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On the Group of Homeomorphisms of the Real Line That Map the Pseudoboundary Onto Itself

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Abstract. In this paper we primarily consider two natural subgroups of the autohomeomorphism group of the real line $\mathbb{R}$, endowed with the compact-open topology. First, we prove that the subgroup of homeomorphisms that map the set of rational numbers $\mathbb{Q}$ onto itself is homeomorphic to the infinite power of $\mathbb{Q}$ with the product topology. Secondly, the group consisting of homeomorphisms that map the pseudoboundary onto itself is shown to be homeomorphic to the hyperspace of nonempty compact subsets of $\mathbb{Q}$ with the Vietoris topology. We obtain similar results for the Cantor set but we also prove that these results do not extend to $\mathbb{R}^n$ for $n \geq 2$, by linking the groups in question with Erdős space.

1 Introduction

All spaces under discussion are separable and metrizable. Some of the results in this paper have been announced in Dijkstra and van Mill [15].

In this paper we are primarily interested in the space of all homeomorphisms of the real line $\mathbb{R}$ endowed with the compact open topology. It is known that this space has two components, each of them homeomorphic to ℓ^2, the Hilbert space of square summable sequences. This is an unpublished result of R. D. Anderson, the proof of which can be found in Bessaga and Pełczyński [5, Proposition VI.8.1]. Similar results are known only in dimensions 2 and ∞; see Luke and Mason [28], Ferry [21], and Toruńczyk [35].

If X is a compact space, then $\mathcal{H}(X)$ denotes the group of autohomeomorphisms of X endowed with the compact-open topology. If X is a noncompact, locally compact space, then $\mathcal{H}(X)$ is endowed with the topology that it inherits from $\mathcal{H}(\alpha X)$, where αX is the one-point compactification. If $A \subset X$, then $\mathcal{H}(X \mid A)$ denotes its subgroup $\{f \in \mathcal{H}(X) : f(A) = A\}$. It is a result of Brouwer [8] that $\mathbb{R}$ is countably dense homogeneous, *i.e.*, if A and B are countable dense subsets of $\mathbb{R}$, then there is a homeomorphism $f \in \mathcal{H}(\mathbb{R})$ such that $f(A) = B$. This result suggests studying the topological group $\mathcal{H}(\mathbb{R} \mid \mathbb{Q})$. Here $\mathbb{Q}$ stands for the rational numbers. We will prove that $\mathcal{H}(\mathbb{R} \mid \mathbb{Q})$ is homeomorphic to $\mathbb{Q}^{\mathbb{N}}$, the infinite power of $\mathbb{Q}$ with the product topology. It is also known, and not difficult to prove, that if E and F are countable dense unions of Cantor sets in $\mathbb{R}$, then there is a homeomorphism $f \in \mathcal{H}(\mathbb{R})$ such

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that $f(E) = F$. These sets play a role in $\mathbb{R}$ that is analogous to that of the pseudoboundary in the Hilbert cube as established by Geoghegan and Summerhill [24]. So this suggests studying the group $\mathcal{H}(\mathbb{R} \mid B)$, where B is any countable dense union of Cantor sets in $\mathbb{R}$. As is to be expected, this is more complex than in the case of a countable dense set. We will prove that $\mathcal{H}(\mathbb{R} \mid B)$ is homeomorphic to the space $\mathcal{K}(\mathbb{Q})$, the hyperspace of nonempty compact subsets of $\mathbb{Q}$ with the Vietoris topology. The space $\mathcal{K}(\mathbb{Q})$ surfaces at many places in the literature. It is the standard example of a coanalytic space that is not Borel. We will also prove corresponding results for the homeomorphism group of the Cantor set. Interestingly, similar results cannot be proved for the Euclidean spaces $\mathbb{R}^n$ for $n \geq 2$, or the Hilbert cube Q, because the corresponding homeomorphism groups contain a copy of Erdős space [20] and are therefore not zero-dimensional.

2 Preliminaries

A *map* is a continuous function. Let $\mathbb{I}$ denote the interval $[0, 1]$. Similarly, $\mathbb{R}$ denotes the real line, and $\mathbb{Q}$ is the space of rational numbers.

By a G_δ*-space* we mean an absolute G_δ, *i.e.*, a space which is a G_δ-subset of any space it is imbedded in. Similarly for $F_{\sigma\delta}$, $G_{\delta\sigma}$, *etc.* It is well known that the G_δ-spaces are precisely the Polish spaces. The Lavrentieff Theorem easily implies that a space is an $F_{\sigma\delta}$-space if and only if it can be imbedded in a Polish space as an $F_{\sigma\delta}$-subset. Similarly for $G_{\delta\sigma}$-space. The collection of $F_{\sigma\delta}$-subsets of a zero-dimensional space X is denoted by $\mathbf{\Pi}^0_3(X)$ and in this paper $\mathbf{\Pi}^0_3$ stands for the class of all zero-dimensional $F_{\sigma\delta}$-spaces. For details, see Kechris [25].

A space is *analytic* if it is a continuous image of the space of irrational numbers. In addition, a space is X *coanalytic* if there is a Polish space $\hat{X}$ which contains X such that $\hat{X} \setminus X$ is analytic. The collection of coanalytic subsets of a zero-dimensional space X is denoted by $\mathbf{\Pi}^1_1(X)$ and $\mathbf{\Pi}^1_1$ stands for the class of all zero-dimensional coanalytic spaces.

If X is a space, then $\mathcal{K}(X)$ denotes the space of nonempty compact subsets of X with the Vietoris topology.

If $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, then $d(x, A) = \inf\{|x - y| : y \in A\}$. If $A_1, A_2 \subset \mathbb{R}$ then the function d_{H} is defined by

$$d_{\mathrm{H}}(A_1, A_2) = \sup\{d(x, A_2), d(y, A_1) : x \in A_1, y \in A_2\}.$$

The restriction of d_{H} to $\mathcal{K}(\mathbb{R})^2$ is called the *Hausdorff metric* and generates the Vietoris topology.

Topologies on Homeomorphism Groups

If X is a topological space, then $\mathcal{H}(X)$ denotes the group of autohomeomorphisms of X and if $A \subset X$, then $\mathcal{H}(X \mid A)$ stands for the subgroup $\{h \in \mathcal{H}(X) : h(A) = A\}$. We denote the identity element of $\mathcal{H}(X)$ by 1_X.

If X is compact then the choice of a topology for $\mathcal{H}(X)$ is straightforward: the compact-open topology coincides with the topology of uniform convergence with

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respect to any compatible metric for X and makes $\mathcal{H}(X)$ into a topological group that is a Polish space. If $A, B \subset X$ then we define $[A, B] = \{h \in \mathcal{H}(X) : h(A) \subset B\}$. Thus a subbasis for the topology on $\mathcal{H}(X)$ consists of the sets $[K, O]$, where K is compact and O is open in X. Note that the topology of point-wise convergence is in general neither metrizable nor compatible with the group structure.

For noncompact spaces the situation is more complex. In general, the topology of uniform convergence depends on the metric that one chooses for X and it is usually much stronger than the compact-open topology. However, for locally compact X a natural choice for a separable metric group topology is available: the topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$, where $\alpha X = X \cup \{\infty\}$ is the one-point compactification. Since $\mathcal{H}(X) = \mathcal{H}(\alpha X \,|\, \{\infty\})$, it is a topological group and a Polish space. Note that the compact-open topology may, even for locally compact spaces, not be compatible with the group structure, in particular with the inverse operation. However, for spaces X with the property that every point has a neighbourhood that is a continuum, the topology that $\mathcal{H}(X)$ inherits from $\mathcal{H}(\alpha X)$ coincides with the compact-open topology, see Dijkstra [14] and Arens [3].

