DISCRETE FREE PRODUCTS OF TWO COMPLEX CYCLIC MATRIX GROUPS

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1. Introduction. All 2-by-2 matrices in this paper are to be viewed as linear fractional transformations on the extended complex plane $\mathbb{C}^*$. Let L^+ and L^- be the open half-planes to the right and left, respectively, of the extended imaginary axis L. Let Λ be the set of complex 2-by-2 matrices A with real trace and determinant ± 1 such that $A(L^+) \subset L^-$. Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$, where

$$\Omega_1 = \{A \in \Lambda : \det A = 1, |\mathrm{tr}\, A| \geq 2\},$$

$$\Omega_2 = \{A \in \Lambda : \det A = 1, |\mathrm{tr}\, A| = 2\cos(\pi/q) \text{ for some integer } q > 2\},$$

$$\Omega_3 = \{A \in \Lambda : \mathrm{tr}\, A = 0\},$$

and

$$\Omega_4 = \{A \in \Lambda : \det A = -1, A^2 \in \Lambda\}.$$

Observe that the elements of $\Omega_1 \cup \Omega_4$ have infinite order, while those of $\Omega_2 \cup \Omega_3$ have finite order.

We will prove that whenever $A \in \Omega$ and $B \in \Omega$ do not both have a fixed point on L, then the group $\langle A, B^t \rangle$ is the discrete free product $\langle A \rangle * \langle B^t \rangle$, where B^t denotes the transpose of B. The case where both A and B have a fixed point on L is also discussed. We show in addition that if $A \in \Omega$ and $B \in \Omega$ are real, then $\langle A, B^t \rangle$ is the discrete free product $\langle A \rangle * \langle B^t \rangle$ if and only if for every real u,

$$\{A, B^t\} \not\subset \left\{ \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} 0 & -u \\ 1/u & 0 \end{vmatrix} \right\}.$$

The significance of these results is discussed in §3. In particular, we show there that the latter result implies that the free products in [**2**, Theorem 1] are all discrete. We thereby fill a gap in [**2**, §4], wherein the discreteness was proved only in a special case.

Discrete free products of two cyclic matrix groups have been extensively studied, along different lines. For example, Purzitsky [**5**] has given necessary and sufficient conditions for any group $\langle A, B \rangle$ generated by *real* linear fractional transformations A and B of determinant 1 to be the discrete free product $\langle A \rangle * \langle B \rangle$. (See also [**6**].)

2. Definitions and notation. Let P denote the set of matrices $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ of determinant ± 1 such that either $a, b \geq 0 \geq c, d$ or $a, b \leq 0 \leq c, d$. We reserve the notation $M = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for a matrix (or transformation) in $\Omega \cap P$. In fact, the symbol M can be used to denote an arbitrary real matrix in Ω, in view of Lemma 5. Given a transformation M, we stipulate

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without loss of generality that $a, b \geq 0 \geq c, d$. If, moreover, $c(a+d) \geq 0$, M is said to be *plussed*.

Let ω_M and ω'_M be the fixed points of M, with $|\omega_M| \leq |\omega'_M|$ if the fixed points are on the extended real line $\mathbb{R}^*$, and with $\operatorname{Im} \omega_M > 0$ otherwise. Note that ω_M is finite, because otherwise $\omega_M = \omega'_M = \infty$ and M would have the form $\begin{vmatrix} 1 & * \\ 0 & 1 \end{vmatrix}$, which contradicts the fact that $M \in P$. Note also that $\operatorname{Re} \omega_M \leq 0$, since $M(L^+) \subset L^-$. Let L_M denote the extended vertical line through ω_M. Let L_M^+ and L_M^- denote the open half-planes to the right and left of L_M, respectively.

If E is a nonsingular 2-by-2 matrix and $\mathscr{A}$ is a set of 2-by-2 matrices, write $\mathscr{A}^E = \{A^E : A \in \mathscr{A}\}$, where $A^E = EAE^{-1}$. A 2-by-2 matrix S is said to be an *L-map* if $S(L^+) = L^+$. Note that if S and S_1 are L-maps, then so are S^{-1} and $S \circ S_1$. In addition, $S(L) = L$, $S(L^-) = L^-$, $\Lambda^S = \Lambda$, and $\Omega^S = \Omega$.

Let $I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, $T_u = \begin{vmatrix} 0 & -u \\ 1/u & 0 \end{vmatrix}$, and $W_r = \begin{vmatrix} 1 & r \\ 0 & -1 \end{vmatrix}$. Write $T = T_1$ and $W = W_0$.

Given $U \subset \mathbb{C}^*$, let $\operatorname{cl} U$ denote the closure of U in $\mathbb{C}^*$ and let $\operatorname{ccl} U$ denote the complement of the closure of U in $\mathbb{C}^*$.

3. Main results. We now present the main theorems. The proofs are postponed until §6.

THEOREM 1. *Let $A, B \in \Omega$ and $C = B^W$. Suppose that A and C do not both have a fixed point on L. Then $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$.*

THEOREM 2. *Let $A, B \in \Omega$ and $C = B^W$. If A and B are real, then $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$ if and only if for every real u, $\{A, C\} \not\subset \{W, T_u\}$.*

Consider the Hecke group $\langle S_\lambda, T \rangle$ generated by $S_\lambda = \begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix}$ and $T = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$. It is well-known that when $\lambda = 2\cos(\pi/q)$ for an integer $q \geq 3$, then $\langle S_\lambda, T \rangle$ is the discrete free product $\langle TS_\lambda \rangle * \langle T \rangle$, and that when λ is a *complex* number of modulus ≥ 2, then $\langle S_\lambda, T \rangle$ is the discrete free product $\langle S_\lambda \rangle * \langle T \rangle$. Theorem 1 is sufficiently general to imply these results. For, if $\lambda = 2\cos(\pi/q)$ for an integer $q \geq 3$, apply Theorem 1 with $A = TS_\lambda$, $B = T$. If $\lambda \in \mathbb{C}$, $|\lambda| \geq 2$, apply Theorem 1 with $A = S_\lambda^G$, $B = T^G$, where $G = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$.

