STOCHASTIC INTEREST RATES AND AUTOREGRESSIVE INTEGRATED MOVING AVERAGE PROCESSES

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ABSTRACT

A practical method is developed for computing moments of insurance functions when interest rates are assumed to follow an autoregressive integrated moving average process.

KEYWORDS

ARIMA (p, d, q)-processes; stochastic interest rates; moments of insurance functions.

1. INTRODUCTION

In most of the insurance literature the theory of life contingencies is developed in a deterministic way. This means that mortality happens according to an a priori known mortality table and that the interest rate is assumed to have a constant value. Nevertheless, the traditional theory of life contingencies implicitly deals with the stochastic nature of mortality and interest rates in that conservative assumptions are taken.

A first step forward was to consider the time until decrement as a random variable, while the interest rate was assumed to be constant. This approach is followed in BOWERS et al. (1987). This (as one could call) "semi-stochastic" approach contains the traditional theory in that most actuarial functions can be considered as the expected values of certain stochastic functions.

It is only since about 1970 that there has been interest in actuarial models which consider both the time until death and the investment rate of return as random variables.

BOYLE (1976) includes the stochastic nature of interest rates in assuming that the force of interest is generated by a white noise series, that is forces of interest in the successive years are normally distributed and uncorrelated.

In the approach of POLLARD (1971) the force of interest in a year is related to the force of interest in the preceding years by using an autoregressive process of order two.

PANJER and BELLHOUSE (1980) and BELLHOUSE and PANJER (1981) develop a general theory including continuous and discrete models. The theory is further worked out for unconditional and conditional autoregressive processes of order one and two.

GIACCOTTO (1986) develops an algorithm for evaluating present value functions when interest rates are assumed to follow an ARIMA $(p, 0, q)$ or an ARIMA $(p, 1, q)$ process.

The goal of this study is to state a methodology for computing in an efficient manner present value functions when the force of interest evolves according to an autoregressive integrated moving average process of order (p, d, q). As will be seen, the method developed here will require less computing time than Giaccotto's method for autoregressive integrated moving average processes of order $(p, 0, q)$ or $(p, 1, q)$.

It should be remarked that we assume that mortality and interest rates posses a certain stochastic nature and that only accidental fluctuations in this mortality and interest rates are considered. Other fluctuations due to mortality improvement, underwriting practice, the choice of a wrong interest model, investment strategy and so on are not considered here.

2. GENERAL THEORY

The theory developed in this section is mainly based on the work of PANJER and BELLHOUSE (1980) and BELLHOUSE and PANJER (1981).

Let D_t be the stochastic variable denoting the discounted value of one dollar payable in t years ($t = 0, 1, 2, \ldots$). The stochastic variable X_t defined by

$$D_t = \exp(-X_t) \qquad t = 0, 1, 2, \ldots \tag{1}$$

can be interpreted as the force of interest over the first t years.

If δ_i is the force of interest in the i-th year ($i = 1, 2, \ldots$), then

$$X_0 = 0$$

$$X_t = \sum_{i=1}^{t} \delta_i \qquad t = 1, 2, \ldots \tag{2}$$

It is assumed that X_t is normally distributed with mean $\mu(t)$ and variance-covariance function $a(t, s)$. The variance of X_t is equal to $a(t, t)$ and is denoted by $\sigma^2(t)$.

It is immediately seen that $E[D_t^k]$ and $E[D_t^k D_s^l]$ are the moment generating functions of the normal distributed variables kX_t and $(kX_t + lX_s)$ calculated for the value (-1). So one finds that

$$E[D_t^k] = \exp\left[-k\mu(t) + \frac{k^2}{2}\sigma^2(t)\right] \qquad t, k \geq 1 \tag{3}$$

and

$$E[D_t^k D_s^l] = \exp\left[-k\mu(t) - l\mu(s) + \frac{k^2}{2}\sigma^2(t) + \frac{l^2}{2}\sigma^2(s) + kla(t, s)\right] \qquad t, s, k, l \geq 1 \tag{4}$$