If X is locally compact and $A \subset X$, then we think of $\mathcal{H}(X \,|\, A)$ as a subspace of $\mathcal{H}(X)$. So $\mathcal{H}(X \,|\, A)$ is a topological group and hence a homogeneous space. If O is an open subset of X then $\mathcal{H}_O(X)$ is the closed subgroup of $\mathcal{H}(X)$ that consists of the elements of $\mathcal{H}(X)$ that are supported on O. We also put $\mathcal{H}_O(X \,|\, A) = \mathcal{H}_O(X) \cap \mathcal{H}(X \,|\, A)$.

If X is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{I}^n$, then we let $\mathcal{H}^+(X)$ denote the component of the identity in $\mathcal{H}(X)$ and we put $\mathcal{H}^+(X \,|\, A) = \mathcal{H}^+(X) \cap \mathcal{H}(X \,|\, A)$. We are primarily interested in $\mathcal{H}^+(\mathbb{R})$, the space of increasing autohomeomorphisms of $\mathbb{R}$, which, according to Anderson (see [5, Proposition VI.8.1]), is homeomorphic to Hilbert space. It is clear that the topology on $\mathcal{H}(\mathbb{R})$ also coincides with the topology that $\mathcal{H}(\mathbb{R})$ inherits from $\mathcal{H}([-\infty, \infty])$. In fact, $\mathcal{H}(\mathbb{R})$ and $\mathcal{H}([-\infty, \infty])$ are identical topological groups. This does not extend to $\mathbb{R}^n$ for $n \geq 2$ because $\mathcal{H}(\mathbb{R}^n)$ does not inherit a topology from $\mathcal{H}([-\infty, \infty]^n)$ (simply observe that autohomeomorphisms of $\mathbb{R}^n$ need not extend to autohomeomorphisms of $[-\infty, \infty]^n$). For $\mathcal{H}(\mathbb{R})$ the situation is particularly nice:

Lemma 2.1 *On $\mathcal{H}(\mathbb{R})$ the compact-open topology, the topology of point-wise convergence, and the topologies inherited from $\mathcal{H}(\alpha\mathbb{R})$ and from $\mathcal{H}([-\infty, \infty])$ coincide.*

Proof It suffices to prove that the topology of pointwise convergence is at least as strong on $\mathcal{H}^+((0, 1))$ as the uniform topology with respect to the standard metric. We assume that each element of $\mathcal{H}^+((0, 1))$ has been extended to an element of $\mathcal{H}^+(\mathbb{I})$. Let $h \in \mathcal{H}^+((0, 1))$ and let $n \in \mathbb{N}$. Put $a_i = h^{-1}(i/n)$ for $i \in \{0, 1, \ldots, n\}$. Then $U = \bigcap_{i=1}^{n-1} \left[\{a_i\}, (\frac{i-1}{n}, \frac{i+1}{n})\right]$ is an open neighbourhood of h in the topology of point-wise convergence. If $f \in U$ and $x \in (0, 1)$, then $x \in [a_i, a_{i+1}]$ for some $i \in \{0, 1, \ldots, n-1\}$. Since h and f are increasing, we have $h(x) \in [\frac{i}{n}, \frac{i+1}{n}]$ and $f(x) \in [f(a_i), f(a_{i+1})] \subset (\frac{i-1}{n}, \frac{i+2}{n})$. Consequently, $|f(x) - h(x)| < \frac{3}{n}$ for each $x \in (0, 1)$. ∎

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Pseudoboundaries

We now discuss absorbers and pseudoboundaries in $\mathbb{R}^n$ and the Hilbert cube. For more background on these sets than we provide here, see Dijkstra [10, 11]. Let $\mathcal{S}$ be a collection of closed subsets of a Polish space X that is invariant under the action of $\mathcal{H}(X)$ and that is closed hereditary. Let $\mathcal{S}_\sigma$ stand for the collection of countable unions of elements of $\mathcal{S}$. If $\mathcal{U}$ is a collection of subsets of X, then two functions $f, g\colon Y \to X$ are called $\mathcal{U}$-*close* if for each $y \in Y$, $f(y) = g(y)$ or there is a $U \in \mathcal{U}$ with $\{f(y), g(y)\} \subset U$. We call an element B of $\mathcal{S}_\sigma$ an $\mathcal{S}$-*absorber* in X if for every $S \in \mathcal{S}$ and every collection $\mathcal{U}$ of open subsets of X there is an $h \in \mathcal{H}(X)$ that is $\mathcal{U}$-close to 1_X while $h(S \cap \bigcup \mathcal{U}) \subset B$. The absorber concept is due to West [36] as a generalization of the capset notion which was introduced by R. D. Anderson (unpublished manuscript) and Bessaga and Pełczyński [4].

Theorem 2.2 (Uniqueness Theorem, West [36]) *If B and B' are $\mathcal{S}$-absorbers and $\mathcal{U}$ is a collection of open subsets of X, then there is an $h \in \mathcal{H}(X)$ that is $\mathcal{U}$-close to 1_X while $h(B \cap \bigcup \mathcal{U}) = B' \cap \bigcup \mathcal{U}$.*

Let $M^0 = C$ be the Cantor set, $M^n = \mathbb{R}^n$ for $n \in \mathbb{N}$, and M^∞ the Hilbert cube Q. Note that countable dense subsets of M^n are absorbers for the collection of finite subsets of M^n (and hence M^n is countably dense homogeneous).

We define the collections of "tame" zero-dimensional compacta in the spaces M^n. For $n = 0$ we have

$$\mathfrak{M}_0^0 = \{K \in \mathcal{K}(C) : \operatorname{int} K = \varnothing\}.$$

For $1 \le n < \infty$,

$$\mathfrak{M}_0^n = \{K \in \mathcal{K}(\mathbb{R}^n) : h(K) \subset (\mathbb{R} \setminus \mathbb{Q})^n \text{ for some } h \in \mathcal{H}(\mathbb{R}^n)\}.$$

This definition corresponds to that of the collection of compacta that are strong Z_{n-2}-sets in $\mathbb{R}^n$ in the sense of Geoghegan and Summerhill [23], *cf.* Dijkstra [10, Theorem 2.1.12]. And finally,

$$\mathfrak{M}_0^\infty = \{K : K \text{ is a zero-dimensional Z-set in } Q\}.$$

According to [24, Proposition 3.1], if $n \in \{1, 2\}$, then $\mathfrak{M}_0^n$ is simply the collection of zero-dimensional compacta in $\mathbb{R}^n$. For higher n there exist wild Cantor sets such as Antoine necklaces that are not part of $\mathfrak{M}_0^n$.

For every $n \in \{0, 1, \dots, \infty\}$ there exists an $\mathfrak{M}_0^n$-absorber B_0^n, see Geoghegan and Summerhill [24] ($n \in \mathbb{N}$), Curtis and van Mill [9] ($n = \infty$), and van Mill [31] ($n = 0$). This absorber B_0^n is called the zero-dimensional pseudoboundary of M^n. If $n \in \{0, 1\}$ then we call B_0^n simply the pseudoboundary of M^n. Every zero-dimensional pseudoboundary is homeomorphic to $\mathbb{Q} \times C$, which follows easily from the following characterization.