If, in Theorem 1, C is defined to be B' instead of B^W, the resulting statement is equivalent. To see this, we need only show that $\Omega^W = \Omega'$. Assume that for some $\tau_1 \in L^+$ and some $A \in \Lambda$, $A^{-1}(\tau_1) = \tau_2 \in \operatorname{cl}(L^+)$. Then $A(\tau_2) \in L^+$, so for some $\tau_3 \in L^+$ close to τ_2, $A(\tau_3) \in L^+$. This contradicts the fact that $A(L^+) \subset L^-$. Thus $A^{-1}(L^+) \subset L^-$ for all $A \in \Lambda$. This proves that $\Lambda = \Lambda^{-1}$. Thus $\Omega = \Omega^{-1}$; so $\Omega' = T\Omega^{-1}T = \Omega^T$. Since $TW = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ is an L-map, $\Omega^{TW} = \Omega$. Therefore, $\Omega' = \Omega^T = \Omega^W$.

Similarly, if C is defined to be B' instead of B^W in Theorem 2, the resulting statement is equivalent. Another equivalent formulation of Theorem 2 is as follows. "Let

$A, B \in \Omega^V$, where $V = 2^{-1/2}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$, and let $C = B^t$. If A and B are real, then $\langle A, C \rangle$ is the discrete free product $\langle A \rangle * \langle C \rangle$ if and only if, for every real pair r, s satisfying $s^2 - r^2 = 1$, we have $\{A, C\} \not\subset \left\{ \begin{vmatrix} r & s \\ -s & -r \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right\}$". In the notation of [**2**], it follows from [**2**, Lemma 9] and Lemma 5 below that $J \subset \Lambda^V$; hence $K \subset \Omega^V$, so Theorem 2 now immediately implies that the free products in [**2**, Theorem 1] are all discrete.

Theorem 2 is, in fact, an extension of a theorem of Newman [**4**, Theorem 15, p. 162]. For if in Theorem 2 the determinants of A and B are restricted to be 1, then Theorem 2 becomes a restatement of Newman's theorem (see Lemma 5 below).

The following simple example shows that the requirement in Theorem 2 that A and B are real cannot be dropped. Let $A = \begin{vmatrix} 1 & i \\ 0 & -1 \end{vmatrix}$ and $C = \begin{vmatrix} 1 & 0 \\ i & -1 \end{vmatrix}$. Then $A \in \Omega$, $C \in \Omega^W$, but $\langle A, C \rangle$ is not the free product $\langle A \rangle * \langle C \rangle$, since $(AC)^3 = I$.

4. Real conjugates of complex matrices. Theorem 4 (below) will enable us to focus attention on those matrices in Ω which are real. First a lemma is proved.

LEMMA 3. *If $N \in \Omega$ fixes ∞, then $N = W_\rho$ for some ρ. If $N \in \Omega$ fixes 0, then $N = W_\rho^t$ for some ρ.*

Proof. Suppose that $N(\infty) = \infty$, so $N = \begin{vmatrix} \alpha & \beta \\ 0 & \delta \end{vmatrix}$. For each $\tau \in L^+$, $N\tau = (\alpha/\delta)\tau + \beta/\delta \in L^-$. Let $\psi = \arg(\alpha/\delta)$ with $-\pi < \psi \le \pi$. Suppose that $\psi \in (-\pi, 0]$. If $\tau \in L^+$ has sufficiently large modulus and $\arg \tau$ is sufficiently close to $\pi/2$, then $N\tau \in L^+$, a contradiction. Suppose then that $\psi \in (0, \pi)$. If $\tau \in L^+$ has sufficiently large modulus and $\arg \tau$ is sufficiently close to $-\pi/2$, then $N\tau \in L^+$, a contradiction. Thus $\psi = \pi$, i.e., $\alpha/\delta < 0$.

Suppose that $\alpha \notin \mathbb{R}$. Since $\operatorname{tr} N$ is real, $\delta = \bar{\alpha}$. Since $\alpha/\bar{\alpha}$ is a negative number of modulus 1, $\alpha = -\bar{\alpha}$. Thus $N = \begin{vmatrix} \alpha & \beta \\ 0 & -\alpha \end{vmatrix}$, i.e., $N = W_\rho$ for some ρ.

Suppose that $\alpha \in \mathbb{R}$. Since $\alpha/\delta < 0$, $\det N < 0$. Thus $N \in \Omega_3$ or $N \in \Omega_4$. For all $\tau \in L^+$, $N^2\tau = (\alpha^2/\delta^2)\tau + \beta(\alpha + \delta)/\delta^2$. If $\tau \in \mathbb{R}$ is sufficiently large, then $N^2\tau \in L^+$. Thus $N^2 \notin \Lambda$, so $N \notin \Omega_4$. Therefore, $N \in \Omega_3$. It follows that $N = \begin{vmatrix} \alpha & \beta \\ 0 & -\alpha \end{vmatrix}$, i.e., $N = W_\rho$ for some ρ. This proves the first assertion.

Suppose that $N \in \Omega$ fixes 0. Then, since $TW = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ is an L-map, $N^{TW} \in \Omega$ and N^{TW} fixes ∞. By the first assertion, $N^{TW} = W_\rho$ for some ρ, so $N = W_{-\rho}^t$.

