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COMPUTING EIGENVALUES OF STURM-LIOUVILLE SYSTEMS OF BESSEL TYPE

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In this paper we shall develop a new method for the computation of eigenvalues of singular Sturm-Liouville problems of the Bessel type. This new method is based on the interpolation of a boundary function in Paley-Wiener spaces. Numerical results are provided to illustrate the method.

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1. Introduction

In this paper, we shall present the theory of a new method for computing eigenvalues of a particular class of singular Sturm-Liouville problems and give numerical results of its use on different problems. The method handles problems that are perturbations of the Liouville normal form of Bessel's equations of the following form:

$$\begin{cases} Ly \equiv -y'' + [\frac{\nu^2-\frac{1}{4}}{x^2} + q(x)]y = \lambda y, \quad 0 < x \leq \pi \\ y(\pi, \lambda) = 0, \end{cases} \tag{1.1}$$

where $q(.)$ is real valued, $q(.) \in L^{1,loc}(0,\pi)$ and $\underline{\lim}_{x\to 0} x^2 q(x) \geq 1 - \nu^2$.

It is well known that the differential expression given by (1.1), generates a symmetric operator regular at $x = \pi$ and in the limit point case at the singular end $x = 0$. Thus only one boundary condition, at $x = \pi$, is needed to generate a self-adjoint extension (see [7, p. 1414]). We recall that, when $q(x) = 0$, the deficiency indices are (1,1) if $\nu \geq 1$ and (2,2) if $0 \leq \nu < 1$. It is also well known that the spectrum of the self-adjoint operator defined by (1.1) is discrete, see [7] and ([**12**, p. 210]).

Due to the importance of eigenvalues in applied mathematics, the problem of computing eigenvalues Sturm Liouville systems has attracted many researchers, [**9**, **3**, **11**, **10**, **13**, **4** and **2**], to cite a few as well as articles based on the well known and powerful codes SLEDGE, SLEIGN, and SLEIGN2.

In [**5**] a new method, based on the analytic properties of the boundary function, was introduced to tackle with success the computation of eigenvalues in the regular case. The question was whether this idea could be extended to include the singular case.

We shall see that for equations of Bessel's type, the Shannon sampling theorem, see [8] and [15], is still applicable and yields satisfactory results.

This paper is organized as follows. In Section 2, we use the semi-classical approximation to obtain asymptotics to the solution of the differential equation. In Section 3, Shannon's sampling theorem is used to interpolate the boundary function which is in a Paley-Wiener space. In Section 4, some numerical results are worked out to illustrate the theory.

2. Preliminaries

It is well known that the semi-classical approximation method may be used to estimate the solution of a differential equation for large real values of the parameter λ. We consider the solutions of the differential equation which are square integrable at $x=0$ and which, in the limit circle case, satisfy the Friedrichs boundary condition at $x=0$. These solutions are all scalar multiples of the particular solution which solves the following integral equation:

$$y(x,\mu^2)=\mu^{-l-1}u(\mu x)+\int_0^x G(x,t,\mu)q(t)y(t,\mu^2)dt \tag{2.1}$$

where

$$\lambda=\mu^2, \quad \nu=l+\frac{1}{2} \quad (l>0)$$

and

$$u(z):=\sqrt{\frac{\pi z}{2}}J_\nu(z) \quad \text{and} \quad \nu^2\geq 1-\underline{\lim}_{x\to 0}x^2q(x).$$

The Green's function is defined by ([6, p. 13])

$$G(x,t,\mu)=\frac{i(-1)^l}{2\mu}\{w(\mu x)w(-\mu t)-w(\mu t)w(-\mu x)\}$$

where $w(z):=i\exp(i\pi l)(\frac{\pi}{2}z)^{\frac{1}{2}}H^{(1)}_{l+\frac{1}{2}}(z)$, (for the definition of the Bessel functions see [1]). The following upper bound

$$|G(x,t,\mu)|<\frac{C}{|\mu|}\left(\frac{|\mu|x}{1+|\mu|x}\right)^{l+1}\left(\frac{|\mu|t}{1+|\mu|t}\right)^{-l}e^{|Im\,\mu|(x-t)} \tag{2.2}$$

is easily proved using

$$|u(z)| < C\left(\frac{|z|}{1+|z|}\right)^{l+1} e^{|Im\, z|} \tag{2.3}$$

where C is a constant, see **[14]** and **[6]**.