Theorem 2.3 (Alexandroff and Urysohn [2]) *A space X is homeomorphic to $\mathbb{Q} \times C$ if and only if X is a zero-dimensional, σ-compact space that is nowhere locally compact and nowhere countable.*

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Lemma 2.4 *If $n \in \{0, 1, 2, \dots, \infty\}$ and $B \in (\mathfrak{M}_0^n)_\sigma$ is a dense set in M^n that is homeomorphic to $\mathbb{Q} \times C$, then B is a zero-dimensional pseudoboundary of M^n.*

This lemma was proved by Curtis and van Mill [9] for $n = \infty$ using the estimated homeomorphism extension theorem for Z-sets. Van Mill [31] noted that the same argument also works for $n = 0$. We observe that for $2 \leq n < \infty$ the lemma can be proved using the Taming Theorem [23, Theorem 2.5]. For $n = 1$ the same method also works because if C_1 and C_2 are two Cantor sets in $\mathbb{R}$, then there exists an order preserving homeomorphism $h\colon C_1 \to C_2$ that can then be extended to an element of $\mathcal{H}^+(\mathbb{R})$.

Observe that a zero-dimensional pseudoboundary in $\mathbb{R}$ or the Cantor set is nothing but a countable dense union of nowhere dense Cantor subsets. In the remaining part of this paper we are interested in, among other things, groups of homeomorphisms that map a given zero-dimensional pseudoboundary onto itself.

Characterizations of $\mathbb{Q}^{\mathbb{N}}$ and $\mathcal{K}(\mathbb{Q})$

We shall need the following characterization of the space $\mathbb{Q}^{\mathbb{N}}$, which follows from a theorem of Steel [34], see also van Engelen [18, Theorem A.2.5].

Theorem 2.5 *A space X is homeomorphic to $\mathbb{Q}^{\mathbb{N}}$ if and only if X is a zero-dimensional, first category $F_{\sigma\delta}$-space with the property that no nonempty clopen subset is a $G_{\delta\sigma}$-space.*

Corollary 2.6 *If X is a homogeneous, zero-dimensional, first category $F_{\sigma\delta}$-space that contains a closed copy of $\mathbb{Q}^{\mathbb{N}}$, then X is homeomorphic to $\mathbb{Q}^{\mathbb{N}}$.*

The fundamental tool for recognizing that zero-dimensional spaces are homeomorphic is the following result of van Engelen [19], which is a zero-dimensional analogue of the Bestvina–Mogilski Uniqueness Theorem [6] for generalized absorbers.

Theorem 2.7 *Let X and Y be two zero-dimensional spaces such that $X = \bigcup_{i=1}^\infty X_i$ and $Y = \bigcup_{i=1}^\infty Y_i$ with X_i (respectively, Y_i) closed and nowhere dense in X (respectively, Y). If every nonempty clopen subset of X (respectively, Y) contains a closed and nowhere dense copy of each Y_i (respectively, X_i), then X and Y are homeomorphic.*

Michalewski [30] observed that we obtain useful characterizations of the space $\mathcal{K}(\mathbb{Q})$ when we combine Theorem 2.7 with results by Hurewicz and Steel, as follows. Note that $\mathcal{K}(\mathbb{Q}) = \bigcup_{q \in \mathbb{Q}} \{K \in \mathcal{K}(\mathbb{Q}) : q \in K\}$ is of the first category in itself. Let C denote the Cantor set and let D be a countable dense subset of C, so D is a topological copy of $\mathbb{Q}$. If X is a subset of a space Y, then we say that the pair (X, Y) *reduces* a subset A of C if there is a map $f\colon C \to Y$ with $f^{-1}(X) = A$. If f is an imbedding, then we say that (X, Y) *H-reduces* A. According to Hurewicz (see [25, Exercise 33.5]), the pair $(\mathcal{K}(D), \mathcal{K}(C))$ reduces every element of $\mathbf{\Pi}_1^1(C)$. If Y is a Polish space and if (X, Y) reduces every element of $\mathbf{\Pi}_1^1(C)$, then it H-reduces every element of $\mathbf{\Pi}_1^1(C)$. This last result is essentially due to Steel [34]; we formulated here the more general version of Louveau and Saint Raymond [27]. Thus we have that $\mathcal{K}(\mathbb{Q})$ (and each of its nonempty clopen subsets) contains a closed copy of every element of $\mathbf{\Pi}_1^1(C)$.

Combining this observation with Theorem 2.7, one obtains a similar characterization as the one in Michalewski [30], as follows.

Theorem 2.8 *A space X is homeomorphic to $\mathcal{K}(\mathbb{Q})$ if and only if X is a zero-dimensional, first category, and coanalytic space with the property that every zero-dimensional coanalytic space admits a closed imbedding in every nonempty clopen subset of X.*

Corollary 2.9 *Let D be a countable dense subset of a Cantor set C. Let X be a homogeneous, zero-dimensional, coanalytic subspace of a space Y such that X is of the first category in itself. If (X, Y) reduces $\mathcal{K}(D)$ (as a subset of the Cantor set $\mathcal{K}(C)$), then X is homeomorphic to $\mathcal{K}(\mathbb{Q})$.*

Proof By completing Y we get to assume that Y is a Polish space. Since (X, Y) reduces $\mathcal{K}(D)$ and $(\mathcal{K}(D), \mathcal{K}(C))$ in turn reduces every element of $\mathbf{\Pi}^1_1(C)$, we have that (X, Y) reduces (and hence H-reduces) every element of $\mathbf{\Pi}^1_1(C)$. Let F be a closed copy of $\mathcal{K}(D) \approx \mathcal{K}(\mathbb{Q})$ in X. If $x \in F$ and if O is a clopen neighbourhood of $x \in X$, then according to Theorem 2.8 $F \cap O$ and hence O contains a closed copy of every zero-dimensional coanalytic space. Since X is homogeneous, this is true for every $x \in X$ and hence $X \approx \mathcal{K}(\mathbb{Q})$. ■

We will now expound upon the analogy between Theorem 2.7 and the Bestvina–Mogilski characterization [6] of certain infinite-dimensional absolute retracts.

Let $\mathcal{S}$ be a class of zero-dimensional spaces that is *topological*, *i.e.,* every space that is homeomorphic to an element of $\mathcal{S}$ is in $\mathcal{S}$, and closed hereditary. We let $\mathcal{S}_\sigma$ stand for the class of all spaces that admit a countable closed cover consisting of elements of $\mathcal{S}$. Note that $(\mathbf{\Pi}^0_3)_\sigma = \mathbf{\Pi}^0_3$ and $(\mathbf{\Pi}^1_1)_\sigma = \mathbf{\Pi}^1_1$. We call a zero-dimensional space X *locally* $\mathcal{S}$*-universal* if every nonempty clopen subset of X contains a closed and nowhere dense copy of each element of $\mathcal{S}$. Observe that for zero-dimensional spaces strong Z-sets are merely closed nowhere dense sets and that local universality is equivalent to strong universality in the sense of Bestvina and Mogilski [6]. A zero-dimensional space X is called an $\mathcal{S}$*-absorbing space* if X is of the first category in itself, an element of $\mathcal{S}_\sigma$, and a locally $\mathcal{S}$-universal space. The following uniqueness theorem is an immediate corollary to Theorem 2.7.