THEOREM 4. *Let $N = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \in \Omega$. Then there is an L-map S such that SNS^{-1} is real.*

Proof. Note that the fixed points ω and ω' of N lie in $\mathrm{cl}(L^-)$, since $N(L^+) \subset L^-$. Define $\omega_1 = \operatorname{Re} \omega$, $\omega_2 = \operatorname{Im} \omega$, $\omega_1' = \operatorname{Re} \omega'$, and $\omega_2' = \operatorname{Im} \omega'$.

Case 1: $\omega, \omega' \in L$. Assume that $\omega=\omega'$. If $\omega \neq \infty$, conjugate N by the L-map $\begin{vmatrix} 0 & 1 \\ 1 & -\omega \end{vmatrix}$. Thus we may assume that both fixed points of N are ∞. Then $N=\begin{vmatrix} 1 & \beta \\ 0 & 1 \end{vmatrix}$ for some β, which contradicts the fact that $N \in \Lambda$. Hence $\omega \neq \omega'$. If neither ω nor ω' is ∞, conjugate N by the L-map $\begin{vmatrix} 1 & -\omega' \\ \omega-\omega' & -\omega(\omega-\omega') \end{vmatrix}$; if one of ω or ω' is ∞, say $\omega=\infty$, then conjugate N by the L-map $\begin{vmatrix} 1 & -\omega' \\ 0 & 1 \end{vmatrix}$ (which fixes $\omega=\infty$). Thus we may assume that $\omega'=0$ and $\omega=\infty$. By Lemma 3, $N=W_\rho$ and $N=W'_\nu$ for some pair ρ, ν. Hence $N=\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$.

Case 2: $\omega' \in L$, $\omega \in L^-$. If $\omega' \neq \infty$, conjugate N by the L-map $\begin{vmatrix} 0 & 1 \\ 1 & -\omega' \end{vmatrix}$. Then we may assume that $\omega'=\infty$. We may moreover assume that $\omega=-1$, for otherwise conjugate N by the L-map $\begin{vmatrix} i & \omega_2 \\ 0 & -i\omega_1 \end{vmatrix}$ (which fixes $\omega'=\infty$). Thus $N=\begin{vmatrix} \alpha & \alpha-\delta \\ 0 & \delta \end{vmatrix}$. By Lemma 3, $N=\begin{vmatrix} 1 & \rho \\ 0 & -1 \end{vmatrix}$ for some ρ. Therefore, $N=\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$.

Case 3: $\omega, \omega' \in L^-$ and $\omega=\omega'$. We may assume that $\omega=-1$, for otherwise conjugate N by the L-map $\begin{vmatrix} i & \omega_2 \\ 0 & -i\omega_1 \end{vmatrix}$. Thus $N=\begin{vmatrix} 1-\gamma & -\gamma \\ \gamma & 1+\gamma \end{vmatrix}$. If $\gamma \in \mathbb{R}$, then N is real, so suppose that $\gamma \notin \mathbb{R}$. Let $\eta=\arg\gamma$. Let $S=\begin{vmatrix} 1 & k \\ k & 1 \end{vmatrix}$, where $k=i\tan(\eta/2)$. Then S is an L-map and $SNS^{-1}=\begin{vmatrix} 1-\gamma_1 & -\gamma_1 \\ \gamma_1 & 1+\gamma_1 \end{vmatrix}$, where $\gamma_1=|\gamma| \in \mathbb{R}$.

Case 4: $\omega, \omega' \in L^-$, $\omega \neq \omega'$. We may assume that $\omega'=-1$, otherwise conjugate N by the L-map $\begin{vmatrix} i & \omega'_2 \\ 0 & -i\omega'_1 \end{vmatrix}$. Assume that $\omega_2 \neq 0$. Let $S'=\begin{vmatrix} xi & 1 \\ 1 & xi \end{vmatrix}$, where x is a solution of the equation

$$\omega_2 x^2 + x(|\omega|^2-1) - \omega_2 = 0.$$

Since the discriminant of this quadratic equation is

$$(\omega_1^2-1)^2+\omega_2^4+2\omega_1^2\omega_2^2+2\omega_2^2,$$

it follows that $x \in \mathbb{R}$. Therefore S' is an L-map. Since $S'(-1)=-1$ and $S'(\omega) \in \mathbb{R}$ by definition of x, we see that $S'N(S')^{-1}$ fixes the two distinct real points -1 and $S'(\omega)$. It may thus be assumed without loss of generality that $\omega_2=0$, i.e., that N fixes -1 and ω, where $\omega<0$, $\omega \neq -1$. Write $N=\begin{vmatrix} \sigma/2-y & \beta \\ \gamma & \sigma/2+y \end{vmatrix}$, where $\sigma=\operatorname{tr} N$. The equalities $N(-1)=-1$ and $N(\omega)=\omega$ imply that $\beta=\gamma\omega$ and $y=(1-\omega)\gamma/2$. Since

$$\det N = \sigma^2/4 - \gamma^2(1+\omega)^2/4,$$

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we have

$$\gamma^2 = 4(\sigma^2/4 - \det N)/(1+\omega)^2.$$

Since σ and $\det N$ are real by the hypothesis $N \in \Omega$, γ is real or purely imaginary. If $\gamma \in \mathbb{R}$, then N is real, so suppose that γ is purely imaginary. Let $S = \begin{vmatrix} i/\sqrt{-\omega} & 1 \\ 1 & i\sqrt{-\omega} \end{vmatrix}$. Then S is an L-map and $SNS^{-1} = \begin{vmatrix} \sigma/2 - z & z(1-\omega)/2\omega \\ z(1-\omega)/2 & \sigma/2 + z \end{vmatrix}$, where $z = i\gamma\sqrt{-\omega}$. Since this matrix is real, the proof is complete.