Thus if $\int_0^\pi x|q(x)|dx < \infty$ successive approximations, see ([**6**, (I.5.5)]) yield

$$|y(x,\mu^2)| \le C_1\left(\frac{x}{1+|\mu|x}\right)^{l+1} e^{|Im\,\mu|x} \tag{2.4}$$

which means that for each fixed x the solution $y(x,\mu^2)$ is an entire function in μ of order one and type x. Furthermore for each fixed x and as $|\mu| \to \infty$ we have

$$|y(x,\mu^2) - \mu^{-l-1}u(\mu x)| = C_2\left(\frac{x}{1+|\mu|x}\right)^{l+1} e^{|Im\,\mu|x}o(1). \tag{2.5}$$

From the above estimates the following asymptotic holds, see ([**6**, (I.5.7)])

$$y(x,\mu^2) = \frac{1}{\mu^{l+1}}\sin\left(\mu x - \frac{1}{2}l\pi\right) + \frac{1}{\mu^{l+1}}e^{|Im\,\mu|x}o(1) \quad |\mu| \to \infty.$$

In the sequel we shall need Paley Wiener spaces, which are defined by

$$PW_a := \left\{F(z) \text{ entire } : |F(z)| < Ce^{a|Im\, z|} \text{ and } \int_{-\infty}^{\infty} |F(x)|^2 dx < \infty\right\}.$$

From now on, we shall assume that $\int_0^\pi x|q(x)|dx < \infty$. In particular, this means that henceforth we are considering limit-point problems only. From (2.4), we deduce

Theorem 1. *If $\int_0^\pi x|q(x)|dx < \infty$ and $\nu > \frac{1}{2}$ then for each fixed x, $y(x,\mu^2) \in PW_x$ as a function of μ.*

The asymptotics in (2.5) will help us to approximate the solution y defined in (2.1) by the solution y_δ of the following initial value problem

$$\begin{cases} -y_\delta'' + [\frac{\nu^2 - \frac{1}{4}}{x^2} + q(x)]y_\delta = \lambda y_\delta, & \delta \le x \le \pi, \\ y_\delta(\delta,\lambda) = \sqrt{\frac{\pi}{2}}\mu^{-\nu}\sqrt{\delta}\,J_\nu(\mu\delta), & y_\delta'(\delta,\lambda) = \sqrt{\frac{\pi}{2}}\mu^{-\nu}\frac{d}{dx}[\sqrt{x}\,J_\nu(\mu x)]_{|x=\delta}. \end{cases} \tag{2.6}$$

The initial conditions are dictated by the integral equation given by (2.1) and δ is taken "small". Recall that

$$\begin{aligned} y(x,\mu^2) &= \sqrt{\frac{\pi}{2}}\mu^{-\nu}\sqrt{x}J_\nu(\mu x) + \int_0^x G(x,t,\mu)q(t)y(t,\mu^2)dt \qquad 0 \le x \le \pi, \\ y_\delta(x,\mu^2) &= \sqrt{\frac{\pi}{2}}\mu^{-\nu}\sqrt{x}J_\nu(\mu x) + \int_\delta^x G(x,t,\mu)q(t)y_\delta(t,\mu^2)dt \qquad \delta \le x \le \pi. \end{aligned} \tag{2.7}$$

This helps to avoid the singularity at $x = 0$ for computational purposes.

Theorem 2. *If* $\int_0^\pi x|q(x)|dx < \infty$, $\mathrm{Im}\mu = 0$ *and* $\nu > \frac{1}{2}$ *then*

$$|y(\pi,\mu^2) - y_\delta(\pi,\mu^2)| < C_2\frac{1}{(1+|\mu|\pi)^{l+1}}\int_0^\delta t|q(t)|dt$$

Proof. Let $\theta(x,\mu^2) := y(x,\mu^2) - y_\delta(x,\mu^2)$, for $\delta \le x \le \pi$, then

$$\theta(x,\mu^2) = \int_\delta^x G(x,t,\mu)q(t)\theta(t,\mu^2)dt + \int_0^\delta G(x,t,\mu)q(t)y(t,\mu^2)dt. \tag{2.8}$$