Theorem 2.10 *If X and Y are two $\mathcal{S}$-absorbing spaces, then X and Y are homeomorphic.*

It follows from Theorems 2.5 and 2.8 that $\mathbb{Q}^{\mathbb{N}}$ is a $\mathbf{\Pi}^0_3$-absorbing space and $\mathcal{K}(\mathbb{Q})$ is a $\mathbf{\Pi}^1_1$-absorbing space. So these spaces are in a sense the maximal elements of their respective classes.

3 Homeomorphisms of $\mathbb{R}$ That Leave $\mathbb{Q}$ Invariant

In this section we will show that if A is any countable dense subset of $\mathbb{R}$ then $\mathcal{H}(\mathbb{R} \mid A)$ is homeomorphic to $\mathbb{Q}^{\mathbb{N}}$.

The following observation is not new, *cf.* Brechner [7, p. 532]. We include a proof for the sake of completeness.

Proposition 3.1 *If A is a zero-dimensional dense subset of $\mathbb{R}$ then $\mathcal{H}(\mathbb{R}\,|\,A)$ is zero-dimensional.*

Proof In view of Lemma 2.1 we may use the topology of point-wise convergence on $\mathcal{H}(\mathbb{R}\,|\,A)$. Note that the map $h \mapsto (h\restriction A, h\restriction(\mathbb{R}\setminus A))$ is an imbedding of $\mathcal{H}(\mathbb{R}\,|\,A)$ into the (non-metric) space $A^A \times (\mathbb{R}\setminus A)^{\mathbb{R}\setminus A}$. The latter space is zero-dimensional as a product of zero-dimensional spaces. ∎

Proposition 3.1 is not valid for $\mathbb{R}^n$ with $n \geq 2$ or for the Hilbert cube, see Corollary 6.3.

Lemma 3.2 *If A is a countable subset of a locally compact space X, then $\mathcal{H}(X\,|\,A)$ is an $F_{\sigma\delta}$-space.*

Proof Observe that

$$\mathcal{H}(X\,|\,A) = \bigcap_{a\in A}\Big(\bigcup_{b\in A}[\{a\},\{b\}] \cap \bigcup_{b\in A}[\{b\},\{a\}]\Big).$$

Since $\mathcal{H}(X)$ is a G_δ-space, the proof is complete. ∎

Theorem 3.3 *$\mathcal{H}(\mathbb{R}\,|\,\mathbb{Q})$ and $\mathcal{H}^+(\mathbb{R}\,|\,\mathbb{Q})$ are homeomorphic to $\mathbb{Q}^{\mathbb{N}}$.*

Proof We will use Corollary 2.6. First, Proposition 3.1 and Lemma 3.2 show that $\mathcal{H}(\mathbb{R}\,|\,\mathbb{Q})$ and $\mathcal{H}^+(\mathbb{R}\,|\,\mathbb{Q})$ are zero-dimensional $F_{\sigma\delta}$-spaces. Since

$$\mathcal{H}(\mathbb{R}\,|\,\mathbb{Q}) = \bigcup_{q\in\mathbb{Q}}[\{0\},\{q\}] \cap \mathcal{H}(\mathbb{R}\,|\,\mathbb{Q}),$$

it is obvious that this space is of the first category in itself.

Consider the Hilbert cube $Q = [\frac{1}{3},\frac{2}{3}]^{\mathbb{Z}}$ and the set $D = \mathbb{Q}\cap[\frac{1}{3},\frac{2}{3}]$. We will construct an imbedding $H\colon Q \to \mathcal{H}^+(\mathbb{R})$ such that $H^{-1}(\mathcal{H}^+(\mathbb{R}\,|\,\mathbb{Q})) = D^{\mathbb{Z}}$. For each $z = (\dots, z_{-1}, z_0, z_1, \dots) \in Q$ and $x \in \mathbb{R}$ we define

$$H_z(x) = \begin{cases} n + 2z_n(x-n), & \text{if } x \in [n, n+\frac{1}{2}],\ n \in \mathbb{Z},\\ n+1+2(1-z_n)(x-n-1), & \text{if } x \in [n+\frac{1}{2}, n+1],\ n \in \mathbb{Z}. \end{cases}$$

Note that for each $z \in Q$ and $n \in \mathbb{Z}$ we have $H_z([n,n+1]) = [n,n+1]$ and that H_z is strictly increasing, so $H_z \in \mathcal{H}^+(\mathbb{R})$. Since $H_z(n+\frac{1}{2}) = n + z_n$ for each $n \in \mathbb{Z}$ and $z \in Q$, $H\colon Q \to \mathcal{H}^+(\mathbb{R})$ is a one-to-one map. If $z, z' \in Q$ and $x \in [n, n+\frac{1}{2}]$ for some $n \in \mathbb{Z}$, then

$$|H_z(x) - H_{z'}(x)| = 2|z_n - z'_n|(x-n) \leq |z_n - z'_n|.$$

If $z, z' \in Q$ and $x \in [n+\frac{1}{2}, n+1]$ for some $n \in \mathbb{Z}$, then

$$|H_z(x) - H_{z'}(x)| = 2|z_n - z'_n|(n+1-x) \leq |z_n - z'_n|.$$

Since $\mathcal{H}(\mathbb{R})$ carries the topology of point-wise convergence (Lemma 2.1), the continuity of H follows and hence H is a closed imbedding of Q into $\mathcal{H}^+(\mathbb{R})$.

We now verify that $H^{-1}(\mathcal{H}^+(\mathbb{R} \mid \mathbb{Q})) = D^{\mathbb{Z}}$. If $H_z \in \mathcal{H}^+(\mathbb{R} \mid \mathbb{Q})$, then for each $n \in \mathbb{Z}$ we consequently have $n + \frac{1}{2} \in \mathbb{Q}$ and hence $n + z_n = H_z(n + \frac{1}{2}) \in \mathbb{Q}$. Thus $z_n \in \mathbb{Q}$ for each $n \in \mathbb{Z}$. Consequently, we have $z \in D^{\mathbb{Z}}$ and $H^{-1}(\mathcal{H}^+(\mathbb{R} \mid \mathbb{Q})) \subset D^{\mathbb{Z}}$. If, on the other hand, $z \in D^{\mathbb{Z}}$ and $x \in \mathbb{R}$, then we see by inspection of the definition of H that $x \in \mathbb{Q}$ if and only if $H_z(x) \in \mathbb{Q}$. ■

We can generalize Theorem 3.3 as follows.

