}} |\zeta_{\mathbf{n}}|_\sigma^{\delta(\sigma)},$$

hence $\prod_{\sigma \in S^-_{\mathbb{K}}} |\zeta_{\mathbf{n}}|_\sigma^{\delta(\sigma)} = \prod_{\sigma \in S_{\mathbb{K}} \setminus S^-_{\mathbb{K}}} |\zeta_{\mathbf{n}}|_\sigma^{-\delta(\sigma)}$. Thus (5.5) implies

$$0 < j\Big(\sum_{\sigma \in S'_{\mathbb{K}}} \delta(\sigma) \log 1/|\zeta_{\mathbf{n}}|_\sigma \Big) - \ell \Big(\sum_{\sigma \in S^-_{\mathbb{K}}} \delta(\sigma) \log 1/|\zeta_{\mathbf{n}}|_\sigma \Big).$$

But this contradicts the definition of κ and $\ell \geq \kappa j$.

Thus we obtain the important conclusion that if $\ell = \lceil \kappa j \rceil$,

$$[x]_{\mathcal{P}^{(0,\ell)}} \cap [x]_{\mathcal{C}_V^j} = x + \phi_{\mathbb{A}}(B'),$$

where

$$B' = B \cap D \subset \prod_{\sigma \in S'_{\mathbb{K}}} \{t \in \mathbb{K}_\sigma : |t|_\sigma \leq \eta\}.$$

It follows that

$$[x]_{\mathcal{P}^{(0,\ell)}} \cap [x]_{\mathcal{C}_V^j} = [x]_{\mathcal{P}^{(0,j-1)}} \cap [x]_{\mathcal{C}_V^j} = [x]_{\mathcal{C}_V}$$

unless there is a $\ell < \ell' < j$ so that

$$\widetilde{T}^{\ell'}(x + \phi_{\mathbb{A}}(B')) \not\subset [x]_{\mathcal{P}}.$$

If $s = \max_{\sigma \in S'_{\mathbb{K}}} |\zeta_{\mathbf{n}}|_\sigma < 1$, then $\widetilde{T}^{\ell'}(x + \phi_{\mathbb{A}}(B')) = \widetilde{T}^{\ell'}(x) + \phi_{\mathbb{A}}(B'')$, where

$$B'' \subset \prod_{\sigma \in S'_{\mathbb{K}}} \{t \in \mathbb{K}_\sigma : |t|_\sigma \leq \eta s^{\ell'}\}.$$

By (3.6), the set of such x has $\widetilde{\mu}$ measure which decays exponentially in ℓ', hence in j, establishing (5.4).

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Once (5.4) has been established, establishing (5.1) is easy. Indeed, if $\mathcal{A} = \{X_j, X_j^{\complement}\}$ for X_j, as in (5.4), then for j large and $\ell = \lceil \kappa j \rceil$,

$$\begin{aligned} h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, V \mid X_{\text{base}} \times \Omega) &= j^{-1} H_{\widetilde{\mu}}(\mathcal{C}_V^{(0,j-1)} \mid \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega) \\ &= j^{-1} H_{\widetilde{\mu}}(\mathcal{P}^{(0,j-1)} \mid \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega) \\ &= j^{-1}(H_{\widetilde{\mu}}(\mathcal{P}^{(0,\ell)} \mid \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega) \\ &\quad + H_{\widetilde{\mu}}(\mathcal{P}^{(0,j-1)} \mid \mathcal{P}^{(0,\ell)} \vee \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega)) \\ &\le j^{-1}(H_{\widetilde{\mu}}(\mathcal{P}^{(0,\ell)} \mid X_{\text{base}} \times \Omega) + H_{\widetilde{\mu}}(\mathcal{A}) \\ &\quad + H_{\widetilde{\mu}}(\mathcal{P}^{(0,j-1)} \mid \mathcal{P}^{(0,\ell)} \vee \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega \vee \mathcal{A})). \end{aligned} \tag{5.6}$$

By definition,

$$\begin{aligned} &H_{\widetilde{\mu}}(\mathcal{P}^{(0,j-1)} \mid \mathcal{P}^{(0,\ell)} \vee \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega \vee \mathcal{A}) \\ &\quad = \widetilde{\mu}(X_j) H_{\widetilde{\mu}|X_j}(\mathcal{P}^{(0,j-1)} \mid \mathcal{P}^{(0,\ell)} \vee \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega) \\ &\qquad + \widetilde{\mu}(X_j^{\complement}) H_{\widetilde{\mu}|X_j^{\complement}}(\mathcal{P}^{(0,j-1)} \mid \mathcal{P}^{(0,\ell)} \vee \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega). \end{aligned}$$

On X_j, the atom $[x]_{\mathcal{P}^{(0,j-1)}} \cap [x]_{\mathcal{C}_V^j} = [x]_{\mathcal{P}^{(0,\ell)}} \cap [x]_{\mathcal{C}_V^j}$ by (5.4), hence

$$H_{\widetilde{\mu}|X_j}(\mathcal{P}^{(0,j-1)} \mid \mathcal{P}^{(0,\ell)} \vee \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega) = 0.$$

On $X_j^{\complement}$, we use the trivial bound

$$H_{\widetilde{\mu}|X_j^{\complement}}(\mathcal{P}^{(0,j-1)} \mid \mathcal{P}^{(0,\ell)} \vee \mathcal{C}_V^j \vee \mathcal{B}_{\text{base}} \vee \mathcal{B}_\Omega) \le j \log(\#\mathcal{P}).$$

Plugging these back in (5.6) and using $\widetilde{\mu}(X_j^{\complement}) \to 0$ as $j \to \infty$, we see that

$$(5.6) \to \kappa h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}} \mid X_{\text{base}} \times \Omega) \quad \text{as } j \to \infty,$$

establishing (5.1). □

6. *Coarse Lyapunov subgroups and the product structure*

A crucial property of the leafwise measures for our argument is their product structure for the coarse Lyapunov subgroups as obtained by the first named author and Anatole Katok [**5**] (see also [**4**, **28**]). For an introduction of the product structure, we also recommend [**10**, §8]. However, all of these papers assumed that the $\mathbb{Z}^d$-action under consideration is semisimple. In our case we do not make this assumption, so our action is given by $\alpha = \alpha_{\text{diag}}\alpha_{\text{uni}}$, with both α_{diag} and α_{uni} defined over $\mathbb{Q}$, that is, these can be thought of as homomorphisms from $\mathbb{Z}^d$ to $\mathrm{GL}_m(\mathbb{Q})$; cf. p. 17. The purpose of this section is to recall the relevant notions and overcome the problems arising from the lack of semi-simplicity in the cases of interest.

6.1. *Lyapunov subgroups over $\mathbb{Q}_\sigma$.*

Fix σ a place of $\mathbb{Q}$, that is, either $\sigma = \infty$ or $\sigma = p$ a prime. We consider the linear maps $\alpha^{\mathbf{n}}$ for $\mathbf{n} \in \mathbb{Z}^d$ on $\mathbb{Q}_\sigma^m$ and are mostly interested in

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the asymptotic behavior of its elements with respect to the norm defined by

$$\|v\|_\sigma = \max_{j=1,\dots,m} |v_j|_\sigma$$

for all $v \in \mathbb{Q}_\sigma^m$. We will also consider this behavior restricted to an α-invariant $\mathbb{Q}_\sigma$-linear subspace V (even when not explicitly stated, V will always be assumed to be α-invariant). For this, we will initially ignore α_{uni} (with polynomial behavior) and focus on α_{diag} (with exponential behavior).

Since α_{diag} is semisimple, $\mathbb{Q}_\sigma^m$ (as well as any α-invariant subspace V) is a direct sum of $\mathbb{Q}_\sigma$-irreducible linear subspaces. On each of these irreducible subspaces, the action of α_{diag} is isomorphic to the action defined by multiplication by d elements $\zeta_1, \dots, \zeta_d \in \mathbb{K}$ on a local field $\mathbb{K}$ (similar to the discussion in the proof of Proposition 2.2). Recall that we extend the the norm on $\mathbb{Q}_\sigma$ to $\mathbb{K}$; this extended norm is denoted by $|\cdot|_\sigma$. We refer to the linear functional $\chi : \mathbb{Z}^d \to \mathbb{R}$ given by

$$\chi : (n_1, \dots, n_d) \mapsto \sum_i n_i \log |\zeta_i|_\sigma \in \mathbb{R}$$

as the *Lyapunov weight* associated with the invariant subspace $\mathbb{K}$, and denote the pairing of a functional χ and a vector $\mathbf{n} \in \mathbb{Z}^d$ (or $\mathbb{R}^d$) by $\chi \cdot \mathbf{n}$. We note that for a vector v in this subspace, we have

$$\|\alpha_{\text{diag}}^{\mathbf{n}} v\|_\sigma \asymp e^{\chi \cdot \mathbf{n}} \|v\|_\sigma \tag{6.1}$$

for all $\mathbf{n} \in \mathbb{Z}^d$, where we write $\asymp$ to indicate that we can bound each of the two terms by a multiple of the other. Here the implicit constants only depend on the action and not on the vector v or on $\mathbf{n}$. We will call $\mathbb{K}$ an $\mathbb{Q}_\sigma$-*irreducible eigenspace* (for α_{diag}) with Lyapunov weight χ.

It follows that $\mathbb{Q}_\sigma^m$ (respectively V) is isomorphic to a finite direct product of local fields $\mathbb{K}$ extending $\mathbb{Q}_\sigma$ so that the linear maps $\alpha_{\text{diag}}^{\mathbf{e}_1}, \dots, \alpha_{\text{diag}}^{\mathbf{e}_d}$ are written in diagonal form using this isomorphism. Moreover, we obtain in this way finitely many Lyapunov weights arising from the action on $\mathbb{Q}_\sigma^m$. Note however that these Lyapunov weights are all functionals into $\mathbb{R}$ (independent of the place σ).

6.2. *Coarse Lyapunov weights and subgroups.* We now apply the above for each place σ of $\mathbb{Q}$. Let S be a finite set of places containing ∞ so that $\alpha_{\text{diag}}^{\mathbf{e}_1}, \dots, \alpha_{\text{diag}}^{\mathbf{e}_d}$ all belong to $\mathrm{GL}_m(\mathbb{Z}_\sigma)$ for $\sigma \notin S$. For $\sigma \notin S$, the only Lyapunov weight for the action of α_{diag} on $\mathbb{Q}_\sigma$ is the zero weight. Hence by varying σ over all places of $\mathbb{Q}$, we only obtain finitely many non-zero Lyapunov weights.

We say that two non-zero Lyapunov weights χ, χ' (possibly arising from different places of $\mathbb{Q}$) are equivalent if there exists some $t > 0$ so that $\chi' = t\chi$. We will denote the equivalence class of a non-zero Laypunov weight χ by $[\chi]$ and will refer to $[\chi]$ as a coarse Lyapunov weight.

For a coarse Lyapunov weight $[\chi]$, we define the *coarse Lyapunov subgroup* $W^{[\chi]}$ to be the S-linear subspace defined as the subgroup generated by all irreducible eigenspaces with Lyapunov weight χ' equivalent to χ. Alternatively, we can use (6.1) to see that the coarse Lyapunov subgroup could also be defined as an intersection of stable horospherical

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subgroups for α_{diag}, namely

$$W^{[\chi]} = \bigcap_{\mathbf{n}\in\mathbb{Z}^d:\chi\cdot\mathbf{n}<0} U^-_{\alpha^{\mathbf{n}}_{\mathrm{diag}}};$$

this relation to the horospherical subgroups is the reason for the dynamical importance of the coarse Lyapunov subgroups.

Because of the polynomial nature of α_{uni}, we have that $U^-_{\alpha^{\mathbf{n}}_{\mathrm{diag}}} = U^-_{\alpha^{\mathbf{n}}}$ for all $\mathbf{n}\in\mathbb{Z}^d$ and hence $W^{[\chi]}$ can be defined directly in terms of the action α by

$$W^{[\chi]} = \bigcap_{\mathbf{n}\in\mathbb{Z}:\chi\cdot\mathbf{n}<0} U^-_{\alpha^{\mathbf{n}}}.$$

In particular, $W^{[\chi]}$ is invariant under $\alpha^{\mathbf{n}}$ for all $\mathbf{n}\in\mathbb{Z}^d$.