5. Lemmas on real matrices in Ω.

Lemma 5. *Let* $N = \begin{vmatrix} w & x \\ y & z \end{vmatrix}$ *have real entries. Then* $N \in \Lambda$ *if and only if* $N \in P$.

Proof. Suppose that $N \in \Lambda$. It was shown in §3 that $\Lambda = \Lambda^{-1}$. Hence $N^{-1}(\infty) \in \mathrm{cl}(L^-)$, so that either $y, z \geq 0$ or $y, z \leq 0$. Further, for all $t > 0$,

$$\frac{wt + x}{yt + z} < 0.$$

It follows that $N \in P$.

Conversely, suppose that $N \in P$. Then $N^{-1}(\infty) \notin L^+$, so for all $\tau \in L^+$,

$$\operatorname{sgn} \operatorname{Re}(N(\tau)) = \operatorname{sgn}\{wy|\tau|^2 + xz + (xy + wz)\operatorname{Re} \tau\} < 0.$$

It follows that $N \in \Lambda$. This completes the proof.

The next lemma characterizes those matrices $M \in \Omega$ which map L_M onto a straight line. (Recall that M always denotes a matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \in \Omega$ with $a, b \geq 0$, and $c, d \leq 0$.)

Lemma 6. *The following are equivalent*:
(i) $\infty \in M(L_M)$;
(ii) $M = W_b$, *or both* $\operatorname{tr} M = 0$ *and* $\det M = 1$;
(iii) $M(L_M) = L_M$.

Proof. We show that (i)$\Rightarrow$(ii)$\Rightarrow$(iii) (note that (iii)$\Rightarrow$(i) is obvious).

Case 1: $\det M = 1$. Suppose that $\infty \in M(L_M)$. If $M(\infty) = \infty$, then $M = \begin{vmatrix} a & b \\ 0 & d \end{vmatrix}$, where $a > 0$ and $d < 0$. This contradicts the fact that $\det M = 1$. Hence $M(\infty) \neq \infty$, i.e., $c < 0$. Then $-d/c \in L_M$, so $-d/c = \operatorname{Re} \omega_M$. By definition of ω_M,

$$-d/c = \operatorname{Re} \omega_M = \operatorname{Re}\{(2c)^{-1}(a - d - \sqrt{(a+d)^2 - 4})\}. \tag{1}$$

It is readily seen that (1) holds if and only if $a + d = 0$. This proves (ii).

To prove that (ii)$\Rightarrow$(iii), suppose that $\operatorname{tr} M = 0$. Then $c \neq 0$ and by the second equality in (1),

$$M(\infty) = a/c = \text{Re } \omega_M = \text{Re } \omega'_M.$$

Thus M maps each of ω_M, ω'_M and ∞ into L_M, so $M(L_M) = L_M$.

Case 2: $\det M = -1$. In this case, $\omega_M, \omega'_M \in \mathbb{R}^*$. Suppose that (i) holds. Assume that $c<0$. Then $-d/c \in L_M$, so $-d/c = \text{Re } \omega_M = \omega_M$. Then

$$-d/c = \omega_M = (a - d - \sqrt{(a+d)^2+4})/2c,$$

which is impossible. Hence $c = 0$, i.e., $M(\infty) = \infty$. By Lemma 3, $M = W_b$. This proves (ii).

To prove that (ii)$\Rightarrow$(iii), note that if $M = W_b$, then M fixes the line $\{z : \text{Re } z = -b/2\} = L_M$. This completes the proof.

Define Π to be the set of (real) $M \in \Omega$ which satisfy (ii) of Lemma 6. Define D_M as follows. If $M \in \Pi$, let $D_M = L_M^-$. If $M \in \Omega_4$, let D_M be the interior of the circle $M^2(L_M)$. For all other $M \in \Omega$, let D_M be the interior of the circle $M(L_M)$. If $M \in \Omega_4$, define D'_M to be the interior of the circle $M(L_M)$. (See Figures 1 through 5.) Note that since M is real and conformal, the circles $M(L_M)$ and $M^2(L_M)$ are orthogonal to the real axis.

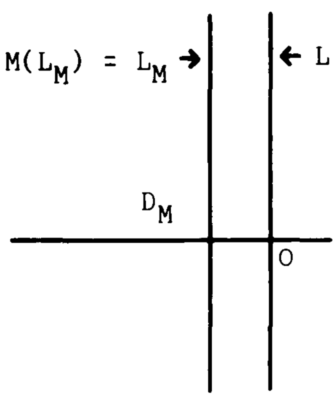

Figure 1. $M \in \Pi$

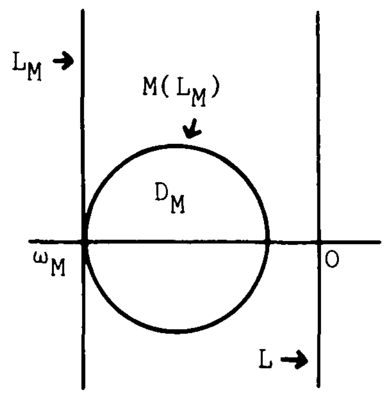

Figure 2. $M \in \Omega_1$, M plussed

Lemmas 7 through 11 (below) show that if M is plussed (recall that this means $c(a+d) \geq 0$), then $M(L_M^+) = D_M$ when $M \notin \Omega_4$, and $M(L_M^+) = D'_M$ and $M^2(L_M^+) = D_M$ when $M \in \Omega_4$. Lemmas 8 through 11 also verify that D_M and D'_M are positioned as suggested by Figures 2 through 5. We note that if $\omega_M \in L$, then $M \in \Omega_3$. For if $\omega_M = ir$, then $M = \begin{vmatrix} 0 & -r \\ 1/r & 0 \end{vmatrix}$ when $r>0$, and M is as given in Lemma 3 when $r = 0$.