Corollary 3.4 *Let X be a locally compact space, let O be an open subset of X, and let D be a countable dense subset of O. If O contains an open set that is homeomorphic to $\mathbb{R}$, then $\mathcal{H}_O(X \mid D)$ is homeomorphic to $\mathbb{Q}^{\mathbb{N}}$ if and only if $\dim \mathcal{H}_O(X \mid D) = 0$.*

Proof We may assume that O contains the space $[-\infty, \infty]$ in such a way that $\mathbb{R}$ is open in X and $D \cap \mathbb{R} = \mathbb{Q}$. Then $\mathcal{H}_{\mathbb{R}}(X \mid D)$ is obviously homeomorphic to $\mathcal{H}^+(\mathbb{R} \mid \mathbb{Q})$ (and hence to $\mathbb{Q}^{\mathbb{N}}$) and it is a closed subspace of the topological group $G = \mathcal{H}_O(X \mid D)$. According to Lemma 3.2, we have that $G = \mathcal{H}(X \mid D) \cap \mathcal{H}_O(X)$ is an $F_{\sigma\delta}$-space. For $y \in D$ consider the closed set $F_y = \big[\{0\}, \{y\}\big] \cap G$ in G and let $h \in F_y$. For $n \in \mathbb{N}$, let $\tau_n \in G$ be the translation $x \mapsto x + 2^{-n}$ of $\mathbb{R}$ which has been extended by the identity on $X \setminus \mathbb{R}$. Then $\lim_{n\to\infty} h \circ \tau_n = h$ and $h(\tau_n(0)) = h(2^{-n}) \neq h(0) = y$ and hence $h \circ \tau_n \notin F_y$ for $n \in \mathbb{N}$. Thus every F_y is nowhere dense and $G = \bigcup_{y \in D} F_y$ is a first category space. Now apply Corollary 2.6. ■

In particular, Theorem 3.3 remains valid if we replace $\mathbb{R}$ by the other one-dimensional manifolds: $\mathbb{I}$, a half open interval, and the circle.

It is possible that $\dim \mathcal{H}_O(X \mid D) = 1$, see Section 6.

4 Homeomorphisms of $\mathbb{R}$ That Leave the Pseudoboundary Invariant

Lemma 4.1 *If A is a Borel subset of a locally compact space X then $\mathcal{H}(X \mid A)$ is a coanalytic space.*

Proof Observe that $A \times \mathcal{H}(X)$ is Borel in $X \times \mathcal{H}(X)$. Define the continuous functions $f \colon A \times \mathcal{H}(X) \to X$ and $g \colon A \times \mathcal{H}(X) \to X$ by

$$f(a, h) = h(a) \quad \text{and} \quad g(a, h) = h^{-1}(a).$$

Put

$$B = f^{-1}(X \setminus A) \cup g^{-1}(X \setminus A).$$

Then B is a Borel subset of $A \times \mathcal{H}(X)$, hence it is a Borel subset of $X \times \mathcal{H}(X)$. We conclude that B is analytic. Since the projection $\pi \colon X \times \mathcal{H}(X) \to \mathcal{H}(X)$ is continuous, it suffices to observe that $\pi(B)$ equals precisely the complement of $\mathcal{H}(X \mid A)$. ■

Theorem 4.2 *There exists a pseudoboundary B of $\mathbb{R}$ that is a subfield of $\mathbb{R}$.*

Proof Let $\mathfrak{H}^1$ stand for the linear Hausdorff measure on the spaces $\mathbb{R}^n$ with the standard metric, see for instance [29, Chapter 4]. For each $n \in \mathbb{N}$, we define

$$G_n = \{K \in \mathcal{K}(\mathbb{I}) : \mathfrak{H}^1(K^n) = 0\},$$

where K^n stands for the Cartesian product. Since $\mathfrak{H}^1$ is evidently upper semicontinuous, we have that each G_n is a G_δ-subset of $\mathcal{K}(\mathbb{I})$. Since each G_n contains all finite sets, we have that it is a dense G_δ and hence $\mathcal{G} = \bigcap_{n=1}^{\infty} G_n$ is a dense G_δ as well. Any dense G_δ in $\mathcal{K}(\mathbb{I})$ contains Cantor sets, so let $C \in \mathcal{G}$ be such a set.

For $n \in \mathbb{N}$, let $\varphi_1^n, \varphi_2^n, \dots$ enumerate all rational functions in n variables with coefficients in $\mathbb{Z}$ and let the open set $O_k^n \subset \mathbb{R}^n$ be the domain of φ_k^n. Write $O_k^n = \bigcup_{i=1}^{\infty} K_{ki}^n$, where each K_{ki}^n is compact. We define the σ-compactum

$$B = \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcup_{i=1}^{\infty} \varphi_k^n(C^n \cap K_{ki}^n).$$

Note that B is invariant under the application of rational functions and hence it is a field. Note also that C is a subset of B and consequently B contains the pseudoboundary $\mathbb{Q} + C$. In order to show that B itself is a pseudoboundary, it suffices to show that $\dim B = 0$, see Theorem 2.3 and Lemma 2.4.

Since every φ_k^n is continuously differentiable, we have that every $\varphi_k^n \restriction K_{ki}^n$ is a Lipschitz map. Since C was chosen with the property $\mathfrak{H}^1(C^n) = 0$ for each $n \in \mathbb{N}$ and Lipschitz functions map sets with measure zero onto sets with measure zero, we have $\mathfrak{H}^1(\varphi_k^n(C^n \cap K_{ki}^n)) = 0$ and hence $\mathfrak{H}^1(B) = 0$ by σ-additivity. So B is zero-dimensional. ■

Theorem 4.3 *If B is the pseudoboundary of $\mathbb{R}$, then $\mathcal{H}(\mathbb{R} \mid B)$ and $\mathcal{H}^+(\mathbb{R} \mid B)$ are homeomorphic to $\mathcal{K}(\mathbb{Q})$.*

Proof In view of Theorem 4.2 and the uniqueness of pseudoboundaries (Theorem 2.2) we may assume that B is a field. We shall apply Corollary 2.9 to $\mathcal{H}(\mathbb{R} \mid B)$ and $\mathcal{H}^+(\mathbb{R} \mid B)$ as subspaces of the space $\mathcal{H}(\mathbb{R})$. Proposition 3.1 and Lemma 4.1 show that the spaces are zero-dimensional and coanalytic. Write $B = \bigcup_{i=1}^{\infty} B_i$, where every B_i is compact. Since $\dim B = 0$, every B_i is nowhere dense in $\mathbb{R}$. If $f \in \big[\{0\}, B_i\big] \cap \mathcal{H}(\mathbb{R} \mid B)$, then we can select a $q \in \mathbb{Q} \setminus \{0\}$, arbitrarily close to 0 such that $f(0) + q \notin B_i$. Then $g \in \mathcal{H}(\mathbb{R} \mid B)$, defined by $g(x) = f(x) + q$, is close to f but not an element of $[\{0\}, B_i]$. Since $\mathcal{H}(\mathbb{R} \mid B) = \bigcup_{i=1}^{\infty} \big[\{0\}, B_i\big] \cap \mathcal{H}(\mathbb{R} \mid B)$ this space (and $\mathcal{H}^+(\mathbb{R} \mid B)$) is of the first category in itself.

Select a Cantor set $C \subset (0, 1)$ such that $D = C \cap B$ is a countable dense subset of C. This can easily be done as follows. Pick a pseudoboundary B' and a Cantor set C' in $\mathbb{R}$ which are disjoint. If D' is a countable dense subset of C', then $B' \cup D'$ is also a pseudoboundary and hence there exists an $f \in \mathcal{H}^+(\mathbb{R})$ with $f(B' \cup D') = B$. Put $C = f(C')$.