Conversely, any stable horospherical group is a product of coarse Lyapunov subgroups: indeed, for any $\mathbf{n}\in\mathbb{Z}^d$, we have

$$U^-_{\alpha^{\mathbf{n}}} = U^-_{\alpha^{\mathbf{n}}_{\mathrm{diag}}} = \bigoplus_{[\chi]:\chi\cdot\mathbf{n}<0} W^{[\chi]}, \tag{6.2}$$

where the direct sum runs over all the coarse Lyapunov subspaces $W^{[\chi]}$ satisfying that $\chi\cdot\mathbf{n}<0$.

Given an α-invariant S-linear subgroup $V < U^-_{\alpha^{\mathbf{n}}}$, we define $V^{[\chi]} = V\cap W^{[\chi]}$ for any non-zero coarse Lyapunov weight $[\chi]$. These also satisfy that V is the direct sum of $V^{[\chi]}$ for all $[\chi]$ with $\chi\cdot\mathbf{n}<0$.

6.3. *Product structure of leafwise measures.* Recall that we assume that μ is as in Theorem 4.1. Let Ω be an arbitrary compact metric space as in §3.1 equipped with an action of $\mathbb{Z}^d$, let $\widetilde{X} = X\times\Omega$, and consider an invariant probability measure $\widetilde{\mu}$ on $\widetilde{X}$ projecting to μ.

THEOREM 6.1. (Entropy and product structure) *Let $X\to X_{\mathrm{pos}}\to X_{\mathrm{base}}$ with $X_{\mathrm{pos}} = X/Y_{\mathrm{pos}}$ and $X_{\mathrm{base}} = X/Y_{\mathrm{base}}$ be as in (4.1) and let $V_{\mathrm{base},\mathbb{A}} < \mathbb{A}^m$ be the rational α-invariant subspace so that Y_{base} is the image of $V_{\mathrm{base},\mathbb{A}}$ modulo $\mathbb{Q}^m$. Let $\mathbf{n}_0\in\mathbb{Z}^d$. Then the leafwise measure on $\widetilde{X}$ for*

$$V^-_{\mathbf{n}_0} = V_{\mathrm{base},\mathbb{A}}\cap U^-_{\alpha^{\mathbf{n}_0}}$$

is, up to proportionality, the product of the leafwise measures for its coarse Lyapunov subgroups $V^{[\chi]} = V^-_{\mathbf{n}_0}\cap W^{[\chi]}$, that is,

$$\widetilde{\mu}_x^{V^-_{\mathbf{n}_0}} \propto \prod_{[\chi]:(\chi\cdot\mathbf{n}_0)<0} \widetilde{\mu}_x^{V^{[\chi]}}$$

for a.e. $x\in\widetilde{X}$. In particular, the relative entropy of $\alpha^{\mathbf{n}_0}$ conditional on the factor $X_{\mathrm{base}}\times\Omega$ is equal to the sum of the entropy contributions of these coarse Lyapunov subgroups, that is,

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}_0}\mid X_{\mathrm{base}}\times\Omega) = h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}_0}, V^-_{\mathbf{n}_0}) = \sum_{[\chi]:\chi\cdot\mathbf{n}_0<0} h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}_0}, V^{[\chi]}). \tag{6.3}$$

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We note that our proof relies on the assumption that X is a zero entropy extension of X_{pos}, and does not extend the product structure to other cases with Jordan blocks. In fact our setup is used to show that Jordan blocks of $\alpha^{\mathbf{n}}$ cannot appear within the subspace $V^{[\chi]} \cap \mathbb{R}^m$ and for ' $\mathbf{n}$ in the kernel of χ' (in the sense described below). Moreover, in the non-Archimedean parts of $V^-_{\mathbf{n}_0}$, we use a different argument.

For any α-invariant S-linear subgroup V, we let $P < V$ be the minimal S-linear α-invariant subspace so that

$$\widetilde{\mu}_x^V(V \setminus P) = 0$$

for a.e. $x \in \widetilde{X}$. We will refer to P as the *supporting subgroup* of V. We note that $\mu_x^V = \mu_x^P$ a.s. and if $W < P$ is another S-linear α-invariant subspace with $W \neq P$, then $\mu_x^V(W) = 0$ a.s. This follows from [11, Lemma 5.2] (as we restrict here to α-invariant subspaces the assumption of class $\mathcal{A}'$ can easily be avoided in the proof of that lemma).

We say that $[\chi]$ is an *exposed coarse Lyapunov weight* for an α-invariant S-linear subgroup $V < U^-_{\alpha^{\mathbf{n}}}$ if $V^{[\chi]}$ is non-trivial, $V = V^{[\chi]} + V'$ for a sum V' of coarse Lyapunov subgroups, and there exists some $\mathbf{n}' \in \mathbb{R}^d$ with $\chi \cdot \mathbf{n}' = 0$ and $\chi' \cdot \mathbf{n}' < 0$ for all Lyapunov weights χ' of V'.

We also recall that $\alpha^{\mathbf{n}}_{\text{uni}}$ is a polynomial map from $\mathbb{Z}^d \to \text{GL}_m(\mathbb{Q})$ and so extends to a homomorphism from $\mathbb{R}^d$ to $\text{GL}_m(\mathbb{R})$ that will also be denoted by α_{uni}.

6.4. *No shearing and proving the product structure.* The following lemma stands in stark contrast to the non-Archimedean case, where α_{uni} takes values in a compact group.

LEMMA 6.2. (Existence of logarithmic sequence for real Jordan blocks) *Let α be a representation of $\mathbb{Z}^d$ on a real vector space P. Suppose that χ is a non-zero Lyapunov weight for α and that all other Lyapunov weights are equivalent to χ. Suppose that $\alpha^{\mathbf{m}}_{\text{uni}}$ is non-trivial for some $\mathbf{m} \in \mathbb{R}^d$ with $\chi \cdot \mathbf{m} = 0$. Then there exists a sequence $\mathbf{n}_k \in \mathbb{Z}^d$ so that $\alpha^{\mathbf{n}_k} \in \text{End}(P)$ converges to a non-zero non-invertible linear map $L \in \text{End}(P)$.*

We will refer to the sequence $\mathbf{n}_k$ as a logarithmic sequence for P since it can be defined using the logarithm map in easy special cases.

Proof of Lemma 6.2. Let $\alpha_{|\cdot|}$ be the representation of $\mathbb{Z}^d$ on P, which has every eigenvector of α_{diag} with eigenvalue λ also as eigenvector but with eigenvalue $|\lambda|$. Note that $\alpha_{|\cdot|}$ extends continuously to all $\mathbf{n} \in \mathbb{R}^d$. Moreover for $\mathbf{n} \in \mathbb{Z}^d$, the map $\alpha^{\mathbf{n}}_{\text{diag}}\alpha^{-\mathbf{n}}_{|\cdot|}$ belongs to a fixed compact subgroup of $\text{GL}(P)$. Using nearest integer vectors, we see that it suffices to construct a sequence $\mathbf{n}_k \in \mathbb{R}^d$ so that $\alpha^{\mathbf{n}_k}_{\text{uni}}\alpha^{\mathbf{n}_k}_{|\cdot|}$ converges to a non-zero non-invertible linear map L.

For this, we apply our assumption and pick a direction $\mathbf{m} \in \mathbb{R}^d$ with $\chi \cdot \mathbf{m} = 0$ so that $\alpha^{\mathbf{m}}_{\text{uni}}$ is non-trivial. Also let $\mathbf{n}_- \in \mathbb{R}^d$ with $\chi \cdot \mathbf{n}_- < 0$. As a non-zero real polynomial, $\alpha^{k\mathbf{m}}_{\text{uni}}$ diverges as $k \to \infty$ and $\alpha^{t\mathbf{n}_-}_{|\cdot|} \to 0$ as $t \to \infty$. Also note that $\alpha^{t\mathbf{n}_-}_{|\cdot|} \to 0$ converges exponentially fast, while $\alpha^{t\mathbf{n}_-}_{\text{uni}}$ can only diverge polynomially fast as $t \to \infty$.

This shows that for each sufficiently large $k \in \mathbb{N}$, we can define $\mathbf{n}_k = k\mathbf{m} + t_k\mathbf{n}_-$, where $t_k > 0$ is chosen (using the intermediate value theorem) minimally so that the

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Hilbert–Schmidt norms satisfy

$$\|\alpha_{\mathrm{uni}}^{\mathbf{n}_k}\alpha_{|\cdot|}^{\mathbf{n}_k}\| = \|\alpha_{\mathrm{uni}}^{k\mathbf{m}+t_k\mathbf{n}_-}\alpha_{|\cdot|}^{t_k\mathbf{n}_-}\| = 1.$$

Since $\alpha_{\mathrm{uni}}^{k\mathbf{m}}$ diverges as $k \to \infty$, we see that also $t_k \to \infty$. If now $v \in P$ is a common eigenvector for α, then $\alpha_{\mathrm{uni}}^{k\mathbf{m}}v = v$, and so

$$\alpha_{\mathrm{uni}}^{\mathbf{n}_k}\alpha_{|\cdot|}^{\mathbf{n}_k}v = \alpha_{\mathrm{uni}}^{t_k\mathbf{n}_-}\alpha_{|\cdot|}^{t_k\mathbf{n}_-}v \to 0$$

as $k \to \infty$ (since $\alpha_{\mathrm{uni}}^{t_k\mathbf{n}_-}$ is polynomial and $\alpha_{|\cdot|}^{t_k\mathbf{n}_-}$ contracts P at exponential rate).

From this and the compactness of the unit ball in finite dimensions, it follows that there exists a converging subsequence of $\alpha_{\mathrm{uni}}^{\mathbf{n}_k}\alpha_{|\cdot|}^{\mathbf{n}_k}$ whose limit is non-zero and non-invertible. As indicated at the beginning of the proof, a subsequence of the integer vectors closest to $\mathbf{n}_k \in \mathbb{R}^d$ will satisfy the conclusions of the lemma. □

We need the following upgrade to the above concerning exposed coarse Lyapunov weights.

LEMMA 6.3. (Properties of logarithmic sequence for adelic action) *Let α be a linear representation of $\mathbb{Z}^d$ on $\mathbb{Q}^m$ defining an adelic action on X_m. Let $\mathbf{n}_- \in \mathbb{Z}^d \setminus \{0\}$ and $V < U^-_{\alpha^{\mathbf{n}_-}}$ be an S-linear subspace. Let $[\chi]$ be an exposed coarse Lyapunov weight $[\chi]$ of V. Suppose that there exists some $\mathbf{m} \in \mathbb{R}^d$ with $\chi \cdot \mathbf{m} = 0$ so that $(\alpha_{\mathrm{uni}}|_{V\cap\mathbb{R}^m})^{\mathbf{m}}$ is non-trivial. Then there exists a sequence $\mathbf{n}_k \in \mathbb{Z}^d$ so that $(\alpha|_V)^{\mathbf{n}_k}$ converges (uniformly within compact subsets) to a non-zero map $L \in \mathrm{End}(V)$ that vanishes on all non-Archimedean subspaces, vanishes on all coarse Lyapunov subgroups $V^{[\chi']}$ with $[\chi'] \neq [\chi]$, and whose restriction to $V^{[\chi]} \cap \mathbb{R}^m$ is non-invertible.*

Proof. This actually follows by the same argument as Lemma 6.2 after choosing $\mathbf{m} \in \mathbb{R}^d$ correctly. Indeed, we first note that the kernel of the homomorphism $\mathbf{n} \in \chi^\perp \mapsto \alpha_{\mathrm{uni}}^{\mathbf{n}} \in \mathrm{GL}(V \cap \mathbb{R}^m)$ is a proper subspace $K < \chi^\perp$, and hence our first constraint on $\mathbf{m}$ is simply $\mathbf{m} \in \chi^\perp \setminus K$.

By definition, $[\chi]$ is an exposed coarse Lyapunov weight for V if there exists some $\mathbf{m} \in \chi^\perp$ with $\chi' \cdot \mathbf{m} < 0$ for all Lyapunov weights χ' of V inequivalent to χ. The latter condition is clearly satisfied by all elements of an open subset of $\chi^\perp$. Hence we can find $\mathbf{m} \in \chi^\perp \setminus K$ with $\mathbf{m} \cdot \chi' < 0$ for all Lyapunov weights χ' of V inequivalent to χ.