PANJER and BELLHOUSE (1980) proved that when the X_t are normally distributed, the moments of and the correlation coefficients between interest, annuity and insurance functions depend upon $E[D_t^k]$ and $E[D_t^k D_s^l]$. For a whole life term insurance, for instance, the moments of the stochastic variable $\underset{\sim}{A}_x$ are given by

$$(5) \qquad E[\underset{\sim}{A}_x^k] = \sum_{t=1}^{\infty} {}_{t-1|}q_x \, E[D_t^k]$$

The second moment for the life annuity $\underset{\sim}{a}_x$ is given by

$$(6) \qquad E[\underset{\sim}{a}_x^2] = \sum_{t=1}^{\infty} {}_{t|}q_x \sum_{r=1}^{t} \sum_{s=1}^{t} E[D_r D_s]$$

Given a model for the yearly forces of interest δ_t, the problem is to find $\mu(t)$, $\sigma^2(t)$ and $a(t, s)$ for $t, s \geq 1$.

3. AUTOREGRESSIVE INTEGRATED MOVING AVERAGE PROCESSES

Assume that the stochastic model governing future forces of interest δ_t ($t = 1, 2, \ldots$) belongs to the class of ARIMA (p, d, q)-processes. Then δ_t is generated by the stochastic difference equation

$$(7) \qquad \nabla^d \delta_t = \mu + b_1(\nabla^d \delta_{t-1} - \mu) + b_2(\nabla^d \delta_{t-2} - \mu) + \ldots + b_p(\nabla^d \delta_{t-p} - \mu) + \xi_t - c_1 \xi_{t-1} - c_2 \xi_{t-2} \ldots - c_q \xi_{t-q}$$

where ∇^d stand for the d-th backward difference operator:

$$(8) \qquad \nabla^1 \delta_t \equiv \nabla \delta_t = \delta_t - \delta_{t-1}$$

$$(9) \qquad \nabla^d \delta_t = \nabla(\nabla^{d-1} \delta_t) \qquad d = 2, 3, \ldots$$

By convention we set $\nabla^0 \delta_t = \delta_t$. Further ξ_t is a normal white noise series with mean zero and variance σ^2. Equation (7) can also be written as

$$(10) \qquad \nabla^d \delta_t = a + b_1 \nabla^d \delta_{t-1} + \ldots + b_p \nabla^d \delta_{t-p} + \xi_t - c_1 \xi_{t-1} - \ldots - c_q \xi_{t-q}$$

with a given by

$$(11) \qquad a = \mu(1 - \sum_{i=1}^{p} b_i)$$

Equation (7) indicates that the process describing δ_t will not necessary be stationary. This means that the force of interest δ_t will not necessary have a constant unconditional mean, variance and autocovariance with any δ_{t-k} for $t \neq k$. The d-th difference of δ_t however follows a stationary autoregressive moving average process. This means that the series describing the interest rate exhibits homogeneity in the sense that, apart from local level, or perhaps local level and trend, one part of the series behaves much like any other part.

In what follows it will implicitly be assumed that the past $(p+d)$ forces of interest $\delta_0, \delta_{-1}, \ldots, \delta_{1-p-d}$ and the past q random disturbances $\xi_0, \ldots, \xi_{1-q}$ are known. Means, variances and covariances will always be considered as conditional on $\delta_0, \delta_{-1}, \ldots, \delta_{1-p-d}, \xi_0, \xi_{-1}, \ldots, \xi_{1-q}$. Remark that if δ_t follows an ARIMA (p, d, q)-process then the X_t given by (2) are normally distributed so that the theory of section 2 can be used.