LEMMA 7. *If* $M \in \Pi$, *then* $M(L_M^+) = D_M$.

Proof. First suppose that $M = W_b$. Then M maps the points $-b/2$, $-b/2+i$, ∞ to $-b/2$, $-b/2-i$, ∞, respectively. Thus M fixes L_M but reverses its orientation. Therefore $M(L_M^+) = L_M^- = D_M$.

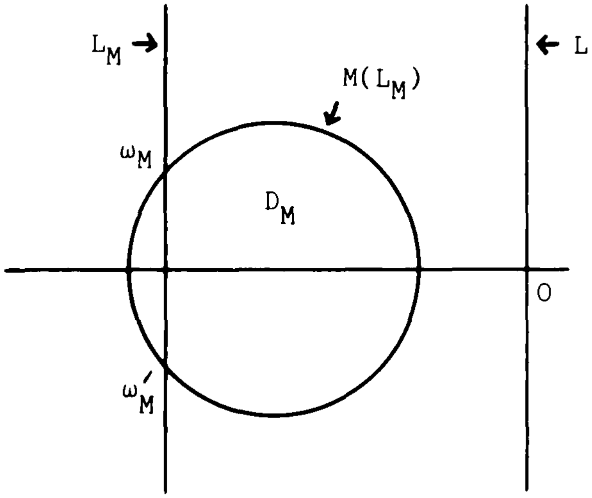

Figure 3. $M \in \Omega_2$, M plussed

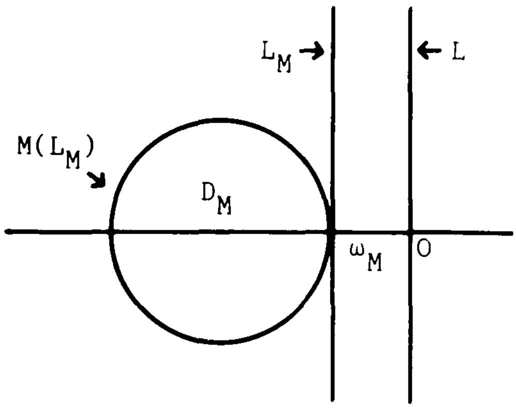

Figure 4. $M \in \Omega_3 \backslash \Pi$

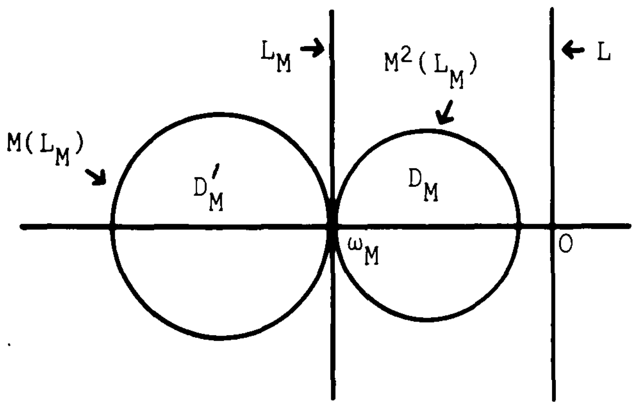

Figure 5. $M \in \Omega_4$, M plussed

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Now suppose that $M=\begin{vmatrix} a & b \\ c & -a \end{vmatrix}$ with $\det M=1$. If $c=0$, then $\det M<0$; hence $c\neq 0$. Therefore, M maps the points $(a-i)/c$, a/c, $(a+i)/c$ to $(a-i)/c$, ∞, $(a+i)/c$, respectively. Consequently, M fixes L_M but reverses its orientation, so $M(L_M^+)=L_M^-=D_M$. This completes the proof.

LEMMA 8. *If $M\in\Omega_1$ is plussed, then $D_M\subset L^-\cap L_M^+$ and $M(L_M^+)=D_M$.*

Proof. It suffices to show that $\omega_M<M(\infty)\leq 0$. For then, clearly $D_M\subset L^-\cap L_M^+$. Also, since the upper half-plane is invariant under M, the sequence $M(\omega_M)$, $M(\omega_M+i)$, $M(\infty)$ will determine a clockwise orientation on $M(L_M)$, so $M(L_M^+)=D_M$.

If $c=0$, then by Lemma 3, $M=W_\rho$ for some ρ. Then $\det M=-1$, which contradicts the fact that $M\in\Omega_1$. Hence $c<0$. As M is plussed, $\sigma=a+d<0$. Hence $\sqrt{\sigma^2-4}<-\sigma$. Dividing by $-2c$ and then adding $(a-d)/2c$, we have

$$\omega_M=(a-d-\sqrt{\sigma^2-4})/2c<a/c=M(\infty).$$

Finally, $a/c\leq 0$ because $M\in P$. This completes the proof.

LEMMA 9. *If $M\in\Omega_2$ is plussed, then $D_M\subset L^-$ and $M(L_M^+)=D_M$.*

Proof. It suffices to show that $\operatorname{Re}\omega_M<M(\infty)\leq 0$. As shown in the proof of Lemma 8, $c<0$, $a/c\leq 0$, and $a+d<0$. Thus,

$$\operatorname{Re}\omega_M=(a-d)/2c<a/c=M(\infty)\leq 0.$$

LEMMA 10. *If $M\in\Omega_3\backslash\Pi$, then $D_M\subset L_M^-$ and $M(L_M^+)=D_M$.*

Proof. It suffices to show that $M(\infty)<\omega_M$. For then, clearly $D_M\subset L_M^-$. Also, since M maps the upper half-plane to the lower half-plane, the sequence $M(\omega_M)$, $M(\omega_M+i)$, $M(\infty)$ will determine a clockwise orientation on $M(L_M)$, so $M(L_M^+)=D_M$.