Using this $\mathbf{m}$ together with $\mathbf{n}_-$, as in the assumptions of the lemma, we now go again through the construction in the proof of Lemma 6.2. Let $\mathbf{n}'_k \in \mathbb{Z}^d$ be a nearest integer approximation of $k\mathbf{m}$ and let $\mathbf{n}_k$ be the nearest integer approximation of $k\mathbf{m} + t_k\mathbf{n}_-$. For all non-Archimedean subspaces V_σ, we note that $\alpha_{\mathrm{uni}}(\mathbb{Z}^d)|_{V_\sigma}$ has compact closure. Hence the restriction $\alpha^{\mathbf{n}'_k}|_{V_\sigma}$ belongs to a compact subset of $\mathrm{Hom}(V_\sigma)$, which implies together with $\mathbf{n}_-$ contracting V and $t_k \to \infty$ that the restriction $\alpha^{\mathbf{n}_k}|_{V_\sigma}$ converges to zero. Similarly, our choice of $\mathbf{m}$ implies that for a coarse Lyapunov weight $[\chi'] \neq [\chi]$, both $\alpha^{\mathbf{n}'_k}|_{V^{[\chi']}}$ and $\alpha^{\mathbf{n}_k}|_{V^{[\chi']}}$ converge to the trivial map. Finally our proof of Lemma 6.2 ensures the claimed properties of the limit map $L|_{V^{[\chi]}_\infty}$. □

We will now combine the above with the setup of §4.4 to prove a restriction concerning the supporting subgroups (cf. §6.3).

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PROPOSITION 6.4. (No shearing on supporting subgroup) *Let P be the supporting subgroup of $V^-_{\mathbf{n}_0} = V_{\mathrm{base},\mathbb{A}} \cap U^-_{\alpha^{\mathbf{n}_0}}$ (with $V_{\mathrm{base},\mathbb{A}}$ as defined in Theorem 6.1). Let χ be an exposed Lyapunov weight of P. Then $(\alpha_{\mathrm{uni}}|_{P^{[\chi]}\cap\mathbb{R}^m})^{\mathbf{n}}$ is trivial for all $\mathbf{n} \in \mathbb{R}^d$ with $\chi \cdot \mathbf{n} = 0$.*

Here we identified $\mathbb{R}^m$ with the corresponding subgroup of $\mathbb{A}^m$, and hence $P^{[\chi]} \cap \mathbb{R}^m$ is the maximal real subspace of the supporting subgroup P.

Proof. We suppose in contradiction that $(\alpha_{\mathrm{uni}}|_{P^{[\chi]}\cap\mathbb{R}^m})^{\mathbf{m}}$ is non-trivial for some $\mathbf{m} \in \mathbb{R}^d$ with $\chi \cdot \mathbf{m} = 0$. Applying Lemma 6.3, we find a logarithmic sequence $\mathbf{n}_k$ and the limit $L \in \mathrm{End}(P)$ of $(\alpha|_P)^{\mathbf{n}_k}$.

Recall that $Y_{\mathrm{base}} < X$ is the adelic subgroup so that $X_{\mathrm{base}} = X/Y_{\mathrm{base}}$, and that $V_{\mathrm{base},\mathbb{A}}$ is the rational subspace of $\mathbb{A}^m$ so that Y_{base} is the image of $V_{\mathrm{base},\mathbb{A}}$ modulo $\mathbb{Q}^m$. Recall also that $Y_{\mathrm{pos}} < Y_{\mathrm{base}}$ is the adelic subgroup so that $X_{\mathrm{pos}} = X/Y_{\mathrm{pos}}$, and let $V_{\mathrm{pos},\mathbb{A}} < V_{\mathrm{base},\mathbb{A}}$ be the rational subspace so that Y_{pos} is the image of $V_{\mathrm{pos},\mathbb{A}}$ modulo $\mathbb{Q}^m$.

By construction we have that the action on $Y_{\mathrm{base}}/Y_{\mathrm{pos}}$ is $\mathbb{A}$-irreducible, or equivalently that the linear representation of $\mathbb{Z}^d$ on $V_{\mathrm{base},\mathbb{A}}/V_{\mathrm{pos},\mathbb{A}}$ (defined over $\mathbb{Q}$) is irreducible over $\mathbb{Q}$. In particular, this representation is semisimple. Unfolding the definitions and restricting to $P^{[\chi]} \cap \mathbb{R}^m$, it follows that for any $v \in P^{[\chi]} \cap \mathbb{R}^m$,

$$\alpha^{\mathbf{m}}_{\mathrm{uni}}(v) \in v + V^{[\chi]},$$

where $V^{[\chi]} = W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}}$. Combining this information with the construction of the logarithmic sequence and its limit L, it follows that

$$L(P^{[\chi]} \cap \mathbb{R}^m) \subseteq V^{[\chi]}.$$

Next recall from (4.1) that X is a zero entropy extension over $X_{\mathrm{pos}} = X/Y_{\mathrm{pos}}$, that is, $h_{\widetilde{\mu}}(\alpha^{\mathbf{n}} \mid X_{\mathrm{pos}} \times \Omega) \le h_{\mu}(\alpha^n \mid X_{\mathrm{pos}}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^d$. This implies by the relationship between the leafwise measures and entropy that the leafwise measures $\widetilde{\mu}^{V^{[\chi]}}_x$ must be trivial almost everywhere—indeed otherwise there would be a positive entropy contribution for the relative entropy over the factor $X_{\mathrm{pos}} \times \Omega$. By the compatibility property (3.1), it follows that there exists a set $X' \subseteq \widetilde{X} = X \times \Omega$ of full measure so that $x,\ x + w \in X'$ for some $w \in V^{[\chi]}$ implies $w = 0$. Using regularity of the Borel probability measure, we choose some compact $K \subseteq X'$ of measure $\widetilde{\mu}(K) > 0.99$.

Our aim in the proof is to use the logarithmic sequence $\mathbf{n}_k$ and the limit L to derive a contradiction to the properties of K. For this, let $\mathcal{A}$ be a σ-algebra that is subordinate to P. We define

$$X_k = \{x \in \alpha^{-\mathbf{n}_k} K \mid \widetilde{\mu}^{\mathcal{A}}_x(\alpha^{-\mathbf{n}_k} K) > 0.9\}$$

and note that $\widetilde{\mu}(X_k) > 0.89$. By the Lemma of Fatou (applied for the probability measure $\widetilde{\mu}$), it follows that

$$\widetilde{\mu}(\limsup_{k\to\infty} X_k) > 0.89.$$

Hence there exists some x_0 and some subsequence $\mathbf{n}'_k$ of $\mathbf{n}_k$ so that $\alpha^{\mathbf{n}'_k} x_0 \in K$ and $\widetilde{\mu}^{\mathcal{A}}_{x_0}(\alpha^{-\mathbf{n}'_k} K) > 0.9$. Applying Lemma of Fatou again—but this time for the probability

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measure $\widetilde{\mu}^{\mathcal{A}}_{x_0}$—we obtain

$$\widetilde{\mu}^{\mathcal{A}}_{x_0}(\limsup_{k\to\infty}\alpha^{-\mathbf{n}'_k}K) > 0.9. \tag{6.4}$$

Also recall that P is the supporting subgroup of $V^-_{\mathbf{n}_0}$ and that $\ker L < P$ is a proper S-linear α-invariant subgroup. Using [11, Lemma 5.2] (cf. p. 112 above) this implies that $\widetilde{\mu}^P_x(\ker L) = 0$, which in turns implies $\widetilde{\mu}^{\mathcal{A}}_x(x + \ker L) = 0$ a.s. We may assume that our x_0 constructed above has this property. Hence there exists some $x_1 \in [x_0]_{\mathcal{A}} \setminus (x_0 + \ker L)$ and another subsequence $\mathbf{n}''_k$ of $\mathbf{n}'_k$ so that $\alpha^{\mathbf{n}''_k}x_1 \in K$.

To summarize, we have found a subsequence $\mathbf{n}''_k$ of the logarithmic sequence $\mathbf{n}_k$, some $x_0 \in X$, and some $x_1 = x_0 + v \in [x]_{\mathcal{A}}$ with $v \in P \setminus \ker L$ so that $\alpha^{\mathbf{n}''_k}x_j \in K$ for $j = 0, 1$. Using compactness of K, we can find yet another subsequence of $\mathbf{n}''_k$ so that $\alpha^{\mathbf{n}''_k}x_0 \to y_0$ and $\alpha^{\mathbf{n}''_k}x_1 \to y_1$ with $y_0, y_1 \in K$. Moreover, since $\alpha^{\mathbf{n}_k}v \to Lv$ as $k \to \infty$, we also have $y_1 = y_0 + Lv$ with $Lv \in V^{[\chi]} \setminus \{0\}$ by the properties of L. However, this contradicts the properties of $K \subset X'$ and so concludes the proof. □

Proof of Theorem 6.1. In view of Proposition 6.4, the product structure of the leafwise measure follows from [4, Theorem 8.2] (or more precisely its proof).

For this, we first recall that if $P < V^-_{\mathbf{n}_0}$ is the corresponding supporting subgroup, then by Proposition 3.2, the leafwise measure $\mu_x^{V^-_{\mathbf{n}_0}}$ coincides with μ^P_x for μ-a.e. x. Next we recall that by Proposition 6.4, we have for any non-trivial coarse Lyapunov weight $[\chi]$ that $\alpha^{\mathbf{m}}_{\text{uni}}|_{P^{[\chi]}\cap\mathbb{R}^m}$ is trivial for any $\mathbf{m} \in \ker\chi$. We also note that $\alpha_{\text{uni}}(\mathbb{Z}^d)$ restricted to the p-adic subspaces $P^{[\chi]} \cap \mathbb{Q}^m_p$ of the coarse Lyapunov subspace $P^{[\chi]}$ has compact closure in $\mathrm{GL}(P^{[\chi]} \cap \mathbb{Q}^m_p)$. Hence we may assume that the metric on $P^{[\chi]} \cap \mathbb{Q}^m_p$ is invariant under $\alpha_{\text{uni}}(\mathbb{Z}^d)$. With these two observations, the inductive argument for [4, Theorem 8.2] applies and proves the product structure for $\widetilde{\mu}_x^{V^-_{\mathbf{n}_0}} = \widetilde{\mu}^P_x$.

The product structure implies now quite directly using (3.9) that the entropy contribution for $V^-_{\mathbf{n}_0}$ equals the sum of the entropy contributions of its coarse Lyapunov subgroups $V^{[\chi]}$, hence the second equality in (6.3).

The first equality in (6.3), that is, the fact that $h(\widetilde{\alpha}^{\mathbf{n}_0}, V^-_{\mathbf{n}_0})$ equals the conditional entropy of $\widetilde{\alpha}^{\mathbf{n}_0}$ over the factor $X_{\text{base}} \times \Omega$ follows e.g. from the proof of [10, Theorem 7.6]. Indeed by conditioning the calculation there on the factor $X_{\text{base}} \times \Omega$, the leafwise measure on the full stable horospherical is automatically supported on $V^-_{\mathbf{n}_0}$, since a displacement by some element of the stable horospherical not belonging to $V_{\text{base},\mathbb{A}}$ would change the point within X_{base}. □

7. *Proof of Theorem 4.1*

7.1. *Rigidity of the entropy function.* In the following, we will again consider entropy contributions for various coarse Lyapunov subgroups with varying definitions of the second factor Ω in the framework of §3. Consistent with our notation so far, we will use e.g. $h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega)$ for the entropy contribution of a coarse Lyapunov subgroup $W^{[\chi]}$ on $X \times \Omega$ and similarly for other factors and foliations.

We now establish the following identity regarding the relation between the entropies of individual elements of the action. This identity is central to our approach.

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THEOREM 7.1. *Let α be a $\mathbb{Z}^d$-action on $X = X_m$ without cyclic factors as in Theorem 4.1. Let μ be an α-invariant probability measure, let X_{pos}, X_{base}, Y_{irred} be as (4.1), and denote the Haar measure on Y_{irred} by λ. Moreover, let Ω be a compact metric space equipped with an action of $\mathbb{Z}^d$, let $\widetilde{\alpha}$ be the corresponding $\mathbb{Z}^d$-action on $X \times \Omega$, and let $\widetilde{\mu}$ be an invariant measure on $X \times \Omega$ projecting to μ. Then there exists a constant $\kappa_{\widetilde{\mu},\Omega} > 0$ with*

$$\mathrm{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) = \kappa_{\widetilde{\mu},\Omega}\mathrm{h}_{\lambda}(\alpha^{\mathbf{n}}_{Y_{\text{irred}}}, W^{[\chi]}) \tag{7.1}$$

for every $\mathbf{n} \in \mathbb{Z}^d$.