The variable Y_t is defined as

(12) $$Y_t = \delta_{1-p-d} + \delta_{2-p-d} + \ldots + \delta_t \qquad t \geq 1-p-d$$

Further we set

(13) $$Y_{-p-d} = 0$$

It follows immediately that

(14) $$\delta_t = Y_t - Y_{t-1} \qquad t \geq 1-p-d$$

So if δ_t follows an ARIMA (p, d, q)-process given by (10) with $\delta_0, \ldots, \delta_{1-p-d}, \xi_0, \ldots, \xi_{1-q}$ known then Y_t follows an ARIMA $(p, d+1, q)$-process given by

(15) $$\nabla^{d+1} Y_t = a + b_1 \nabla^{d+1} Y_{t-1} + \ldots + b_p \nabla^{d+1} Y_{t-p} + \xi_t - c_1 \xi_{t-1} - \ldots - c_q \xi_{t-q}$$

with $Y_{-p-d}, Y_{1-p-d}, \ldots, Y_0$ and $\xi_0, \xi_{-1}, \ldots, \xi_{1-q}$ known.

Now it is easy to see that the ARIMA $(p, d+1, q)$-process describing Y_t can be written as an ARIMA $(l, 0, q)$-process with $l = p+d+1$:

(16) $$Y_t = a + \phi_1 Y_{t-1} + \ldots + \phi_l Y_{t-l} + \xi_t - c_1 \xi_{t-1} - \ldots - c_q \xi_{t-q}$$

with $\phi_1, \phi_2, \ldots, \phi_l$ suitable functions of $b_1, \ldots, b_p$.

Examples

(1) If δ_t follows an ARIMA $(p, 0, q)$-process then

(17) $$\delta_t = \mu + b_1(\delta_{t-1} - \mu) + \ldots + b_p(\delta_{t-p} - \mu) + \xi_t - c_1 \xi_{t-1} - \ldots - c_q \xi_{t-q}$$

Y_t can then be written as an ARIMA $(p+1, 0, q)$-process given by

(18) $$Y_t = a + \phi_1 Y_{t-1} + \ldots + \phi_{p+1} Y_{t-p-1} + \xi_t - c_1 \xi_{t-1} - \ldots - c_q \xi_{t-q}$$

with

(19) $$a = \mu(1 - \sum_{i=1}^{p} b_i)$$

and

(20) $$\phi_i = b_i - b_{i-1} \qquad i = 1, \ldots, p+1$$

with $b_0 = -1$ and $b_{p+1} = 0$

(2) If δ_t follows an ARIMA $(p, 1, q)$-process then

(21) $\nabla\delta_t = \mu + b_1(\nabla\delta_{t-1} - \mu) + \ldots + b_p(\nabla\delta_{t-p} - \mu) + \xi_t - c_1\xi_{t-1} - \ldots - c_q\xi_{t-q}$

Y_t can then be written as an ARIMA $(p+2, 0, q)$-process given by

(22) $Y_t = a + \phi_1 Y_{t-1} + \ldots + \phi_{p+2} Y_{t-p-2} + \xi_t - c_1\xi_{t-1} - \ldots - c_q\xi_{t-q} \quad t \geq 1$

with

(23) $$a = \mu(1 - \sum_{i=1}^{p} b_i)$$

and

(24) $$\phi_i = b_i - 2b_{i-1} + b_{i-2} \qquad i = 1, \ldots, p+2$$

with $b_{-1} = b_{p+1} = b_{p+2} = 0$ and $b_0 = -1$

In the next lemma we derive an expression for the Y_t in terms of known values plus a function of future error terms ξ_t.

Lemma 1

Assume that Y_t moves according to an ARIMA $(l, 0, q)$-process given by (16) and with $Y_0, Y_{-1}, \ldots, Y_{1-l}$ and $\xi_0, \xi_{-1}, \ldots, \xi_{1-q}$ known. The Y_t can be written as