We will construct a map $H\colon \mathcal{K}(C) \to \mathcal{H}^+(\mathbb{R})$ such that $H^{-1}(\mathcal{H}^+(\mathbb{R} \mid B)) = \mathcal{K}(D)$. Since B is an absorber for the zero-dimensional compacta in $\mathbb{R}$, there is a $\varphi \in \mathcal{H}^+(\mathbb{R})$

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such that $\varphi(C) \subset B$ and $\varphi(x) = x$ for $x \leq 0$ or $x \geq 1$. If $A \in \mathcal{K}(C)$, then we define $\hat{A} = A \cup (-\infty, 0] \cup [1, \infty)$. For each $A \in \mathcal{K}(C)$ let

$$\mathcal{F}_A = \{(a, b) \subset \mathbb{R} \setminus \hat{A} : a, b \in \hat{A}\}.$$

For each $A \in \mathcal{K}(C)$ and we define $H_A : \mathbb{R} \to \mathbb{R}$ as follows

$$H_A(x) = \begin{cases} \varphi(x) & \text{if } x \in \hat{A}, \\ \varphi(a) + \frac{\varphi(b)-\varphi(a)}{b-a}(x-a) & \text{whenever } x \in (a,b) \in \mathcal{F}_A. \end{cases}$$

Note that $H_A([a, b]) = [\varphi(a), \varphi(b)]$ for each $(a, b) \in \mathcal{F}_A$ so H_A is a surjection. Since H_A is obviously strictly increasing, we have that it is an element of $\mathcal{H}^+(\mathbb{R})$.

Claim 1 $H^{-1}(\mathcal{H}^+(\mathbb{R} \mid B)) = \mathcal{K}(D)$.

Proof If $A \in \mathcal{K}(C)$, then $H_A \restriction A = \varphi \restriction A$ so $H_A(A) = \varphi(A) \subset \varphi(C) \subset B$. If $H_A \in \mathcal{H}(\mathbb{R} \mid B)$, then we have that $A \subset H_A^{-1}(B) \cap C = B \cap C = D$. Thus $H^{-1}(\mathcal{H}^+(\mathbb{R} \mid B)) \subset \mathcal{K}(D)$.

In order to show that $\mathcal{K}(D) \subset H^{-1}(\mathcal{H}^+(\mathbb{R} \mid B))$, let $A \in \mathcal{K}(D) = \mathcal{K}(C \cap B)$. We first prove that $H_A(B) \subset B$. If $x \in B$, then there are three cases to consider. If $x \in A$, then $H_A(x) = \varphi(x) \in \varphi(C) \subset B$. If $x \in \hat{A} \setminus A$, then $H_A(x) = \varphi(x) = x \in B$. If $x \in (a, b) \in \mathcal{F}_A$, then $a, b \in A \cup \{0, 1\} \subset B$. Consequently, $\varphi(a), \varphi(b) \in \varphi(C) \cup \{0, 1\} \subset B$. Since x is also in B and B is a field, we have that $H_A(x) = \varphi(a) + \frac{\varphi(b)-\varphi(a)}{b-a}(x - a)$ is in B. Thus $H_A(B) \subset B$.

If, on the other hand, $H_A(x) \in B$, then we have again three cases. If $x \in A$, then $x \in B$. If $x \in \hat{A} \setminus A$, then $x = H_A(x) \in B$. If $x \in (a, b) \in \mathcal{F}_A$, then as above $a, b, \varphi(a), \varphi(b) \in B$ and $H_A(x) = \varphi(a) + \frac{\varphi(b)-\varphi(a)}{b-a}(x - a) = c$ for some $c \in B$. Thus $x = a + \frac{b-a}{\varphi(b)-\varphi(a)}(c - \varphi(a)) \in B$ and hence $H_A^{-1}(B) \subset B$. In conclusion, $H_A(B) = B$ and $\mathcal{K}(D) \subset H^{-1}(\mathcal{H}^+(\mathbb{R} \mid B))$. ∎

We now need to prove that H_A is continuous in $A \in \mathcal{K}(C)$. Select an $\varepsilon \in \left(0, \frac{1}{2}\right)$. Since φ is the identity outside $(0, 1)$, the map is uniformly continuous and we can find a $\delta \in (0, \varepsilon)$ such that $|\varphi(x) - \varphi(y)| < \varepsilon$ whenever $|x - y| < \delta$.

Claim 2 *For any $x \in \mathbb{R}$ and $A \in \mathcal{K}(C)$, if $d(x, \hat{A}) < \delta^2$, then $|H_A(x) - \varphi(x)| < 3\varepsilon$.*

Proof If $x \in \hat{A}$, then $H_A(x) = \varphi(x)$.

So we may assume that $x \notin \hat{A}$. Then there is an interval $(a, b) \in \mathcal{F}_A$ with $a < x < b$. Note that $d(x, \hat{A}) = \min\{x - a, b - x\}$. By symmetry we may assume that $d(x, \hat{A}) = x - a < \delta^2$ and hence $|\varphi(x) - \varphi(a)| < \varepsilon$. We have

$$|H_A(x) - \varphi(x)| \leq |\varphi(a) - \varphi(x)| + \frac{|\varphi(b) - \varphi(a)| \cdot |x - a|}{|b - a|}.$$

If $b - x < \delta$, then

$$|\varphi(b) - \varphi(a)| \frac{|x - a|}{|b - a|} \leq |\varphi(b) - \varphi(x)| + |\varphi(x) - \varphi(a)| < 2\varepsilon$$

thus $|H_A(x) - \varphi(x)| < 3\varepsilon$. If $b - x \geq \delta$, then

$$\frac{|\varphi(b) - \varphi(a)| \cdot |x - a|}{|b - a|} < \frac{\delta^2}{\delta} < \varepsilon$$

thus $|H_A(x) - \varphi(x)| < 2\varepsilon$. ∎

Let A_1 and A_2 be elements of $\mathcal{K}(C)$ such that $d_{\mathrm{H}}(A_1, A_2) < \delta^3$. Note that then also $d_{\mathrm{H}}(\hat{A}_1, \hat{A}_2) < \delta^3$.

Claim 3 *For each $x \in \mathbb{R}$ we have $|H_{A_1}(x) - H_{A_2}(x)| < 6\varepsilon$.*

Proof If $d(x, \hat{A}_1) < \delta^2/2$, then $d(x, \hat{A}_2) < \delta^2/2 + \delta^3 < \delta^2$. So

$$|H_{A_1}(x) - H_{A_2}(x)| \leq |H_{A_1}(x) - \varphi(x)| + |\varphi(x) - H_{A_2}(x)| < 6\varepsilon$$

by Claim 2.