We recall that we use λ to denote the Haar measure on the appropriate adelic quotient (that should hopefully be clear from the context; e.g. in (7.1), λ is the Haar measure on Y_{irred}). While the proof of Theorem 7.1 is much more complicated than in the case considered by Rudolph, this theorem plays a similar role in our proof as [**34**, Theorem 3.7] did in Rudolph's proof in [**34**].

As a first step towards the theorem, we consider just one coarse Lyapunov subgroup. Note that unlike Theorem 7.1, which uses in an essential way the irreducibility of Y_{irred}, the next lemma only uses the fact that $W^{[\chi]}$ is a coarse Lyapunov group.

LEMMA 7.2. *Using the same notation as in Theorem 7.1, let $[\chi]$ be a coarse Lyapunov weight for Y_{irred}. Then there exists some $\kappa_{\widetilde{\mu},\Omega,[\chi]} \geq 0$ with*

$$\mathrm{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) = \kappa_{\widetilde{\mu},\Omega,[\chi]}\mathrm{h}_{\lambda}(\alpha^{\mathbf{n}}_{Y_{\text{irred}}}, W^{[\chi]})$$

for every $\mathbf{n} \in \mathbb{Z}^d$.

Proof. We first note that for $\mathbf{n} \in \mathbb{Z}^d$ with $\chi \cdot \mathbf{n} < 0$ and $k \in \mathbb{N}$, Proposition 3.4 implies

$$\mathrm{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{k\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) = k\mathrm{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega). \tag{7.2}$$

Moreover, since $W^{[\chi]}$ is a coarse Lyapunov subgroup, for all $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^d$ with $\chi \cdot \mathbf{n} < \chi \cdot \mathbf{m} < 0$, we have that

$$\mathrm{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) \geq \mathrm{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{m}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega).$$

In conjunction with (7.2), this implies elementarily that there is a constant $c \geq 0$ depending only on $\tilde{\mu}$, $W^{[\chi]}$, and α so that $\mathrm{h}_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]}) = c|\chi \cdot \mathbf{n}|$ for all $\mathbf{n} \in \mathbb{Z}^d$.

Next notice that for similar reasons $\mathrm{h}_{\lambda}(\alpha^{\mathbf{n}}, W^{[\chi]})$ is given by a similar formula for a constant $c_{\lambda} \geq 0$. As $[\chi]$ is assumed to be a coarse Lyapunov weight for Y_{irred}, we have $c_{\lambda} > 0$ and obtain the lemma with $\kappa_{\widetilde{\mu},\Omega,[\chi]} = c/c_{\lambda}$. □

LEMMA 7.3. *We again use the notation in Theorem 7.1. For any coarse Lyapunov weight $[\chi]$ of X, we have*

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\text{pos},\mathbb{A}} \mid X_{\text{base}} \times \Omega) = 0,$$

where $V_{\text{pos},\mathbb{A}} < \mathbb{A}^m$ is the rationally defined subspace so that Y_{pos} is the image of $V_{\text{pos},\mathbb{A}}$ modulo $\mathbb{Q}^m$. Moreover, if $[\chi]$ is not a coarse Lyapunov weight for Y_{irred}, then

$$\mathrm{h}_{\widetilde{\mu}}(\alpha^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) = 0.$$

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Proof. We claim for the entropy contributions for $W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}}$ that

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}} \mid X_{\mathrm{base}} \times \Omega) = h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}} \mid X_{\mathrm{pos}} \times \Omega). \tag{7.3}$$

To see this, recall that $Y_{\mathrm{pos}} < Y_{\mathrm{base}}$, and correspondingly $X_{\mathrm{base}} = X/Y_{\mathrm{base}}$ is a factor of $X_{\mathrm{pos}} = X/Y_{\mathrm{pos}}$. By (3.8′), the conditional entropy contribution

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}} \mid X_{\mathrm{base}} \times \Omega)$$

is given by

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}} \mid X_{\mathrm{base}} \times \Omega) = H_{\widetilde{\mu}}(\mathcal{C} \mid \widetilde{\alpha}^{-\mathbf{n}}\mathcal{C} \vee \mathcal{B}_{\mathrm{base}} \vee \mathcal{B}_{\Omega})$$

for $\mathcal{C}$ a $W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}}$-subordinate σ-algebra for $X \times \Omega$ (and $\mathcal{B}_{\mathrm{base}}$ and $\mathcal{B}_{\Omega}$ the σ-algebras of Borel measurable sets on X_{base} and Ω, respectively). However, since each atom of $\mathcal{C}$ is contained in a single orbit of Y_{pos}, its image under the projection from X to $X_{\mathrm{pos}} = X/Y_{\mathrm{pos}}$ consists of a single point, so modulo $\widetilde{\mu}$,

$$\widetilde{\alpha}^{-\mathbf{n}}\mathcal{C} \vee \mathcal{B}_{\mathrm{base}} = \widetilde{\alpha}^{-\mathbf{n}}\mathcal{C} \vee \mathcal{B}_{\mathrm{pos}}$$

(with $\mathcal{B}_{\mathrm{pos}}$ the Borel σ-algebra on X_{pos}). It follows that

$$\begin{aligned} h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}} \mid X_{\mathrm{base}} \times \Omega) &= H_{\widetilde{\mu}}(\mathcal{C} \mid \widetilde{\alpha}^{-\mathbf{n}}\mathcal{C} \vee \mathcal{B}_{\mathrm{base}} \vee \mathcal{B}_{\Omega}) \\ &= H_{\widetilde{\mu}}(\mathcal{C} \mid \widetilde{\alpha}^{-\mathbf{n}}\mathcal{C} \vee \mathcal{B}_{\mathrm{pos}} \vee \mathcal{B}_{\Omega}) \\ &= h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}} \mid X_{\mathrm{pos}} \times \Omega) \end{aligned}$$

as claimed.

Using (7.3) and the relation between entropy contributions and entropy in Proposition 3.4, we now obtain

$$\begin{aligned} h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}} \mid X_{\mathrm{base}} \times \Omega) &\le h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}} \mid X_{\mathrm{pos}} \times \Omega) \\ &\le h_{\mu}(\alpha_X^{\mathbf{n}} \mid X_{\mathrm{pos}}) = 0, \end{aligned}$$

where the last inequality follows from the choice of X_{pos} in §4.4.

So suppose now that $[\chi]$ is not a coarse Lyapunov weight for Y_{irred}. Let $V_{\mathrm{base},\mathbb{A}} < \mathbb{A}^m$ be the rational subspace corresponding to Y_{base}. Since we consider entropy contribution conditional on $X_{\mathrm{base}} \times \Omega$, we may use the above argument again and replace the coarse Lyapunov subgroup $W^{[\chi]}$ with intersection $W^{[\chi]} \cap V_{\mathrm{base},\mathbb{A}}$. However, since $[\chi]$ is not a coarse Lyapunov weight for $Y_{\mathrm{irred}} = Y_{\mathrm{base}}/Y_{\mathrm{pos}}$, it follows that $W^{[\chi]} \cap V_{\mathrm{base},\mathbb{A}} = W^{[\chi]} \cap V_{\mathrm{pos},\mathbb{A}}$. Now the first part of the lemma implies that the entropy contribution vanishes. Since the entropy contribution for the Haar measure λ on Y_{irred} vanishes too, this proves the lemma. □

Proof of Theorem 7.1. By the Abromov–Rokhlin entropy addition formula, we have

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}} \mid X_{\mathrm{base}} \times \Omega) = h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}} \mid X_{\mathrm{pos}} \times \Omega) + h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}_{X_{\mathrm{pos}} \times \Omega} \mid X_{\mathrm{base}} \times \Omega)$$

for all $\mathbf{n} \in \mathbb{Z}^d$. However, by properties of $X_{\mathrm{pos}} = X/Y_{\mathrm{pos}}$ in (4.1), we have

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}} \mid X_{\mathrm{pos}} \times \Omega) \le h_{\mu}(\alpha^{\mathbf{n}} \mid X_{\mathrm{pos}}) = 0. \tag{7.4}$$

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Therefore,

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}} \mid X_{\text{base}} \times \Omega) = h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}_{X_{\text{pos}} \times \Omega} \mid X_{\text{base}} \times \Omega) \quad \text{for all } \mathbf{n} \in \mathbb{Z}^d. \tag{7.5}$$

We claim that the entropy contributions for all coarse Lyapunov subgroups $W^{[\chi]}$ satisfy a similar equation, namely

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) = h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}_{X_{\text{pos}} \times \Omega}, W^{[\chi]}_{\text{pos}} \mid X_{\text{base}} \times \Omega) \tag{7.6}$$

for all $\mathbf{n} \in \mathbb{Z}^d$ with $\chi \cdot \mathbf{n} < 0$, where $W^{[\chi]}$ and $W^{[\chi]}_{\text{pos}}$ denote the coarse Lyapunov subgroups for X and X_{pos}, respectively. Applying Proposition 3.5 to $W^{[\chi]}$ and $W^{[\chi]}_{\text{pos}}$ (and with $X_{\text{base}} \times \Omega$ playing the role of Ω in Proposition 3.5), we conclude that

$$\begin{aligned} h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) &\le h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}_{X_{\text{pos}} \times \Omega}, W^{[\chi]}_{\text{pos}} \mid X_{\text{base}} \times \Omega) \\ &\quad + h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\text{pos},\mathbb{A}} \mid X_{\text{base}} \times \Omega) \end{aligned}$$

with $V_{\text{pos},\mathbb{A}} < \mathbb{A}^m$ as in Lemma 7.3. By Lemma 7.3,

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \cap V_{\text{pos},\mathbb{A}} \mid X_{\text{base}} \times \Omega) = 0,$$

hence

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) \le h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}_{X_{\text{pos}} \times \Omega}, W^{[\chi]}_{\text{pos}} \mid X_{\text{base}} \times \Omega). \tag{7.7}$$

Now fix some $\mathbf{n} \in \mathbb{Z}^d$ and take the sum over all coarse Lyapunov weights $[\chi]$ with $(\chi \cdot \mathbf{n}) < 0$. By the second claim in Theorem 6.1 (applied both to X and to X_{pos}), this leads to an inequality between the two terms in (7.5). However, since in (7.5) equality holds, equality for the entropy contributions in (7.7) must hold as well. Varying $\mathbf{n} \in \mathbb{Z}^d$ gives (7.6) for all coarse Lyapunov weights $[\chi]$.

Next we are going to combine Theorem 5.1 and Theorem 6.1. By Theorem 5.1,

$$h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}_{X_{\text{pos}} \times \Omega}, W^{[\chi]}_{\text{pos}} \mid X_{\text{base}} \times \Omega) \le \frac{h_{\widetilde{\mu}}(\alpha^{\mathbf{n}} | X_{\text{base}} \times \Omega)}{h_\lambda(\alpha^{\mathbf{n}}_{Y_{\text{irred}}})} h_\lambda(\alpha^{\mathbf{n}}_{Y_{\text{irred}}}, W^{[\chi]}_{Y_{\text{irred}}}). \tag{7.8}$$

Set $\kappa_{\mathbf{n},\widetilde{\mu},\Omega} = h_{\widetilde{\mu}}(\alpha^{\mathbf{n}} | X_{\text{base}} \times \Omega) / h_\lambda(\alpha^{\mathbf{n}}_{Y_{\text{irred}}})$; note that it does not depend on the coarse Lyapunov weight $[\chi]$. Taking the sum over all coarse Lyapunov weights $[\chi]$ with $\chi \cdot \mathbf{n} < 0$ gives, by (6.3) of Theorem 6.1 on the left-hand side, the conditional entropy $h_{\widetilde{\mu}}(\alpha^{\mathbf{n}} | X_{\text{base}} \times \Omega)$ and on the right-hand side, we obtain $\kappa_{\mathbf{n},\widetilde{\mu},\Omega} h_\lambda(\alpha^{\mathbf{n}}_{Y_{\text{irred}}})$, which in view of the definition of $\kappa_{\mathbf{n},\widetilde{\mu},\Omega}$ also equals $h_{\widetilde{\mu}}(\alpha^{\mathbf{n}} | X_{\text{base}} \times \Omega)$. This shows that in fact

$$h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}_{X_{\text{pos}} \times \Omega}, W^{[\chi]}_{\text{pos}} \mid X_{\text{base}} \times \Omega) = \kappa_{\mathbf{n},\widetilde{\mu},\Omega}\, h_\lambda(\alpha^{\mathbf{n}}_{Y_{\text{irred}}}, W^{[\chi]}_{Y_{\text{irred}}}), \tag{7.8$'$}$$

for all coarse Lyapunov weights $[\chi]$ with $\chi \cdot \mathbf{n} < 0$.