If $c=0$, then $M=W_\rho$ for some ρ, which contradicts the fact that $M\notin\Pi$. Hence $c<0$ and

$$M(\infty)=a/c<(a-1)/c=\omega_M.$$

This completes the proof.

LEMMA 11. *If $M\in\Omega_4$ is plussed, then $D'_M\subset L_M^-$ $D_M\subset L^-\cap L_M^+$, $M(L_M^+)=D'_M$, and $M(D'_M)=D_M$.*

Proof. As in the proof of Lemma 10, it suffices to show that

$$M(\infty)<\omega_M<M^2(\infty)\leq 0.$$

Write $M^2=\begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix}$. Since M is plussed, $c_2=c(a+d)\geq 0$. Also, $a_2+d_2=(a+d)^2+2\geq 2$. This proves that M^2 is plussed and that $M^2\in\Omega_1$. Thus, the inequality $\omega_M<M^2(\infty)\leq 0$ follows exactly as in the proof of Lemma 8.

If $c=0$, then by Lemma 3, $M=W_\rho$, which contradicts the fact that $M\in\Omega_4$. Hence

$c<0$. Clearly,

$$-(d+a)<\sqrt{(a+d)^2+4}.$$

Dividing by $-2c$ and then adding $(a-d)/2c$, we have

$$M(\infty)=a/c<(a-d-\sqrt{(a+d)^2+4})/2c=\omega_M.$$

This completes the proof.

LEMMA 12. *Define $\mathcal{O}_M = L_M^- \cup D_M$. If M is plussed, then $M^n(\operatorname{ccl}\mathcal{O}_M)\subset\mathcal{O}_M$ for all n such that $M^n\neq I$.*

Proof. Case 1: $M\in\Omega_1$. We have $M(L_M^+)=D_M\subset L_M^+$ by Lemma 8, and hence

$$M^n(L_M^+)\subset D_M\subset\mathcal{O}_M \qquad (n>0). \tag{2}$$

Also, $M^{-1}(\operatorname{ccl} D_M)=L_M^-\subset \operatorname{ccl} D_M$, and hence

$$M^{-n}(\operatorname{ccl} D_M)\subset L_M^-\subset\mathcal{O}_M \qquad (n>0). \tag{3}$$

Since $\operatorname{ccl}\mathcal{O}_M=L_M^+\cap\operatorname{ccl} D_M$, the desired result follows from (2) and (3).

Case 2: $M\in\Omega_2$. Let q be the order of M. Fix $\tau\in\operatorname{ccl}\mathcal{O}_M$. Let $K_\tau=\{z\in\mathbb{C}^*:|t_1(z)|=|t_1(\tau)|\}$ where $t_1=\begin{vmatrix}1 & -\omega_M\\ 1 & -\omega_M'\end{vmatrix}$. If $\tau\in\mathbb{R}$, then $K_\tau=\mathbb{R}^*$. If $\tau\notin\mathbb{R}$, then K_τ is a circle through τ such that $M(K_\tau)=K_\tau$. (This can be shown using the formula $t(W(\tau))=\rho^{-2}t(\tau)$ which occurs in [**1**, p. 112].) If $\operatorname{Im}\tau>0$, then $K_\tau\subset\{z:\operatorname{Im} z>0\}$ and ω_M is inside K_τ; if $\operatorname{Im}\tau<0$, then $K_\tau\subset\{z:\operatorname{Im} z<0\}$ and ω_M' is inside K_τ. Thus $K_\tau\cap\mathcal{O}_M$ is an arc or a ray. (See Figure 6.) By Lemma 9, $M(L_M^+)=D_M$ and hence also $M^{-1}(\operatorname{ccl} D_M)=L_M^-$. Consequently, $M(\tau)\in\mathcal{O}_M$ and $M^{-1}(\tau)\in\mathcal{O}_M$. We claim that the points $\tau, M(\tau),\ldots,M^{q-1}(\tau)$ occur in that cyclic order on K_τ. To see this, choose a complex linear fractional transformation Y such that $Y(\infty)$ and $Y(0)$ are the fixed points of M. Then $X=Y^{-1}MY=\begin{vmatrix}u & 0\\ 0 & 1/u\end{vmatrix}$ for some complex $u\neq 0$. We have $\alpha=X^q(\alpha)=u^{2q}\alpha$ for all α, so $u^{2q}=1$. Since

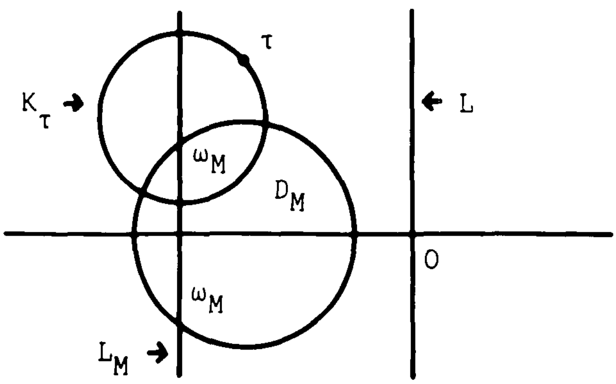

Figure 6

$M\in\Omega_2$, $u^2=\exp(\pm 2\pi i/q)$. Let $\alpha=Y^{-1}(\tau)$. The points $\alpha, X(\alpha), X^2(\alpha),\ldots, X^{q-1}(\alpha)$ occur in that cyclic order on $Y^{-1}K_\tau$. This proves the claim. It follows that the points $M(\tau),\ldots,M^{q-1}(\tau)$ all lie on $K_\tau\cap\mathcal{O}_M$. Thus $M^n(\tau)\in\mathcal{O}_M$ for all n such that $M^n\neq I$.