Thus we may assume that, say $d(x, \hat{A}_1) \geq \delta^2/2$, and by symmetry also that $d(x, \hat{A}_2) \geq \delta^2/2$. So we find that $x \in (a_1, b_1) \in \mathcal{F}_{A_1}$ and $x \in (a_2, b_2) \in \mathcal{F}_{A_2}$. Note that

$$\begin{aligned} d(x, \hat{A}_1) &= \min\{x - a_1, b_1 - x\} \geq \delta^2/2, \\ d(x, \hat{A}_2) &= \min\{x - a_2, b_2 - x\} \geq \delta^2/2, \\ \min\{b_1 - a_1, b_2 - a_2, &b_1 - a_2, b_2 - a_1\} \geq \delta^2. \end{aligned}$$

We may assume without loss of generality that $a_2 \geq a_1$ and hence $a_2 \in [a_1, b_1]$. This leads to

$$\min\{a_2 - a_1, b_1 - a_2\} = d(a_2, \hat{A}_1) \leq d_{\mathrm{H}}(\hat{A}_1, \hat{A}_2) < \delta^3.$$

Since $b_1 - a_2 \geq \delta^2 > \delta^3$, we have $|a_2 - a_1| < \delta^3$ and by symmetry also $|b_2 - b_1| < \delta^3$. Consider

$$\begin{aligned} &H_{A_1}(x) - H_{A_2}(x) \\ &\quad = \varphi(a_1) - \varphi(a_2) + \frac{(\varphi(b_1) - \varphi(a_1)) \cdot (x - a_1)}{b_1 - a_1} - \frac{(\varphi(b_2) - \varphi(a_2)) \cdot (x - a_2)}{b_2 - a_2} \\ &\quad = \varphi(a_1) - \varphi(a_2) + (\varphi(b_1) - \varphi(b_2) + \varphi(a_2) - \varphi(a_1))\frac{x - a_1}{b_1 - a_1} \\ &\qquad + (\varphi(b_2) - \varphi(a_2))\Big(\frac{x - a_2}{b_2 - a_2} \cdot \frac{b_2 - b_1 + a_1 - a_2}{b_1 - a_1} + \frac{a_2 - a_1}{b_1 - a_1}\Big). \end{aligned}$$

Consequently,

$$|H_{A_1}(x) - H_{A_2}(x)| \leq \varepsilon + (\varepsilon + \varepsilon) \cdot 1 + 1 \cdot \Big(1 \cdot \frac{2\delta^3}{\delta^2} + \frac{\delta^3}{\delta^2}\Big) < 6\varepsilon,$$

where we used $\{\varphi(a_2), \varphi(b_2)\} \subset \mathbb{I}$. ∎

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So $H\colon \mathcal{K}(C) \to \mathcal{H}^+(\mathbb{R})$ is a continuous function and hence both $(\mathcal{H}^+(\mathbb{R} \mid B), \mathcal{H}^+(\mathbb{R}))$ and $(\mathcal{H}(\mathbb{R} \mid B), \mathcal{H}(\mathbb{R}))$ reduce $\mathcal{K}(D)$. This completes the proof of Theorem 4.3 ■

Again we can generalize as follows.

Corollary 4.4 *Let X be a locally compact space, let O be an open subset of X, and let U be an an open subset of O which is a copy of $\mathbb{R}$. If B is a Borel subset of O such that $B \cap U$ is the pseudoboundary of U, then $\mathcal{H}_O(X \mid B)$ is homeomorphic to $\mathcal{K}(\mathbb{Q})$ if and only if* $\dim \mathcal{H}_O(X \mid B) = 0$.

Proof We may assume that O contains the space $[-\infty, \infty]$ in such a way that $\mathbb{R}$ is open in X and $B \cap \mathbb{R}$ is a subfield and a pseudoboundary of $\mathbb{R}$. Let G denote the group $\mathcal{H}_O(X \mid B)$. In view of Corollary 2.9, let D be a countable dense subset of a Cantor set C. We have that the pair $(\mathcal{H}_{\mathbb{R}}(X \mid B), \mathcal{H}_{\mathbb{R}}(X))$ is identical to $(\mathcal{H}^+(\mathbb{R} \mid B \cap \mathbb{R}), \mathcal{H}^+(\mathbb{R}))$ and hence $(\mathcal{H}_{\mathbb{R}}(X \mid B), \mathcal{H}_{\mathbb{R}}(X))$ reduces $\mathcal{K}(D)$. Since $G \cap \mathcal{H}_{\mathbb{R}}(X) = \mathcal{H}_{\mathbb{R}}(X \mid B)$, we have that also $(G, \mathcal{H}(X))$ reduces $\mathcal{K}(D)$. According to Lemma 4.1 we have that $G = \mathcal{H}(X \mid B) \cap \mathcal{H}_O(X)$ is a co-analytic space. As in the proof of Theorem 4.3, write $B \cap \mathbb{R} = \bigcup_{i=1}^{\infty} B_i$, where every B_i is a nowhere dense compactum. Consider the set $V = \bigcup_{h \in G} h(\mathbb{R})$. Since every $h(\mathbb{R})$ is open in X and X is separable metric, we can find a countable subset $\{h_j\colon j \in \mathbb{N}\}$ of G such that $V = \bigcup_{j=1}^{\infty} h_j(\mathbb{R})$. For $i, j \in \mathbb{N}$ consider the closed set $F_{ij} = [\{0\}, h_j(B_i)] \cap G$ in G and let $f \in F_{ij}$. Since $\mathbb{Q} \setminus \{0\}$ is dense in a neighbourhood of 0 in X, we can select a $q \in \mathbb{Q} \setminus \{0\}$, arbitrarily close to 0 such that $h_j^{-1}(f(q)) \notin B_i$. Then $g \in G$, defined by $g(x) = f(x + q)$ if $x \in \mathbb{R}$ and $g(x) = f(x)$ if $x \in X \setminus \mathbb{R}$, is close to f but not an element of F_{ij}. Thus every F_{ij} is nowhere dense in G. If $f \in G$, then $f(0) \in B \cap V$. Let $j \in \mathbb{N}$ be such that $f(0) \in h_j(\mathbb{R})$ and hence $h_j^{-1}(f(0)) \in B \cap \mathbb{R}$. Select an $i \in \mathbb{N}$ such that $h_j^{-1}(f(0)) \in B_i$ and note that $f \in F_{ij}$. Thus we have that $G = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} F_{ij}$ and hence it is a first category space. Now apply Corollary 2.9. ■

Michalewski [30] proved that $\mathcal{K}(\mathbb{Q})$ is a (boolean) topological group, answering a question of Fujita and Taniyama [22]. Recall that a group is *boolean* if every element equals its own inverse. By showing that $\mathcal{K}(\mathbb{Q})$ is $\mathcal{H}(\mathbb{R} \mid B)$, we have found a radically different group structure for the same space.

5 The Cantor Set

In this section we show that the results we obtained in Sections 3 and 4 for $\mathbb{R}$ are also valid for the Cantor set.

Let C be $\{0, 1\}^{\mathbb{N}}$, equipped with the standard boolean group structure (denoted by $+$). Let D be the countable dense subgroup

$$\{x = (x_1, x_2, \dots) \in C : \text{there is an } n \in \mathbb{N} \text{ with } x_i = 0 \text{ for all } i > n\}.$$

We have the following "norm" on C:

$$|x| = \max\{2^{-i}|x_i| : i \in \mathbb{N}\}.$$

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Note that this function and the metric it generates assume only values in $\{0\} \cup \{2^{-i} : i \in \mathbb{N}\}$ and that we have the non-archimedean triangle inequality $|x + y| \leq \max\{|x|, |y|\}$. This implies that if $|x| > |y|$, then $|x + y| = |x|$ for all $x, y \in C$. If $x \in C$ and $A \in \mathcal{K}(C)$, then $d(x, A) = \min\{|x + y| : y \in A\}$. The Hausdorff metric on $\mathcal{K}(C)$ is given by $d_{\mathrm{H}}(A_1, A_2) = \max\{d(x, A_2), d(y, A_1) : x \in A_1, y \in A_2\}$.

It is well known that $\mathcal{H}(C)$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{N}^{\mathbb{N}}$, see [5, p. 206].