We now choose $\mathbf{n}_0$ so that $\chi \cdot \mathbf{n}_0 \neq 0$ for all coarse Lyapunov weights $[\chi]$. Since

$$\kappa_{\mathbf{n}_0,\widetilde{\mu},\Omega} = \kappa_{-\mathbf{n}_0,\widetilde{\mu},\Omega},$$

equation (7.8′), together with (7.6), implies that

$$h_{\widetilde{\mu}}(\widetilde{\alpha}^{\mathbf{n}_0}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) = \kappa_{\mathbf{n}_0,\widetilde{\mu},\Omega} h_\lambda(\alpha^{\mathbf{n}_0}_{Y_{\text{irred}}}, W^{[\chi]})$$

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for all coarse Lyapunov weights $[\chi]$. It follows that the constants appearing in Lemma 7.2, which may depend on $[\chi]$ but not on $\mathbf{n}_0$, agree with $\kappa_{\mathbf{n}_0,\widetilde{\mu},\Omega}$, which may depend on $\mathbf{n}_0$ but not on $[\chi]$. This gives the theorem. □

7.2. *Rigidity of the entropy function implies invariance.* We are now ready to prove Theorem 4.1. As before, we work with the setup explained in §4.4, specifically (4.1): $X_{\text{pos}} = X/Y_{\text{pos}}$ is a factor of X so that $h_\mu(\alpha^{\mathbf{n}}|X_{\text{pos}}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^d$, Y_{irred} is an α-invariant $\mathbb{A}$-irreducible subgroup of X_{pos}, and $X_{\text{base}} = X/Y_{\text{base}} = X_{\text{pos}}/Y_{\text{irred}}$ satisfies that for some $\mathbf{n} \in \mathbb{Z}^d$ we have that $h_\mu(\alpha^{\mathbf{n}}_{X_{\text{pos}}}|X_{\text{base}}) > 0$.

Applying Theorem 7.1 with Ω being the trivial factor, we obtain a constant $\kappa_\mu > 0$ so that $h_\mu(\alpha^{\mathbf{n}}|X_{\text{base}}) = \kappa_\mu h_\lambda(\alpha^{\mathbf{n}}_{Y_{\text{irred}}}) > 0$ for all $\mathbf{n} \in \mathbb{Z}^d$.

Next we choose a coarse Lyapunov weight $[\chi]$ of Y_{irred} and consider it as a coarse Lyapunov weight for X. Let $V_{\text{base},\mathbb{A}} < \mathbb{A}^m$ be the rational α-invariant subspace so that Y_{base} is the image of $V_{\text{base},\mathbb{A}}$ modulo $\mathbb{Q}$. We also set $W = W^{[\chi]} \cap V_{\text{base},\mathbb{A}}$. Let f_W be a positive function on W, which is integrable with respect to μ_x^W for every x in a set of full measure as in §3.1. We take

$$\Omega = \left\{ [\nu] : \nu \text{is a locally finite measure on } W \text{ with } \int f_W \, d\nu < \infty \right\},$$

where $[\nu]$ denotes the equivalence class of ν in the space of locally finite measures with respect to proportionality. One can equip Ω with the structure of a compact metric space in a standard way. The map (defined for a.e. $x \in X$) that takes $x \in X$ and maps it to the proportionality class of its leafwise measure $[\mu_x^W]$ is, by (3.4), a factor map of the $\mathbb{Z}^d$-action α on X to the action of $\mathbb{Z}^d$ on elements of Ω by pushforward with respect to the linear action corresponding to α on W. Taking the product of Ω with X_{base}, we get a factor of X and we apply Theorem 7.1 once more over this factor $X_{\text{base}} \times \Omega$ to obtain a constant $\kappa_{\widetilde{\mu},\Omega} \geq 0$ so that

$$h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) = \kappa_{\widetilde{\mu},\Omega} h_\lambda(\alpha^{\mathbf{n}}_{Y_{\text{irred}}}, W^{[\chi]}) > 0 \tag{7.9}$$

for all coarse Lyapunov weights $[\chi]$ and $\mathbf{n} \in \mathbb{Z}^d$.

LEMMA 7.4. *Y_{irred} has at least two linearly independent Lyapunov weights.*

Proof. Since Y_{irred} is irreducible, we may apply Proposition 2.2 and describe the action on Y_{irred} using a global field $\mathbb{K}$ and its elements. Also recall that the eigenspaces for Y_{irred} correspond to the completions of $\mathbb{K}$.

Suppose in contradiction that α has no two linearly independent Lyapunov weights. Then every non-zero Lyapunov weight must be a multiple of χ. We now define the hyperplane $H < \mathbb{R}^d$ as the kernel of χ. Suppose $\mathbf{n} \in \mathbb{Z}^d$ is close to H, that is, satisfies $\chi' \cdot \mathbf{n} \in (-\epsilon, \epsilon)$ for all Lyapunov weights χ' of Y_{irred} and some $\epsilon > 0$ to be determined later. For the algebraic number $\zeta_{\mathbf{n}}$ corresponding to $\mathbf{n}$, this becomes the inequality

$$e^{-\epsilon} < |\zeta_{\mathbf{n}}|_\sigma < e^{\epsilon} \tag{7.10}$$

for all places σ of $\mathbb{K}$ (and $|\zeta_{\mathbf{n}}|_\sigma = 1$ for places σ lying over finite primes p not in S, with S as in (3.5). However, for small enough $\epsilon > 0$, it follows from (7.10) applied to all finite

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places that $\zeta_{\mathbf{n}}$ must be an algebraic unit. Applying (7.10) also to all infinite places, we get that this unit satisfies that all its real and complex embeddings have absolute value close to one. It follows from Dirichlet's unit theorem that ζ must be a root of unity and $\mathbf{n} \in H$. However, this implies that the action of $\alpha(\mathbb{Z}^d)$ on Y_{irred} is virtually cyclic. By the Jordan decomposition over $\mathbb{Q}$, X has a factor isomorphic to Y_{irred}; the fact that $\alpha(\mathbb{Z}^d)$ on Y_{irred} is virtually cyclic now contradicts our standing assumption that X has no virtually cyclic factors. □

PROPOSITION 7.5. *In fact, $\kappa_{\widetilde{\mu},\Omega} = \kappa_\mu$ (hence $\kappa_{\widetilde{\mu},\Omega} > 0$).*

Proof. Let χ' be a Lyapunov weight of Y_{irred} that is linearly independent to χ. The existence of χ' follows from Lemma 7.4. Choose some $\mathbf{n} \in \mathbb{Z}^d$ so that $\chi \cdot \mathbf{n} < 0$ and $\chi' \cdot \mathbf{n} < 0$. The product structure of the leafwise measures in Theorem 6.1 (see also [**10**, Corollary 8.6]) now implies that for a set $X' \subset X$ of full measure, we have the following property: for any $x \in X'$ and any $w' \in W^{[\chi']}$ with $x + w' \in X'$, we have $[\mu_{x+w'}^{W^{[\chi]}}] = [\mu_x^{W^{[\chi]}}]$. For a σ-algebra $\mathcal{A}$ subordinate to $W^{[\chi']}$, this means that, on the complement of a null set, all points in a given atom of $\mathcal{A}$ are contained in the same fiber of the factor map from X to Ω. In other words, the map $x \mapsto (x, [\mu_x^{W^{[\chi]}}])$ from X to $X \times \Omega$ maps σ-algebras subordinate to $W^{[\chi']}$ on X to σ-algebras subordinate to $W^{[\chi']}$ on $X \times \Omega$. From this it follows that the leafwise measures for X and for $X \times \Omega$ with respect to the subgroup $W^{[\chi']}$ agree. In particular, we have

$$h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}, W^{[\chi']} \mid X_{\text{base}} \times \Omega) = h_\mu(\alpha^{\mathbf{n}}, W^{[\chi']} \mid X_{\text{base}}),$$

which together with Theorem 7.1 proves the proposition. □

We continue working under the assumptions stated at the beginning of §7.2.

COROLLARY 7.6. *For any subset $X'' \subset X$ of full measure, there exist $x \in X''$ and a non-zero $v \in W$ with $x + v \in X''$ and $\mu_x^W \propto \mu_{x+v}^W$.*

Proof. By Proposition 7.5, we have $\kappa_{\widetilde{\mu},\Omega} > 0$. We now apply this to the coarse Lyapunov weight $[\chi]$ and the subgroup $W = W^{[\chi]} \cap V_{\text{base},\mathbb{A}}$ that was used to define the factor Ω. Choose $\mathbf{n} \in \mathbb{Z}^d$ with $\chi \cdot \mathbf{n} < 0$. It now follows from the definition of $\kappa_{\widetilde{\mu},\Omega}$ in Theorem 7.1 that

$$h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega) = \kappa_{\widetilde{\mu},\Omega} h_\lambda(\alpha^{\mathbf{n}}_{Y_{\text{irred}}}, W^{[\chi]}) > 0.$$

We also note that $h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}, W \mid X_{\text{base}} \times \Omega) = h_{\widetilde{\mu}}(\alpha^{\mathbf{n}}, W^{[\chi]} \mid X_{\text{base}} \times \Omega)$ (c.f. e.g. the first lines in the proof of Theorem 5.1). We note that positive entropy contribution, as in (3.9), shows in particular that the leafwise measure $\widetilde{\mu}_x^W$ gives zero mass to $0 \in W$.

By the characterizing properties of leafwise measures in terms of W-subordinate σ-algebras, the fact that the leafwise measure $\widetilde{\mu}_x^W$ gives zero mass to $0 \in W$ implies that for any subset $X'' \subset \widetilde{X}$ of full measure for $\widetilde{\mu}$,

$$\text{there exist } x \in X'' \text{ and } v \in W \setminus \{0\} \text{ so that} x + v \in X''. \tag{7.11}$$

Since the action of W on $\widetilde{X} = X \times \Omega$ was defined to be trivial on Ω and $\widetilde{\mu}$ was defined as the push forward of μ under the map $x \mapsto (x, [\mu_x^W])$, (7.11) translates to the following

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statement: for every subset $X'' \subset X$ of full measure for μ, there exists $x \in X''$ and $v \in W \setminus \{0\}$ with $x + v \in X''$ and $[\mu_x^W] = [\mu_{x+v}^W]$. However, this is precisely the claim in the corollary. □

Proof of Theorem 4.1. We first show how the statement in Corollary 7.6 implies invariance of μ under translation by all elements of a non-trivial adelic subgroup. We note that by (3.1), the claim in Corollary 7.6 amounts to saying that for any set $X'' \subseteq X'$ of full measure, there exists some $x \in X''$ and $v \in W \setminus \{0\}$ with $\mu_x^W + v \propto \mu_x^W$. Moreover, by our discussion concerning (3.2)–(3.3), there exists X'' of full measure so that for any $x \in X''$, there is some $v \in W$ with $\mu_x^W + v = \mu_x^W$ (see also [4, Lemma 5.10]). Hence Corollary 7.6 implies that for all $x \in X''$, the closed subgroup

$$W_x = \{v \in W : \mu_x^W + v = \mu_x^W\}$$

is non-trivial. We define $\widetilde{W}_x$ as the maximal S-linear subgroup of W_x. We will show below (using the equivariance formula (3.4) and Poincarè recurrence) that a.s. $\widetilde{W}_x$ is non-trivial. In fact, [4, Proposition 6.2] shows that $W_x = \widetilde{W}_x$ is itself S-linear, at least if the action is semisimple.

We define the dimension D_x of $\widetilde{W}_x$ as the sum of the dimensions of the maximal subspaces over $\mathbb{Q}_\sigma$ contained in $\widetilde{W}_x$ for all $\sigma \in S$. Even though $\widetilde{W}_x$ may not be normalized by α, the equivariance formula (3.4) implies that both W_x and $\widetilde{W}_x$ are equivariant for the action. Hence the dimension of $\widetilde{W}_x$ is invariant under α. Therefore, D_x is constant (say equal to D) for a.e. x.