(25) $$Y_t = \sum_{i=1}^{l} Y_{i-l} \sum_{j=\max(0,\, i-t)}^{i-1} \phi_{l-j} a_{j-i+t} - \sum_{i=1}^{q} \xi_{i-q} \sum_{j=\max(0,\, i-t)}^{i-1} c_{q-j} a_{j-i+t} + a \sum_{i=0}^{t-1} a_i + \sum_{i=0}^{t-1} \beta_i \xi_{t-i} \quad t \geq 1$$

where the coefficients a_i and β_i are given by

(26) $$a_0 = 1, \qquad \beta_0 = 1$$

(27) $$a_i = \sum_{j=1}^{\min(i,\, l)} \phi_j a_{i-j} \qquad i \geq 1$$

(28) $$\beta_i = a_i - \sum_{j=1}^{\min(i,\, q)} c_j a_{i-j} \qquad i \geq 1$$

Proof

For arbitrary constants a_i $(i = 0, 1, \ldots, t-1)$ we find for $t \geq 1$

$$\sum_{i=0}^{t-1} a_i Y_{t-i} = \sum_{j=1}^{l} \phi_j \sum_{i=j}^{t+j-1} a_{i-j} Y_{t-i} - \sum_{j=1}^{q} c_j \sum_{i=j}^{t+j-1} a_{i-j}\xi_{t-i} + \sum_{i=0}^{t-1} (a + \xi_{t-i}) a_i$$

By interchanging the order of summation in the second member of this equation and by using the a_i and β_i defined in (26), (27) and (28) we find

$$Y_t = \sum_{i=t}^{t+l-1} Y_{t-i} \sum_{j=i-t+1}^{\min(i,l)} \phi_j a_{i-j} - \sum_{i=t}^{t+q-1} \xi_{t-i} \sum_{j=i-t+1}^{\min(i,q)} c_j a_{i-j}$$
$$+ \text{a} \sum_{i=0}^{t-1} a_i + \sum_{i=0}^{t-1} \beta_i \xi_{t-i}$$

After some straightforward calculation (25) is obtained.

Remark that the first, the second and the thirth term in the right member of (25) are constants while the fourth term is stochastic.

In the following theorem expressions are derived for computing $\mu(t)$, $\sigma^2(t)$ and $a(t, s)$.

Theorem 1

If Y_t follows an ARIMA $(l, 0, q)$-process given by (16) then $\mu(t)$, $\sigma^2(t)$ and $a(t, s)$ can be computed by

(29) $$\mu(t) = a - Y_0(1 - \sum_{i=1}^{l} \phi_i) + \sum_{i=1}^{l} \phi_i \mu(t-i) - \sum_{i=1}^{q} c_i \eta(t-i) \qquad t \geq 1$$

where $\mu(0) = 0$ and $\mu(-i) = -(\delta_0 + \ldots + \delta_{1-i}) \quad i = 1, \ldots, l-1$

and $$\eta(i) = \begin{cases} \xi_i & i \leq 0 \\ 0 & i > 0 \end{cases}$$

(30) $$\sigma^2(t) = \sigma^2 \sum_{i=0}^{t-1} \beta_i^2 = \sigma^2(t-1) + \beta_{t-1}^2 \qquad t \geq 1$$

with $\sigma^2(0) = 0$ and the β_i defined in (26), (27) and (28).

(31) $$a(t, s) = \sigma^2 \sum_{i=1}^{s} \beta_{t-i} \beta_{s-i} \qquad t > s \geq 1$$

Proof

From (2), (12) and (16) we obtain

$$X_t = -Y_0 + a + \phi_1 Y_{t-1} + \ldots + \phi_l Y_{t-l} + \xi_t - c_1 \xi_{t-1} - \ldots - c_q \xi_{t-q} \qquad t \geq 1$$

Taking the expected value of both members gives (29).
(30) and (31) follow immediately from (25).

The results obtained in lemma 1 and theorem 1 become much simpler if Y_t follows an ARIMA $(l, 0, 0)$-process. The expressions to compute $\mu(t)$, $\sigma^2(t)$ and $a(t, s)$ for this case are stated in the following theorem.