Case 3: $M\in\Omega_3$. By Lemmas 7 and 10,

$$M(\mathrm{ccl}\ \mathcal{O}_M)=M(L_M^+)=D_M\subset L_M^-=\mathcal{O}_M.$$

Since M is an involution, this is the desired result.

Case 4: $M\in\Omega_4$. By Lemma 11, $M^2(L_M^+)=D_M\subset L_M^+$, and hence

$$M^{2n}(L_M^+)\subset D_M\subset\mathcal{O}_M\qquad(n>0).\tag{4}$$

Also, $M^{-2}(\mathrm{ccl}\ D_M)=L_M^-\subset\mathrm{ccl}\ D_M$, and hence

$$M^{-2n}(\mathrm{ccl}\ D_M)\subset L_M^-\subset\mathcal{O}_M\qquad(n>0).\tag{5}$$

For all $n>0$, we have, by (5) and Lemma 11,

$$M^{-2n+1}(L_M^+)=M^{-2n}(D'_M)\subset M^{-2n}(\mathrm{ccl}\ D_M)\subset\mathcal{O}_M.\tag{6}$$

For all $n>0$, we have, by (4) and Lemma 11,

$$M^{2n-1}(L_M^+)\subset M^{-1}(D_M)=D'_M\subset L_M^-\subset\mathcal{O}_M.\tag{7}$$

The desired result now follows from (4), (5), (6) and (7). This completes the proof.

6. Proofs of Theorems 1 and 2.

Proof of Theorem 1. By Theorem 4, there exist L-maps Q_1 and Q_2 such that $X=Q_1^{-1}AQ_1$ and $Y=Q_2^{-1}BQ_2$ are real. Since $A\in\Omega$ and Q_1 is an L-map, $X\in\Omega$. Similarly, $Y\in\Omega$. We may assume without loss of generality that X is plussed, otherwise replace A by A^{-1}. Similarly, assume that Y is plussed. Let $Z=Y^W$ (recall that $W=\left(\begin{vmatrix}1&0\\0&-1\end{vmatrix}\right)$. Note that $Z=Q_3^{-1}CQ_3$, where $Q_3=Q_2^W$. Define $\mathcal{O}_X$ and $\mathcal{O}_Y$ as in Lemma 12. Note that $\mathcal{O}_X\subset L^-$ and $W(\mathcal{O}_Y)\subset L^+$ (see Figures 1 through 5). Since Q_1 and Q_3 are L-maps, $\mathcal{O}_1=Q_1(\mathcal{O}_X)\subset L^-$ and $\mathcal{O}_3=Q_3(W(\mathcal{O}_Y))\subset L^+$. Define $F=\mathrm{ccl}(\mathcal{O}_1\cup\mathcal{O}_3)$. Since, by hypothesis, at least one of A and C fixes no point of L, we have $F\neq\varnothing$. To see this, suppose, for example, that A fixes no point of L. Then X fixes no point of L, so $L^-\cap\mathrm{ccl}(\mathcal{O}_X)\neq\varnothing$ (see Figures 1 through 5). Thus $L^-\cap\mathrm{ccl}(\mathcal{O}_1)\neq\varnothing$, so $F\neq\varnothing$.

To show that $\langle A, C\rangle$ is the discrete free product $\langle A\rangle*\langle C\rangle$, we will show that for every nontrivial reduced word U in $\langle A, C\rangle$, $U(F)\cap F=\varnothing$. To show this, it suffices to prove that for all integers n such that $A^n\neq I$,

$$A^n(F\cup\mathcal{O}_3)\subset\mathcal{O}_1,\tag{8}$$

and that for all integers n such that $C^n\neq I$,

$$C^n(F\cup\mathcal{O}_1)\subset\mathcal{O}_3.\tag{9}$$

We give an example to illustrate why it suffices to prove (8) and (9). Let $\tau\in F$ and let U

be the reduced word $A^5C^{-3}A^4$. By (8), $\tau_1=A^4\tau\in\mathcal{O}_1$. By (9), $\tau_2=C^{-3}\tau_1\in\mathcal{O}_3$. By (8), $U(\tau)=A^5\tau_2\in\mathcal{O}_1$, so $U(\tau)\notin F$.

Putting $M=X$ in Lemma 12, we have $X^n(\mathrm{ccl}\,\mathcal{O}_X)\subset\mathcal{O}_X$ for all n such that $X^n\neq I$. Since

$$Q_1(\mathrm{ccl}\,\mathcal{O}_X)=\mathrm{ccl}(Q_1(\mathcal{O}_X))=\mathrm{ccl}(\mathcal{O}_1),$$

it follows that $A^n(\mathrm{ccl}\,\mathcal{O}_1)\subset\mathcal{O}_1$, for all n such that $A^n\neq I$. This yields (8). Putting $M=Y$ in Lemma 12, we have $WZ^nW(\mathrm{ccl}\,\mathcal{O}_Y)\subset\mathcal{O}_Y$ for all n such that $Z^n\neq I$. Hence $C^n(\mathrm{ccl}\,\mathcal{O}_3)\subset\mathcal{O}_3$ for all n such that $C^n\neq I$. This yields (9) and completes the proof.

Before proving Theorem 2, we prove the following lemma.