We claim that Corollary 7.6 implies a.s. that $\widetilde{W}_x$ is non-trivial, or equivalently that $D_x \geq 1$ a.s. For this, we apply Luzin's theorem and let $K \subset X'$ be a compact subset of measure close to 1, on which all almost sure properties of the leafwise measures hold, and on which the map $x \in K \mapsto \mu_x^W$ is continuous. To obtain the almost sure conclusion, we apply the following argument for an increasing sequence of such Luzin sets that cover almost all of the space.

By Poincaré recurrence, we see that for a.e. $x \in K$, there exists two increasing sequences $n_k^-, n_k^+ \in \mathbb{N}$ with $T^{-n_k^-} x, T^{n_k^+} x \in K$ converging to x as $k \to \infty$. Suppose now $v \in W_x \setminus \{0\}$ for one such x so that $\mu_x^W + v = \mu_x^W$. If v has a non-trivial real component, we are going to use the sequence n_k^+. Indeed applying (3.4) for these powers, we see that the leafwise measure at $T^{n_k^+} x$ has translation invariance under $\mathbb{Z}(T^{n_k^+} v)$. Note that $T^{n_k^+} v$ converges to 0 as $k \to \infty$. However, since the unit ball in $\mathbb{R}^m$ is compact, we may choose a subsequence and assume in addition that the direction of $T^{n_k^+} v$ converges in projective space to $\mathbb{R}\widetilde{v}$ for some $\widetilde{v} \neq 0$. Note that this implies that the subgroups $\mathbb{Z}(T^{n_k^+} v)$ converge in the Chabauty topology to $\mathbb{R}\widetilde{v}$. Combined with continuity of the leafwise measures restricted to K, this now implies that μ_x^W is invariant under translation by $\mathbb{R}\widetilde{v}$, which implies $\widetilde{v} \in \widetilde{W}_x$ and $D_x \geq 1$ as desired.

So suppose now v has trivial real component and let us write v_p for the p-adic component of v for all $p \in S$. In this case, the invariance of μ_x^W under $\mathbb{Z}v$ implies invariance under $\overline{\mathbb{Z}v}$, which by the Chinese Remainder Theorem is the product of the compact subgroups $\mathbb{Z}_p v_p$ for all primes $p \in S$. To obtain a non-trivial S-linear subgroup, we fix a prime $p \in S$ with $v_p \neq 0$ and we are going to use the sequence n_k^-. Indeed, as

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above, $T^{-n_k^-}x \in K$ has invariance under $\mathbb{Z}_p T^{-n_k^-} v_p$, where $T^{-n_k^-} v_p$ diverges to infinity but projectively converges to some $\mathbb{Q}_p \widetilde{v}$. By continuity of the leafwise measures on K, this again implies $\widetilde{v} \in \widetilde{W}_x$ and $D_x \geq 1$ as desired.

We define the (measurable) factor map as

$$\phi : x \mapsto \widetilde{W}_x,$$

from X to the space $\mathcal{F}$ of closed subgroups of W in the Chabauty topology. We also decompose μ over ϕ, that is, we consider the conditional measures $\mu_x^{\phi^{-1}\mathcal{B}_\mathcal{F}}$. By the compatibility condition (3.1) and the definition of $\widetilde{W}_x$, we have $\phi(x) = \phi(x+v)$ whenever $v \in W$ and both $x, x+v$ belong to X''. In particular, this shows for a W-subordinate σ-algebra $\mathcal{A}$ that $\mathcal{A}$ contains $\phi^{-1}\mathcal{B}_\mathcal{F}$ modulo null sets. Hence, after fixing a choice of the conditional measures $\mu_x^{\mathcal{A}}$, we have that the (doubly) conditional measure $(\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}})_x^{\mathcal{A}}$ agrees with $\mu_x^{\mathcal{A}}$ for $\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}}$-a.e. x and μ-a.e. x_0. This in turn implies for the leafwise measures of μ and $\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}}$ with respect to W by the characterizing properties that

$$(\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}})_x^W = \mu_x^W \tag{7.12}$$

for $\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}}$-a.e. x and for μ-a.e. x_0.

Fix some x_0. Then essentially by definition of the factor ϕ, we have that $\widetilde{W}_x = \widetilde{W}_{x_0}$ for $\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}}$-a.e. x. In other words, by definition of $\widetilde{W}_x$ and (7.12), we have that for $\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}}$-a.e x, the leafwise measures $(\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}})_x^W$ are invariant under the group $\widetilde{W}_{x_0}$, hence by the standard properties of leafwise measures that $\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}}$ is itself invariant under the action of $\widetilde{W}_{x_0}$ by translations. Note that, as far as we know at this point, the group $\widetilde{W}_x$ may depend on x, and hence we have not established the invariance of μ itself under any translations yet.

As $X = \mathbb{A}^m/\mathbb{Q}^m$ is an abelian group and $\mathbb{Q}^m$ acts trivially, this implies that $\mu_{x_0}^{\phi^{-1}\mathcal{B}_\mathcal{F}}$ is in fact invariant under the closure G_x of $\widetilde{W}_{x_0} + \mathbb{Q}^m$ in X. Since $\widetilde{W}_{x_0}$, as an S-linear subgroup, is invariant under multiplication by $\mathbb{Q}$, we have that its annihilator $G_x^\perp$ in the Pontryagin dual $\mathbb{Q}^m$ to X is a vector space over $\mathbb{Q}$. Hence G_x is an adelic subgroup of X.

The equivariance formula (3.4) implies a similar equivariance formula for W_x, for $\widetilde{W}_x$, and hence also for G_x. In other words, $x \mapsto G_x$ is a (measurable) factor map for α with values in the countable set of all adelic subspaces of X_m. Hence there exists an adelic subspace G so that $G_x = G$ on a set of positive measure. By Poincaré recurrence, we may conclude that there is a finite index subgroup Λ of $\mathbb{Z}^d$ so that $\alpha(\Lambda)$ normalizes G. However, the assumption in Theorem 4.1 now implies that G is actually normalized by $\mathbb{Z}^d$. Ergodicity under α now implies that $G_x = G$ a.s. Therefore, μ is invariant under G. □

8. *Disjointness*

We will deduce in this section the rigidity of joinings and in particular prove Corollary 1.4. We say a measure on a solenoid X is *homogeneous* if it is the Y-invariant measure on a coset $Y + y$ of a closed subgroup $Y < X$.

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PROPOSITION 8.1. *Let $d, r \geq 2$ and let α_i be $\mathbb{Z}^d$-actions on solenoids X_i (equipped with Haar measure λ_{X_i}) with no virtually cyclic factors for $i = 1, \dots, r$. Suppose there exists a non-trivial joining μ between α_i for $i = 1, \dots, r$. Then there exists a finite index subgroup $\Lambda \subseteq \mathbb{Z}^d$ such that there exists also a homogeneous non-trivial joining λ_G between $\alpha_{i,\Lambda}$ for $i = 1, \dots, r$. In fact, the subgroup G can be chosen to be any of the groups in the conclusion of Theorem 1.3 when applied to an ergodic component of the positive entropy measure μ on $X_1 \times \cdots \times X_r$.*

Proof. Let $X = X_1 \times \cdots \times X_r$ and $\alpha = \alpha_1 \times \cdots \times \alpha_r$ be the product group and action, and let μ be a non-trivial joining. Without loss of generality, we can assume μ is α-ergodic (since a.e. ergodic component of a joining is again a joining). We apply Theorem 1.3 and obtain a finite index subgroup $\Lambda \subset \mathbb{Z}^d$ and an α_Λ-invariant closed subgroup $G = G_1 \subset X$. We claim that the Haar measure of G is an homogeneous non-trivial joining for the Λ-action.

To see this, let μ_j for $j = 1, \dots, J$ be as in Theorem 1.3 and let $\pi_i : X \to X_i$ for $i = 1, \dots, r$ be the coordinate projection map. First notice that the Haar measure of G cannot be the trivial joining. If it were, it would follow that $G = X$, $\mu_1 = \lambda_X$, $\mu_j = (\alpha^{\mathbf{n}})_* \mu_1 = \lambda_X$ for all j and some $\mathbf{n}$ (that depends on j), and so $\mu = \lambda_X$ would be the trivial joining. To show the claim, we only need to prove that $\pi_i(G) = X_j$ for $i = 1, \dots, r$.

Fix some i. Clearly $\tilde{\lambda}_j = (\pi_i)_* \mu_j$ defines a measure on X_i that is invariant under $\alpha_{i,\Lambda}$. By assumption, $\lambda_{X_i} = (1/M)(\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_J)$. Note that $\alpha_{i,\Lambda}$ acts ergodically on X_i with respect to λ_{X_i} by the assumption that there are no virtually cyclic factors. Therefore, $\tilde{\lambda}_j = \lambda_{X_i}$ for all j, in other words, μ_1 is a joining. Consider now the group $Y = X_i / \pi_i(G) \cong X/(G + \ker \pi_i)$ endowed with the measure ν induced by μ_1. However, by the above, the projection of μ_1 to X_i is λ_{X_i}, and so $\nu = \lambda_Y$. Since Y is a factor of X/G, the entropy (with respect to ν) of every element on Y of the action must vanish. However, the action on X_i contains elements with completely positive entropy. Therefore, $Y = \{0\}$ and $\pi_j(G) = X_i$ for all i. □

Proof of Corollary 1.4, simpler case. Assume first that α_1 and α_2 are totally irreducible not virtually cyclic actions and let $\alpha = \alpha_1 \times \alpha_2$. Suppose $\alpha_{1,\Lambda}$ and $\alpha_{2,\Lambda}$ are algebraically weakly isomorphic for some finite index subgroup $\Lambda \subset \mathbb{Z}^d$. So there exists a finite-to-one algebraic factor map $\varphi : X_1 \to X_2$ for the two subactions. Clearly, the graph G of φ is α_Λ-invariant and so is its Haar measure λ_G. The average μ over the elements $\alpha_*^{\mathbf{n}} \lambda_G$ in the (finite) orbit of λ_G under the action of α defines a non-trivial joining between α_1 and α_2.

Let μ be a joining between α_1 and α_2. We have to show that either $\mu = \lambda_{X_1} \times \lambda_{X_2}$ is the trivial joining, or find a finite index subgroup $\Lambda \subset \mathbb{Z}^d$ such that $\alpha_{1,\Lambda}$ and $\alpha_{2,\Lambda}$ are algebraically weakly isomorphic. Assume that μ is not the trivial joining, then there exists a finite index subgroup Λ and a homogeneous non-trivial joining λ_G between $\alpha_{1,\Lambda}$ and $\alpha_{2,\Lambda}$ by Proposition 8.1. Here $G \subset X$ is a proper closed subgroup with $\pi_1(G) = X_1$ and $\pi_2(G) = X_2$.

Next we study the factors $X_1' = X_1/\{x_1 : (x_1, 0) \in G\}$ and $X_2' = X_2/\{x_2 : (0, x_2) \in G\}$. Suppose X_1' is trivial, then $X_1 \times \{0\} \subset G$ and $\pi_2(G) = X_2$ implies that $G = X_1 \times X_2$, a contradiction to G being proper.

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So assume now X_1' and X_2' are non-trivial, and let $G' \subset X_1' \times X_2'$ be the subgroup defined by G. Then $G' \cap (X_1' \times \{0\}) = G' \cap (\{0\} \times X_2') = \{0\}$, so G' is the graph of an isomorphism between X_1' and X_2'. Since G' is closed, compactness shows the isomorphism is continuous. Since all of the above are invariant subgroups, it follows that $\alpha_{1,\Lambda}$ and $\alpha_{2,\Lambda}$ have a common non-trivial factor $X_1' \cong X_2'$. By assumption $\alpha_{1,\Lambda}$ and $\alpha_{2,\Lambda}$ are irreducible, so the kernel of the above factor maps must be finite. Let N be the order of the kernel, then multiplication by N defines a factor map ψ_N from X_1 to X_1 that can be extended to a factor map ψ_N from X_1' to X_1, that is, the two actions on X_1 and X_1' are weakly algebraically isomorphic. □

Proof of Corollary 1.4, general case. Let $X = \prod_{=1}^{r} X_j$ and $\alpha = \alpha_1 \times \cdots \times \alpha_r$ be the product group and action. Suppose $j \neq k \in \{1, \ldots, r\}$ and $\alpha_{j,\Lambda}$ and $\alpha_{k,\Lambda}$ have a common non-trivial factor β on a solenoid Y, where $\Lambda \subset \mathbb{Z}^d$ is a finite index subgroup. Let $\varphi_j : X_j \to Y$ and $\varphi_k : X_k \to Y$ be the corresponding group homomorphisms. Then $G = \{x \in X : \varphi_j(x_j) = \varphi_k(x_k)\}$ is a non-trivial closed α_Λ-invariant subgroup of X such that $\pi_i(G) = X_i$ for $i = 1, \ldots, r$. The Haar measure λ_G on G has finite orbit under α, and the average μ over the elements in the (finite) orbit of λ_G is a non-trivial joining.