It is easily seen that (2.2) and (2.4) imply

$$\begin{aligned} \left|\left(\frac{x}{1+|\mu|x}\right)^{-l-1}\theta(x,\mu^2)\right| &\le C\int_\delta^x\left(\frac{t}{1+|\mu|t}\right)|q(t)|\left(\frac{t}{1+|\mu|t}\right)^{-l-1}\theta(t,\mu^2)dt \\ &\quad + CC_1\int_0^\delta\left(\frac{t}{1+|\mu|t}\right)|q(t)|dt; \end{aligned}$$

using Gronwall's inequality

$$\left|\left(\frac{x}{1+|\mu|x}\right)^{-l-1}\theta(x,\mu^2)\right| \le CC_1\int_0^\delta\left(\frac{t}{1+|\mu|t}\right)|q(t)|dt\; e^{C\int_\delta^x\left(\frac{t}{1+|\mu|t}\right)|q(t)|dt}$$

$$|\theta(\pi,\mu^2)| \le CC_1\left(\frac{\pi}{1+|\mu|\pi}\right)^{l+1}\int_0^\delta\left(\frac{t}{1+|\mu|t}\right)|q(t)|dt\; e^{C\int_\delta^\pi\left(\frac{t}{1+|\mu|t}\right)|q(t)|dt}$$

hence the result.

3. Sampling

It follows from Theorem 1 that $y(\pi,\mu^2) \in PW_\pi$, as a function of μ and so it can be recovered from a sequence only of its values at the sampling points. We now recall the well known W.S.K. (Whittaker-Shannon-Kotel'nikov) sampling Theorem ([**8**] and [**15**]).

If $f \in PW_\Omega$ then

$$f(t) = \sum_{n \in Z} f(nT) \frac{\sin[\Omega(t - nT)]}{\Omega(t - nT)}, \quad \text{where } T = \frac{\pi}{\Omega}$$

and the convergence is uniform in any compact subset of the complex plane and also in L^2_{dx}.

Clearly the eigenvalues λ_n of the problem are the zeros of

$$B(\mu) := y(\pi, \mu^2) \in PW_\pi$$

and therefore by W.S.K. theorem we obtain

$$B(\mu) = \sum_{n \in Z} B(n) \frac{\sin[\pi(\mu - n)]}{\pi(\mu - n)}.$$

Similarly let

$$B_{N,\delta}(\mu) := \sum_{|n| \le N} B_\delta(n) \frac{\sin[\pi(\mu - n)]}{\pi(\mu - n)} \tag{3.1}$$

where $B_\delta(n) = y_\delta(\pi, n^2)$.

Theorem 1. *For all $\varepsilon > 0$ and $\rho > 0$ there exist $\delta > 0$ and a positive integer N such that*

$$|B(\mu) - B_{N,\delta}(\mu)| < \varepsilon, \quad \forall \mu \in [-\rho, \rho].$$

Proof. Given $\varepsilon > 0$, and $k > 0$, it follows from Theorem 2 that there exits a $\delta > 0$ such that

$$\sup_{-k \le \mu \le k} |B(\mu) - B_\delta(\mu)| \le \frac{\varepsilon}{2}.$$

Using an estimate of truncation error, see ([**15**, p. 93]), there is an integer N such that

$$|B_\delta(\mu) - B_{N,\delta}(\mu)| \le \frac{E_l |\sin \pi\mu|}{\pi\sqrt{1 - 4^{-k}}(N+1)^l} \left[\frac{1}{\sqrt{N - \mu}} + \frac{1}{\sqrt{N + \mu}}\right] \le \frac{\varepsilon}{2} \quad \forall \mu \in [-N, N],$$

where $E_l = \int_{-\infty}^{\infty} |\mu^l B_\delta(\mu)|^2 d\mu$. Thus the result where $\rho := \min(N, k)$.

Remark 2. The truncation error is zero at the sampling points and bounded over $[-N, N]$.

Corollary 3. *(error estimate) For any given $\varepsilon > 0$, there exist a $\delta > 0$ and $N \in \mathbf{N}$ such that if $\overline{\mu}$ and $\mu_{N,\delta}$ denote the roots of B and $B_{N,\delta}$ respectively, then the following estimates hold:*

$$-\varepsilon < B_{N,\delta}(\overline{\mu}) < \varepsilon \quad \textit{and} \quad |\mu_{N,\delta} - \overline{\mu}| \leq \frac{\varepsilon}{\inf_{\mu_{N,\delta} \leq \zeta \leq \overline{\mu}} |B'_{N,\delta}(\zeta)|}.$$

Proof. From the above theorem and the fact that $\overline{\mu}$ is a zero of B, we conclude that $|B_{N,\delta}(\overline{\mu})| < \varepsilon$, which can be written as $|B_{N,\delta}(\overline{\mu}) - B_{N,\delta}(\mu_{N,\delta})| < \varepsilon$ and by the mean value theorem, $|\overline{\mu} - \mu_{N,\delta}|.|B'_{N,\delta}(\zeta)| < \varepsilon$ for some ζ between $\overline{\mu}$ and $\mu_{N,\delta}$; hence the result. Note that the zeros of $B_{N,\delta}$ are simple for ε small enough and therefore $B'_{N,\delta}(\zeta) \neq 0$ around $\overline{\mu}$.