LEMMA 13. *Let A and C be linear fractional transformations of order 2 such that $A\neq C$ and AC is not elliptic. Then $\langle A, C\rangle$ is the discrete free product $\langle A\rangle*\langle C\rangle$.*

Proof. The reduced words in $\langle A, C\rangle$ are the words alternating in the symbols A and C, e.g., $ACACACA$. Since $A\neq C$, $AC\neq I$. Since AC is not elliptic, AC has infinite order. Hence $\langle A, C\rangle=\langle A\rangle*\langle C\rangle$. Moreover, by [**3**, Theorem 1E, p. 87], $\langle AC\rangle$ is discrete. Since $[\langle A, C\rangle:\langle AC\rangle]=2$, it follows that $\langle A, C\rangle$ is discrete. This completes the proof.

Proof of Theorem 2. If $\{A, C\}\subset\{W, T_u\}$ for some u, clearly $\langle A, C\rangle$ is not the free product of $\langle A\rangle$ and $\langle C\rangle$. Conversely, assume that for every u, $\{A, C\}\not\subset\{W, T_u\}$. It may be assumed that both A and C have a fixed point on L, otherwise $\langle A, C\rangle$ is the discrete free product $\langle A\rangle*\langle C\rangle$ by Theorem 1. As $W(L)=L$, B also has a fixed point on L. Let $M\in\{A, B\}$ (where we write $M=\begin{vmatrix} a & b \\ c & d\end{vmatrix}$ with $a, b\geq 0$ and $c, d\leq 0$). If M fixes 0 or ∞, then $M=W_b$ (with $b\geq 0$) or $M=W_c^t$ (with $c\leq 0$) by Lemma 3. If M fixes a point ui $(0<u<\infty)$, then an easy calculation shows that $M=\begin{vmatrix} a & -cu^2 \\ c & a\end{vmatrix}$. But since the diagonal elements of M cannot both be positive or negative, $a=0$. Hence $M=T_u$. Since for every u, $\{A, C\}\not\subset\{W, T_u\}$, $\langle A, C\rangle$ is one of the following groups or their transposes:

$$G_1=\langle W_r, W_s\rangle \qquad (r\leq 0, s>0), \tag{10}$$

$$G_2=\langle W_r^t, W_s\rangle \qquad (r>0, s>0), \tag{11}$$

$$G_3=\langle W_s, T_u\rangle \qquad (s\neq 0), \tag{12}$$

or

$$G_4=\langle T_u, T_v\rangle \qquad (u>v>0). \tag{13}$$

Let U_i and V_i be the first and second given generator of G_i, respectively $(i=1,2,3,4)$. It remains to show that each G_i is the discrete free product $\langle U_i\rangle*\langle V_i\rangle$. It is easily checked that U_i and V_i are involutions and that U_iV_i is not elliptic $(i=1,2,3,4)$. (Note, for example, that U_3V_3 is loxodromic.) Hence the result follows from Lemma 13 and the proof is complete.

7. Transformations with fixed points on L. We discuss the structure of $\langle A, C\rangle$ where $A\in\Omega$, $C\in\Omega^W$, and both A and C have a fixed point on L. The case where these fixed points are equal is treated in Theorem 14; the remaining case is treated in Theorem 15.

THEOREM 14. *Let $A\in\Omega$ and $C\in\Omega^W$. Suppose that A and C both fix $\omega\in L$. Then $\langle A, C\rangle$ is the discrete free product $\langle A\rangle*\langle C\rangle$ unless $A=C$.*

Proof. If $\omega\neq\infty$, conjugate A and C by the L-map $\begin{vmatrix}0 & 1\\ 1 & -\omega\end{vmatrix}$. Thus we may assume without loss of generality that $\omega=\infty$. By Lemma 3, $A=W_\rho$ and $C=W_\nu^W=W_{-\nu}$ for some pair ρ, ν. Suppose that $A\neq C$. Then $AC=\begin{vmatrix}1 & u\\ 0 & 1\end{vmatrix}$ for some $u\neq 0$. The result thus follows from Lemma 13.

THEOREM 15. *Let $A\in\Omega$ and $C\in\Omega^W$. Suppose that A fixes $\omega_A\in L$ and C fixes $\omega_C\in L$, where $\omega_A\neq\omega_C$. Then there exists an L-map S such that $A^S=W_\rho$ and $C^S=W_\nu^t$ for some pair ρ, ν. Also $\langle A, C\rangle$ is the discrete free product $\langle A\rangle*\langle C\rangle$ if and only if $A\neq C$ and AC is not elliptic.*

Proof. We begin by proving the first assertion. If $\omega_A\neq\infty$, conjugate A and C by the L-map $\begin{vmatrix}0 & 1\\ 1 & -\omega_A\end{vmatrix}$. Thus it may be assumed without loss of generality that $\omega_A=\infty$. Let S be the L-map $\begin{vmatrix}1 & -\omega_C\\ 0 & 1\end{vmatrix}$ (note that $\omega_C\neq\omega_A=\infty$). Then $A^S(\infty)=\infty$ and $C^S(0)=0$. Since $(C^S)^W\in\Omega$ and $(C^S)^W(0)=0$, it follows from Lemma 3 that $A^S=W_\rho$ for some ρ and $C^{WS}=W_{-\nu}^t$ for some ν. Hence $C^S=W_\nu^t$.

To prove the second assertion, assume that AC is elliptic. If AC has finite order, then $\langle A, C\rangle$ is not the free product $\langle A\rangle*\langle C\rangle$. If AC has infinite order, then $\langle AC\rangle$, and hence $\langle A, C\rangle$, is not discrete. Conversely, assume that $A\neq C$ and that AC is not elliptic. By the first assertion of this theorem, A and C are involutions. Thus $\langle A, C\rangle$ is the discrete free product $\langle A\rangle*\langle C\rangle$ by Lemma 13.

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