Suppose now that μ is a non-trivial joining between α_i for $i = 1, \ldots, r$, and apply Proposition 8.1. We obtain a finite index subgroup $\Lambda \subset \mathbb{Z}^d$ and an α_Λ-invariant proper closed subgroup $G < X$ that satisfies $\pi_i(G) = X_i$ for $i = 1, \ldots, r$.

Next we factor X_i by the subgroup

$$H_i = \pi_i(G \cap (\{0\}^{i-1} \times X_i \times \{0\}^{r-i}))$$

to get $X_i' = X_i/H_i$ for $i = 1, \ldots, r$ and the factor $X' = \prod_i X_i'$ of X. If $X' = \{0\}$, then $G = X$ which contradicts G being a proper subgroup.

So assume that X' is non-trivial, and therefore infinite. Let $G' < X'$ be the image of G. Clearly $\pi_i(G') = X_i'$ for $i = 1, \ldots, r$. Let i be minimal such that $H = G' \cap Z_i$ is infinite, where $Z_i = X_1' \times \cdots \times X_i' \times \{0\}^{r-i}$. Then the kernel $\{x \in H : x_i = 0\}$ of $\pi_i|_H$ is finite, and so $\pi_i(H)$ is an infinite closed α_Λ-invariant subgroup of X_i'. Let β be an irreducible component of $\alpha_\Lambda|_H$. Since $\pi_i|_H$ is finite-to-one, β is also an irreducible component of $\alpha_{i,\Lambda}|_{\pi_i(H)}$. Since an irreducible component of a subgroup is also an irreducible component of the whole group, we see that $\alpha_\Lambda|_H$ and $\alpha_{i,\Lambda}$ share β as an irreducible component. By construction of X', we have $\{z \in H : z_k = 0 \text{ for all } k \neq i\} = \{0\}$. Therefore H is isomorphic to a subgroup of $X_1' \times \cdots \times X_{i-1}'$. We conclude that there exists $j < i$ such that $\alpha_{i,\Lambda}$ and $\alpha_{j,\Lambda}$ have β as a common factor. □

9. *Invariant σ-algebras and measurable factors*

In addition to the characterization of factors stated in Corollary 1.5, we prove in this section a generalization of the isomorphism rigidity [**21**] for higher rank actions. Indeed we characterize when two actions by automorphisms of solenoids have a common measurable factor.

Suppose α_1, α_2 are $\mathbb{Z}^d$-actions by automorphisms of the solenoids X_1 and X_2. Let Γ_1 respectively Γ_2 be finite groups of affine automorphisms of X_1 and X_2 that are normalized by the respective actions. We say that the two factors of X_1 and X_2 arising from Γ_1

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respectively Γ_2 are *isomorphic* if there exists an affine isomorphism $\Phi : X_1 \to X_2$ such that $\Phi \circ \alpha_1^{\mathbf{n}} \circ \Gamma_1 = \alpha_2^{\mathbf{n}} \circ \Gamma_2 \circ \Phi$ for all $\mathbf{n} \in \mathbb{Z}^d$.

We claim that this is essentially the only way common measurable factors of higher rank actions on solenoids can arise.

COROLLARY 9.1. (Classification of common factors) *Let $d \geq 2$ and let α_1, α_2 be $\mathbb{Z}^d$-actions by automorphisms of the solenoids X_1 and X_2 without virtually cyclic factors. Suppose α_1 and α_2 have a common measurable factor. Then there exist closed invariant subgroups $X_1' \subseteq X_1$ and $X_2' \subseteq X_2$, finite groups Γ_1 of affine automorphisms of X_1/X_1' and Γ_2 of affine automorphisms of X_2/X_2' that are normalized by the corresponding actions such that this common measurable factor can be described alternatively as the factor of X_1/X_1' by the orbit equivalence relation of Γ_1, or similarly using X_2/X_2' and Γ_2. Moreover, the isomorphism between these two realizations of the factor is algebraic in the following sense: there exists an affine isomorphism $\Phi : X_1/X_1' \to X_2/X_2'$ such that*

$$\Phi \circ \alpha^{\mathbf{n}}_{X_1/X_1'} \circ \Gamma_1 = \alpha^{\mathbf{n}}_{X_2/X_2'} \circ \Gamma_2 \circ \Phi \tag{9.1}$$

for all $\mathbf{n} \in \mathbb{Z}^d$.

Note that we have more rigidity for the factors than we had for joinings in Corollary 1.4. In particular, there is no need to consider finite index subactions in order to classify when two $\mathbb{Z}^d$ actions on tori have a common factor; we illustrate this point with the following example.

Example 9.2. Let α_1 be the $\mathbb{Z}^2$-action by automorphisms of the solenoid X_1 dual to $\mathbb{Z}[1/2, 1/3]$ generated by multiplication by 2 and by 3 on X_1 (i.e. the α_1 action on X_1 is the invertible extension of the $\times 2$, $\times 3$ action on $\mathbb{T}$). We define a $\mathbb{Z}^2$-action α_2 on $X_2 = X_1^2$ by

$$\begin{aligned} \alpha_2^{\mathbf{e}_1}(x_1, x_2) &= (-2x_2, 2x_1), \\ \alpha_2^{\mathbf{e}_2}(x_1, x_2) &= (3x_1, 3x_2) \quad \text{for } (x_1, x_2) \in X_2. \end{aligned}$$

Then $\alpha_2^{4\mathbf{e}_1}(x_1, x_2) = (16x_1, 16x_2)$ for $(x_1, x_2) \in X_2$ and so the restriction $\alpha_{2,\Lambda}$ of α_2 to $\Lambda = (4\mathbb{Z}) \times \mathbb{Z}$ is identical to the action $\alpha_{1,\Lambda} \times \alpha_{1,\Lambda}$ on $X_2 = X_1^2$. By Theorem 1.4 α_1 and α_2 are not disjoint. In fact, let

$$Z = \{(x_1, (x_1, x_2) : x_1, x_2 \in X\} \subseteq X_1 \times X_2.$$

Then $\pi_1(Z) = X_1$ and $\pi_2(Z) = X_2$. Therefore, the Haar measure m_Z of Z satisfies $(\pi_1)_* m_Z = m_{X_1}$ and $(\pi_2)_* m_Z = m_{X_2}$. Since Z is only invariant under $(\alpha_1 \times \alpha_2)_\Lambda$, m_Z is not a joining. However, it is easy to check that $\mu = \frac{1}{4} \sum_{j=0}^{3} (\alpha_1 \times \alpha_2)^{j\mathbf{e}_1} m_Z$ is a non-trivial joining between α_1 and α_2.

We note that α_1 and α_2 are both irreducible actions. Suppose now that α_1 and α_2 have a common measurable factor. By Corollary 9.1, there exist closed invariant subgroups $X_1' \subseteq X_1$ and $X_2' \subseteq X_2$ such that X_1/X_1' and X_2/X_2' are isomorphic as groups. By irreducibility, X_1' and X_2' are either finite or everything. However, if X_1' and X_2' are finite, then X_1/X_1' is still one-dimensional while X_2/X_2' is still two-dimensional, and so these groups cannot be

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isomorphic. Therefore, $X_1' = X_1$, $X_2' = X_2$, and the common measurable factor has to be the trivial factor.

Before we start with the proofs of Corollary 1.5 and Corollary 9.1, we recall some basic facts about conditional measures and the construction of the relatively independent joining.

9.1. *The relatively independent joining.* Let $\alpha_1, \alpha_2, \beta$ be $\mathbb{Z}^d$-actions on the standard Borel probability spaces $(X_1, \mathcal{B}_1, m_1)$, $(X_2, \mathcal{B}_2, m_2)$, and $(Y, \mathcal{B}_Y, \rho)$, respectively, and let $\psi_1 : X_1 \to Y$ and $\psi_2 : X_2 \to Y$ be factor maps. Then $\mathcal{A}_1 = \psi_1^{-1}\mathcal{B}_Y$ and $\mathcal{A}_2 = \psi_2^{-1}\mathcal{B}_Y$ are invariant σ-algebras with conditional measures $m_{1,x}^{\mathcal{A}_1}$ for $x \in X_1$ and $m_{2,x}^{\mathcal{A}_2}$ for $x \in X_2$. By the basic properties of conditional measures, there is some null set $N_1 \subset X_1$ so that $m_{1,x}^{\mathcal{A}_1} = m_{1,x'}^{\mathcal{A}_1}$ for every $x, x' \in X_1 \setminus N_1$ with $\psi_1(x) = \psi_1(x')$, and so we can remove a nullset from X_1 and Y and define $m_{\psi_1^{-1}y} = m_{1,x}^{\mathcal{A}_1}$ for $x \in \psi_1^{-1}y$. We do this similarly for $\mathcal{A}_2$ to define $m_{\psi_2^{-1}y}$. The relatively independent joining $m_1 \times_{(Y,\rho)} m_2$ of m_1 and m_2 over the common factor (Y, ρ) is defined by

$$m_1 \times_{(Y,\rho)} m_2 = \int_Y m_{\psi_1^{-1}y} \times m_{\psi_2^{-1}y}\, d\rho(y). \tag{9.2}$$

It is well known (and easy to verify directly) that $m_1 \times_{(Y,\rho)} m_2$ projects to m_1 on X_1 and to m_2 on X_2 and that $m_1 \times_{(Y,\rho)} m_2$ is invariant under $\alpha_1 \times \alpha_2$ (hence is a joining between α_1 and α_2). Furthermore, the relatively independent joining $m_1 \times_{(Y,\rho)} m_2$ gives full measure to the set

$$D_Y = \{(x_1, x_2) : \psi_1(x_1) = \psi_2(x_2)\} \subset X_1 \times X_2 \tag{9.3}$$

and moreover

$$\psi_1^{-1}C \times X_2,\ X_1 \times \psi_2^{-1}C,\ \text{and } \psi_1^{-1}C \times \psi_2^{-1}C$$

are equal up to a $m_1 \times_{(Y,\rho)} m_2$-nullset for any $C \in \mathcal{B}_Y$.

9.2. *Proofs of Corollary 1.5 and Corollary 9.1.* Let $\alpha_1, \alpha_2, X_1, X_2$ be as in Corollary 9.1, and suppose the $\mathbb{Z}^d$-action β on $(Y, \mathcal{B}_Y, \rho)$ is a common factor of α_1 and α_2. Let ψ_1 and ψ_2 be the corresponding factor maps, $\mathcal{A}_1 = \psi_1^{-1}\mathcal{B}_Y$ and $\mathcal{A}_2 = \psi_2^{-1}\mathcal{B}_Y$ the corresponding invariant σ-algebras, and let $\nu = \lambda_{X_1} \times_{(Y,\rho)} \lambda_{X_2}$ be the relatively independent joining.

The main idea for the proof is to use Theorem 1.3 to study ν. In the following example, we see how the algebraic construction of the factor is encoded in the relatively independent joining, and that the latter does not have to be ergodic.

Example 9.3. Let α be a $\mathbb{Z}^d$-action on a solenoid X, and let $\mathcal{A} = \mathcal{B}_X^{\{\mathrm{Id}, -\mathrm{Id}\}}$ be the α-invariant σ-algebra of measurable subsets A satisfying $A = -A$. Then the relatively independent joining $\nu = \lambda_X \times_{\mathcal{A}} \lambda_X$ of λ_X over the factor described by $\mathcal{A}$ is $\nu = \frac{1}{2}(\lambda_D + \lambda_{D_-})$, where $D = \{(x, x) : x \in X\}$ and $D_- = \{(x, -x) : x \in X\}$. The ergodic components of ν are λ_D and λ_{D_-}.