Observe that from $-\varepsilon < B_{N,\delta}(\overline{\mu}) < \varepsilon$ and a local inversion of $B_{N,\delta}(.)$ we obtain an enclosure for the eigenvalues

$$\overline{\mu} \in (B_{N,\delta}^{-1}(\mp\varepsilon), B_{N,\delta}^{-1}(\pm\varepsilon)) \tag{3.2}$$

depending on the sign of $B'_{N,\delta}$. Recall that the coefficients $y_\delta(\pi, n^2)$ are computed numerically from (2.6) and thus the function $B_{N,\delta}(.)$ is known from (3.1). Therefore (3.2) provides a practical error bound for the obtained eigenvalues.

4. Numerical results

We shall consider the eigenvalue problems given in the examples below where $\nu > 1$ (limit point case at $x = 0$). In each example, we shall compute the sampling values $y_\delta(\pi, \mu^2)$ at the points $\mu = 0, 1, 2, \ldots, 20$, and where $\delta = 10^{-5}$. These sampling values are then used to interpolate $B_\delta(\mu)$, and then solve for its zeros. The outputs of SLEIGN2 have been obtained using a tolerance of 10^{-6}

Example 1. Bessel's Equation ($\nu = 5/2$)

$$\begin{cases} -y'' + \frac{\nu^2 - \frac{1}{4}}{x^2} y = \lambda y, & 0 < x \leq \pi \\ y(\pi, \lambda) = 0. \end{cases}$$

Here the solution is given by $y(x, \mu^2) = \sqrt{x} J_\nu(x\mu)$ and the eigenvalues satisfy $J_\nu(\pi\sqrt{\lambda_n}) = 0$

μ (Eigenvalues μ^2)		
$\sqrt{\lambda_n}$	Sampling $N = 20$	SLEIGN2
1.834566041	1.834566109	1.834567524
2.895032021	2.895032290	2.895034369
3.922513938	3.922514613	3.922517049
4.938451519	4.938452886	4.938453199
5.948905035	5.948907470	5.948909144

Example 2. $(\nu = 2)$

$$\begin{cases} -y'' + \left(\frac{\nu^2 - \frac{1}{4}}{x^2} + x^2\right) y = \lambda y, & 0 < x \le \pi \\ y(\pi, \lambda) = 0. \end{cases}$$

μ (Eigenvalues μ^2)	
Sampling $N = 20$	SLEIGN2
2.462949030	2.462950016
3.288339398	3.288353392
4.149833151	4.149866239
5.063634795	5.063671968
6.007577378	6.007585813

Example 3. $(\nu = 3/2)$

$$\begin{cases} -y'' + \left(\frac{\nu^2 - \frac{1}{4}}{x^2} + \sin x\right) y = \lambda y, & 0 < x \le \pi \\ y(\pi, \lambda) = 0. \end{cases}$$

μ (Eigenvalues μ^2)	
Sampling $N = 20$	SLEIGN2
1.699674822427	1.699655059
2.604506077325	2.604389293
3.570068095387	3.569732553
4.553053525686	4.552323791
5.543261224280	5.541903482

Example 4. ($\nu = 5/2$)

$$\begin{cases} -y'' + \left(\frac{\nu^2-\frac{1}{4}}{x^2} + \frac{1}{x}\right)y = \lambda y, & 0 < x \le \pi \\ y(\pi, \lambda) = 0. \end{cases}$$

μ (Eigenvalues μ^2)	
Sampling $N = 20$	SLEIGN2
1.970274439470	1.970276075
3.004360435708	3.004362749
4.015153641332	4.015156884
5.019347630098	5.019351323
6.021006315094	6.021010173

5. Conclusion

In this paper we succeeded in generalizing our approach for the localization and approximation of eigenvalues for regular Sturm-Liouville problems [5], to the singular case. The approach is based on the well established technique: Shannon's sampling theorem. The results obtained are promising and agree with the ones obtained by the code SLEIGN2 to a high precision. Extension of this work to the limit circle case will be dealt with in a forthcoming paper where the codes SLEIGN2 and SLEDGE will be used to check our results.

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