Theorem 2

If Y_t follows an ARIMA $(l, 0, 0)$-process given by (16) with $c_1 = c_2 = \ldots = c_q = 0$ then $\mu(t)$, $\sigma^2(t)$ and $a(t, s)$ can be computed by

$$(32) \qquad \mu(t) = a - Y_0 (1 - \sum_{i=1}^{l} \phi_i) + \sum_{i=1}^{l} \phi_i \mu(t-i) \qquad t \geq 1$$

where $\mu(0) = 0$ and $\mu(-i) = -(\delta_0 + \ldots \delta_{1-i}) \qquad i = 1, \ldots, l-1$

$$(33) \qquad \sigma^2(t) = \sigma^2 \sum_{i=0}^{t-1} a_i^2$$

with $\sigma^2(0) = 0$ and the a_i defined in (26) and (27)

$$(34) \qquad a(t, s) = \sigma^2 \sum_{i=1}^{s} a_{t-i} a_{s-i} \qquad t > s \geq 1$$

The proof follows immediately from theorem 1 by deleting the terms in c_i $(i = 1, \ldots, q)$.

4. REMARKS

The method described by GIACCOTTO (1986) for ARIMA $(p, 0, q)$- and ARIMA $(p, 1, q)$-processes requires for the computation of $\sigma^2(t)$ values of $x_i(t)$ and $y_i(t)$ $(i = 1, \ldots, t)$, which can be computed recursively but that depend on t. In the method developed here for computing $\sigma^2(t)$, the algorithm is written so that the α_i- and β_i-values are independent of t.

We remark from theorem 1 and 2 that $\sigma^2(t)$ and $a(t, s)$ are independent of the past forces of interest $\delta_0, \delta_{-1}, \ldots, \delta_{1-l}$. So it follows that when the same interest rate model is used from year to year with only the past l forces of interest and the past q disturbances changing, the $\sigma^2(t)$ and $a(t, s)$ remain the same. Only the $\mu(t)$ will have to be recomputed every year.

5. EXAMPLE

To use our results the following procedure should be followed:

1) Choose an ARIMA (p, d, q) interest rate model and estimate the parameters involved. (see e.g. BOX and JENKINS (1970)).
2) Write Y_t as an ARIMA $(p+d+1, 0, q)$-process.
3) Compute the α_i's and the β_i's.
4) Compute $\mu(t)$, $\sigma^2(t)$, $a(t, s)$.
5) Compute the moments of actuarial functions.

To illustrate the procedure assume that we have the following model for the interest rate:

$$\delta_t = 0.08 + 0.6(\delta_{t-1} - 0.08) - 0.3(\delta_{t-2} - 0.08) + \xi t \qquad t \geq 1$$

https://doi.org/10.1017/S0515036100008618 Published online by Cambridge University Press

where ξ_t is a white noise series with variance 0.0016 and $\delta_0 = 0.06$ and $\delta_{-1} = 0.07$.

Using (18), (19) and (20) Y_t can be written as

$$Y_t = 0.056 + 1.6\, Y_{t-1} - 0.9\, Y_{t-2} + 0.3\, Y_{t-3} + \xi_t \qquad t \geq 1$$

The a_t, $\mu(t)$, $\sigma^2(t)$ and $a(t,s)$ can then be computed by using theorem 2 and formula (26) and (27).

In table 1 a_t, $\mu(t)$, $\sigma^2(t)$, $E[D_t]$ and Var $[D_t]$ are given for $t = 0, 1, \ldots, 5$. In the last column the discounted value of 1 \$ payable in t years computed with a constant force of interest equal to the unconditional expected value of δ_t is given. In the example described here the stochastic approach leads to higher single premiums. This fact could be expected by observing δ_0 and δ_{-1}.

TABLE 1

MEAN AND VARIANCE OF A PAYMENT OF 1 \$ DUE IN t YEARS

t	a_t	$\mu(t)$	$\sigma^2(t)$	$E[D_t]$	Var $[D_t]$	$\exp(-0.08\,t)$
0	1	0	0	1	0	1
1	1.6000	0.0710	0.0016	0.9322	0.0014	0.9231
2	1.6600	0.1516	0.0057	0.8618	0.0042	0.8521
3	1.5160	0.2347	0.0101	0.7948	0.0064	0.7866
4	1.4116	0.3163	0.0138	0.7339	0.0075	0.7261
5		0.3964	0.0170	0.6784	0.0080	0.6703

ACKNOWLEDGEMENT

The author wishes to thank the anonymous referees for their helpful comments.

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