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This example shows that for the relatively independent joining over a factor, we have to take ergodic components in order to apply Theorem 1.3. However, in general, we also have to take ergodic components as in Theorem 1.3 with respect to a finite index subgroup to obtain measures invariant under translation by elements of certain subgroups.

LEMMA 9.4. *The set*

$$X_1' = \{x' \in X_1 : \psi_1(t) = \psi_1(t + x') \text{ for } \lambda_{X_1}\text{-a.e. } t \in X_1\}$$

is a closed α-invariant subgroup of X_1. Furthermore, ψ_1 descends (on the complement of a nullset) to a well-defined factor map from X_1/X_1' to Y.

Proof. Since $\mathcal{A}_1 = \psi_1^{-1}\mathcal{B}_Y$ is countably generated, it is easily seen that

$$X_1 = \{x' \in X_1 : \lambda_{X_1}(A \triangle (A + x')) = 0 \text{ for all } A \in \mathcal{A}_1\}.$$

Furthermore, it is well known that $\lambda_{X_1}(A \triangle (A + x'))$ depends continuously on $x' \in X$ for any $A \in \mathcal{B}_{X_1}$. From this, it follows that X_1' is a closed subgroup. Moreover, X_1' is α_1-invariant since $\mathcal{A}_1$ is invariant (equivalently, since ψ_1 is a factor map).

Note that $\psi_1(t) = \psi_1(t + x')$ for $\lambda_{X_1} \times \lambda_{X_1'}$-a.e. $(t, x') \in X_1 \times X_1'$ by definition of X_1' and Fubini's theorem. Therefore, for λ_{X_1}-a.e. $t \in X_1$, we know that $\psi_1(t) = \psi_1(t + x')$ for $\lambda_{X_1'}$-a.e. $x' \in X_1'$. Therefore, $\psi_1(t)$ is (outside some nullset) independent of the representative $t \in X_1$ of the coset $t + X_1'$, which implies the second part of the lemma. □

By Lemma 9.4, we can replace X_1 by X_1/X_1' and similarly X_2 by X_2/X_2' for the remainder of the proof. Hence we assume that

$$X_1' = \{x' \in X_1 : \psi_1(t) = \psi_1(t + x') \text{ for } \lambda_{X_1}\text{-a.e. } t \in X_1\} = \{0\} \tag{9.4}$$

and similarly for X_2, and will show the existence of finite groups Γ_1 and Γ_2 of affine automorphisms of X_1 and X_2, and the existence of the affine isomorphism $\Phi : X_1 \to X_2$ satisfying (9.1).

Since Theorem 1.3 assumes ergodicity, we need to study the ergodic components of the relatively independent joining. The following easy consequence of the definition of the ergodic decomposition gives all the properties we will need for the ergodic components.

LEMMA 9.5. *Almost every ergodic component μ of the relatively independent joining ν of X_1 and X_2 over Y is still a joining between α_1 and α_2 that satisfies $\mu(D_Y) = 1$, where D_Y is defined as in (9.3).*

In the next lemma, we analyze what translation invariance for a measure μ as above tells us about the factor maps.

LEMMA 9.6. *Let μ be a measure on $X_1 \times X_2$ that projects to the Haar measure λ_{X_1} respectively λ_{X_2} and satisfies $\mu(D_Y) = 1$ where D_Y is defined in (9.3). Suppose μ is translation invariant under elements of $G \subset X_1 \times X_2$ and that G projects surjectively to X_1 and X_2. Then G is the graph of a group isomorphism $\phi : X_1 \to X_2$. Moreover, there exists some $w_\phi \in X_2$ such that the affine isomorphism $\Phi(x) = \phi(x) + w_\phi$ satisfies*

$$\psi_1(x) = \psi_2(\Phi(x)) \quad \text{for a.e. } x \in X_1 \tag{9.5}$$

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and that μ is the Haar measure of the graph of Φ. The element $w_\phi \in X_2$ is uniquely determined by ϕ and (9.5).

Proof. Suppose $(x', 0) \in G$, $C \in \mathcal{B}_Y$, and $A = \psi_1^{-1}C \in \mathcal{A}_1$. Then $A \times X_2 = X_1 \times \psi_2^{-1}C$ (modulo μ) and

$$\begin{aligned}(A + x') \times X_2 &= A \times X_2 + (x', 0)\\ &= X_1 \times \psi_2^{-1}C + (x', 0) \quad \text{(modulo } \mu)\\ &= X_1 \times \psi_2^{-1}C\\ &= A \times X_2 \qquad\qquad \text{(modulo } \mu),\end{aligned}$$

where we used invariance of μ under translation by $(x', 0)$ in the transition from the first to the second line. It follows that

$$\lambda_{X_1}((A + x') \triangle A) = \mu(((A + x') \times X_2) \triangle (A \times X_2)) = 0$$

for any $A \in \mathcal{A}_1$ and so $x' \in X_1'$. Therefore, $x' = 0$ by assumption (9.4). The same holds for elements of the form $(0, x') \in G$. Together with our assumption that G projects onto X_1 and onto X_2, this implies that G is the graph of a group isomorphisms $\phi : X_1 \to X_2$.

We show next that w_ϕ is uniquely determined. So suppose $w, w' \in X_2$ are such that (9.5) holds independently of whether Φ is defined using w or using w'. Let $v = \phi^{-1}(w - w')$. Then

$$\psi_1(x) = \psi_2(\phi(x) + w) = \psi_2(\phi(x + v) + w') = \psi_1(x + v)$$

for λ_{X_1}-a.e. $x \in X$. Therefore, $v \in X_1 = \{0\}$ (by (9.4) again) and $w = w'$.

It remains to show the existence of w_ϕ and that μ is the Haar measure of the graph of Φ. Since μ is invariant under translation by elements of G and since $\mu(D_Y) = 1$, it follows that

$$\mu(D_Y - (x, \phi(x))) = 1 \quad \text{for every } x \in X_1.$$

In other words, we know for $\mu \times \lambda_{X_1}$-a.e. $((z_1, z_2), x) \in (X_1 \times X_2) \times X_1$ that $(z_1, z_2) + (x, \phi(x)) \in D_Y$. By Fubini's theorem, this shows for μ-a.e. $(z_1, z_2) \in X_1 \times X_2$ that

$$(z_1, z_2) + (x, \phi(x)) \in D_Y \quad \text{for } \lambda_{X_1}\text{-a.e. } x \in X_1.$$

However, by definition of D_Y, this is equivalent to

$$\psi_1(z_1 + x) = \psi_2(z_2 + \phi(x)) \quad \text{for } \lambda_{X_1}\text{-a.e. } x \in X_1.$$

We define $w = z_2 - \phi(z_1)$, then (9.5) follows for $\Phi(x) = \phi(x) + w$. However, by uniqueness $w = w_\phi$ is independent of (z_1, z_2). Therefore, $z_2 = \phi(z_1) + w_\phi$ and (z_1, z_2) belong to the graph of $\Phi(x) = \phi(x) + w_\phi$. This holds for μ-a.e. (z_1, z_2), and together with invariance of μ under translation by elements of G, it follows that μ is the Haar measure of the graph of Φ. □

Finally, we can describe the structure of the relatively independent joining.

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LEMMA 9.7. *The relatively independent joining is a convex combination*

$$\nu = \sum_{j\in J} a_j \lambda_{\Phi_j} \quad \text{with } a_j > 0$$

of at most countably many Haar measures λ_{Φ_j} on graphs of affine isomorphisms Φ_j that satisfy (9.5).

Proof. Let μ be an ergodic component of ν as in Lemma 9.5. By Theorem 1.3, there exist $\mu_1, \dots, \mu_M$ and $G_1, \dots, G_M$, such that $\mu = 1/M \sum_{i=1}^{M} \mu_i$ and μ_i is invariant under translation by elements of G_i. By Proposition 8.1, each G_i projects surjectively to X_1 and X_2. Therefore, Lemma 9.6 shows that μ_i is the Haar measure λ_Φ of the graph of an affine isomorphism $\Phi(x) = \phi(x) + w_\phi$ that satisfies (9.5).

We claim that there are at most countably many group isomorphisms $\phi : X_1 \to X_2$. Since ϕ uniquely determines w_ϕ, Φ, μ_i, and by ergodicity also μ, the above claim implies that there are at most countably many ergodic components μ, each of which is a convex combinations of Haar measures.

To prove the claim, it is enough to notice that every ϕ as above is uniquely determined by its dual $\hat{\phi} : \hat{X}_2 \to \hat{X}_1$ that is the restriction of a $\mathbb{Q}$-linear map from $\mathbb{Q}^{n_2} \supseteq \hat{X}_2$ to $\mathbb{Q}^{n_1} \supseteq \hat{X}_1$. □

Proof of Corollary 1.5. It is well known that every invariant σ-algebra $\mathcal{A}$ can be realized as $\psi^{-1}\mathcal{B}_Y$ for some factor map $\psi : X \to Y$ and some $\mathbb{Z}^d$-action on a standard Borel probability space $(Y, \mathcal{B}_Y, \rho)$. Recall that by Lemma 9.4, we may assume that (9.4) holds.

Let ν be the relatively independent joining of λ_X and λ_X over (Y, ρ). Let

$$\Gamma = \{\gamma : \gamma \text{ is an affine automorphism such that } \psi(\gamma(x)) = \psi(x) \text{ for } \lambda_X\text{-a.e. } x \in X\}. \tag{9.6}$$

Then Γ is a group normalized by α that is at most countable (see proof of Lemma 9.7) and satisfies

$$\nu = \sum_{\gamma\in\Gamma} a_\gamma \lambda_\gamma \quad \text{with } a_\gamma \geq 0$$

by Lemma 9.7. From the construction (9.2) of the relatively independent joining, it follows that the conditional measures of ν with respect to the σ-algebra $\mathcal{C} = \mathcal{B}_X \times \{\emptyset, X\}$ are

$$\nu^{\mathcal{C}}_{(x,x')} = \delta_x \times (\lambda_X)^{\mathcal{A}}_x \quad \text{for } \nu\text{-a.e. } (x, x') \in X \times X.$$

However, from the above decomposition of ν, it is also easy to calculate the conditional measures of ν with respect to $\mathcal{C}$, which shows that

$$(\lambda_X)^{\mathcal{A}}_x = \sum_{\gamma\in\Gamma} a_\gamma \delta_{\gamma(x)} \quad \lambda_X\text{-almost everywhere.}$$

Since every $\gamma_0 \in \Gamma$ preserves λ_X and $\mathcal{A}$, it follows that $(\lambda_X)^{\mathcal{A}}_{\gamma_0 x} = (\gamma_0)_*(\lambda_X)^{\mathcal{A}}_x$ almost everywhere, and hence

$$(\lambda_X)^{\mathcal{A}}_{\gamma_0 x} = (\gamma_0)_* \sum_{\gamma\in\Gamma} a_\gamma \delta_{\gamma(x)} = \sum_{\gamma\in\Gamma} a_\gamma \delta_{\gamma_0(\gamma(x))} \quad \text{almost everywhere.}$$

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However, by definition γ_0 also preserves a.e. atom of $\mathcal{A}$ and so $(\lambda_X)^{\mathcal{A}}_{\gamma_0 x} = (\lambda_X)^{\mathcal{A}}_x$ almost everywhere, hence

$$(\lambda_X)^{\mathcal{A}}_{\gamma_0 x} = \sum_{\gamma\in\Gamma} a_\gamma \delta_{\gamma(x)} \quad \lambda_X\text{-almost everywhere.}$$

By comparing the above two displayed formulæ we conclude that Γ is finite, and all the coefficients a_{γ_0} are equal to each other, that is to say

$$(\lambda_X)^{\mathcal{A}}_x = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\delta_{\gamma(x)}.$$

Since the conditional measures determine the σ-algebra (modulo λ_X), the corollary follows. □

Proof of Corollary 9.1. We already constructed X_1' and X_2' in Lemma 9.4. Applying Corollary 1.5, we find Γ_1 and Γ_2. Finally, let $\Phi : X_1 \to X_2$ be an affine isomorphism as in (9.5) that exists by Lemma 9.7. Then $\Phi^{-1}\circ\gamma_2\circ\Phi$ belongs to Γ_1 (defined as in (9.6)) for any $\gamma_2 \in \Gamma_2$. By symmetry, $\Gamma_2\circ\Phi = \Phi\circ\Gamma_1$ which concludes the proof. □

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