Theorem 5.1 *$\mathcal{H}(C \mid D)$ is homeomorphic to $\mathbb{Q}^{\mathbb{N}}$.*

Proof We will use Corollary 2.6. Lemma 3.2 shows that $\mathcal{H}(C \mid D)$ is an $F_{\sigma\delta}$-space. Since $\mathcal{H}(C \mid D) = \bigcup_{x \in D} \big[\{0\}, \{x\}\big] \cap \mathcal{H}(C \mid D)$, it is obvious that this space is of the first category in itself.

Consider the Cantor set $\mathfrak{C} = \alpha(C \times \mathbb{N})$ with $\mathfrak{D} = D \times \mathbb{N}$ as a dense topological copy of $\mathbb{Q}$. We shall construct an imbedding $H \colon C^{\mathbb{N}} \to \mathcal{H}(\mathfrak{C})$ such that $H^{-1}(\mathcal{H}(\mathfrak{C} \mid \mathfrak{D})) = D^{\mathbb{N}}$. Let ρ be the metric on $\mathfrak{C}$ that satisfies the following properties:

$$\rho((x, n), (y, n)) = 2^{-n}|x + y|,$$

$$\rho((x, n), \infty) = \rho(\infty, (x, n)) = 2^{-n},$$

$$\rho((x, n), (y, m)) = \max\{2^{-n}, 2^{-m}\} \quad \text{if } n \neq m.$$

We let $\hat{d}$ be the following product metric on $C^{\mathbb{N}}$:

$$\hat{d}(z, z') = \max\{2^{-n}|z_n + z'_n| : n \in \mathbb{N}\},$$

where $z = (z_1, z_2, \dots) \in C^{\mathbb{N}}$ and $z' = (z'_1, z'_2, \dots) \in C^{\mathbb{N}}$.

For every $z \in C^{\mathbb{N}}$ we put $H_z(\infty) = \infty$ and

$$H_z(x, n) = (x + z_n, n),$$

whenever $(x, n) \in C \times \mathbb{N}$. Note that $H_z \,|\, C \times \{n\} \in \mathcal{H}(C \times \{n\})$ for each $z \in C^{\mathbb{N}}$ and $n \in \mathbb{N}$, thus $H_z \in \mathcal{H}(\mathfrak{C})$. If $z, z' \in C^{\mathbb{N}}$ and $(x, n) \in C \times \mathbb{N}$, then

$$\rho\big(H_z(x, n), H_{z'}(x, n)\big) = \rho\big((x + z_n, n), (x + z'_n, n)\big) = 2^{-n}|z_n + z'_n|.$$

This means that H is an isometry from $(C^{\mathbb{N}}, \hat{d})$ to $(\mathcal{H}(\mathfrak{C}), \hat{\rho})$, where $\hat{\rho}$ denotes the sup metric that is associated with ρ.

Finally, we verify that $H^{-1}(\mathcal{H}(\mathfrak{C} \mid \mathfrak{D})) = D^{\mathbb{N}}$. If $H_z \in \mathcal{H}(\mathfrak{C} \mid \mathfrak{D})$, then by observing that for each $n \in \mathbb{N}$ we have $(0, n) \in \mathfrak{D}$, we get $(z_n, n) = H_z(0, n) \in \mathfrak{D} = D \times \mathbb{N}$. This means that $z \in D^{\mathbb{N}}$ and $H^{-1}(\mathcal{H}(\mathfrak{C} \mid \mathfrak{D})) \subset D^{\mathbb{N}}$. If, on the other hand, $z \in D^{\mathbb{N}}$ and $(x, n) \in C \times \mathbb{N}$, then we have $H_z(x, n) = (x + z_n, n) \in D \times \mathbb{N}$ if and only if $x \in D$ because D is a group that contains z_n. ■

Theorem 5.2 *If B is the pseudoboundary of a Cantor set C, then $\mathcal{H}(C \mid B)$ is homeomorphic to $\mathcal{K}(\mathbb{Q})$.*

Proof Note that by Lemma 2.4 $B = D \times C$ is a pseudoboundary in the Cantor set C^2. We shall apply Corollary 2.9 to $\mathcal{H}(C^2 \mid B)$ as a subspace of the zero-dimensional space $\mathcal{H}(C^2)$. Lemma 4.1 shows that $\mathcal{H}(C^2 \mid B)$ is coanalytic. Since $\mathcal{H}(C^2 \mid B) = \bigcup_{x \in D} \big[\{(0,0)\}, \{x\} \times C\big] \cap \mathcal{H}(C^2 \mid B)$, it is obvious that this space is of the first category in itself.

We will construct a map $H\colon \mathcal{K}(C) \to \mathcal{H}(C^2)$ such that $H^{-1}(\mathcal{H}(C^2 \mid B)) = \mathcal{K}(D)$. If $(x_1, y_1), (x_2, y_2) \in C^2$ and $A \in \mathcal{K}(C^2)$, then we define

$$\rho\big((x_1, y_1), (x_2, y_2)\big) = \max\big\{ |x_1 + x_2|, |y_1 + y_2| \big\}$$

and $\rho((x_1, y_1), A) = \min\{\rho((x_1, y_1), z) : z \in A\}$. Let ρ_{H} denote the associated Hausdorff metric on $\mathcal{K}(C^2)$. Note that ρ is also non-archimedean:

$$\rho(z_1, z_3) \le \max\{\rho(z_1, z_2), \rho(z_2, z_3)\}$$

and hence

$$\rho(z_1, z_3) = \max\{\rho(z_1, z_2), \rho(z_2, z_3)\}$$

whenever $\rho(z_1, z_2) \ne \rho(z_2, z_3)$. If $n \in \mathbb{N} \cup \{\infty\}$, then we define $G_n \in \mathcal{H}(C^2)$ by

$$G_n(x, y) = ((y_1, \dots, y_n, x_{n+1}, x_{n+2}, \dots), (x_1, \dots, x_n, y_{n+1}, y_{n+2}, \dots)),$$

where $(x, y) \in C^2$. Note that $G_\infty(x, y) = (y, x)$ and that $\rho(G_n(x, y), G_\infty(x, y)) \le 2^{-n-1}$. This last fact is also valid for $n = \infty$ if we put $2^{-\infty} = 0$. Since G_n for finite n switches only finitely many coordinates, we have that $G_n(D \times C) = D \times C$, thus $G_n \in \mathcal{H}(C^2 \mid B)$ for each $n \in \mathbb{N}$. Also note that every G_n is an isometry with respect to ρ.

If $z \in C^2$ and $A \in \mathcal{K}(C)$, then we put

$$n_A(z) = -\log_2(\rho(z, A \times \{0\})) \quad \text{and} \quad H_A(z) = G_{n_A(z)}(z).$$

Note that we use the convention $\log_2 0 = -\infty$, so $H_A(x, 0) = (0, x)$ whenever $x \in A$.

Claim 4 *For each $A \in \mathcal{K}(C)$, H_A is an isometry with respect to ρ.*

Proof Let $z, z' \in C^2$. If $j = n_A(z) = n_A(z')$, then $H_A(z) = G_j(z)$ and $H_A(z') = G_j(z')$. Since G_j is an isometry, we may assume that, say, $j = n_A(z) < n_A(z') = k$ and hence $i = -\log_2(\rho(z, z')) < \infty$. Observe that $i \le j$ because if $i > j$ then

$$2^{-j} = \rho(z, A \times \{0\}) \le \max\big\{ \rho(z, z'), \rho(z', A \times \{0\}) \big\} = \max\{2^{-i}, 2^{-k}\} < 2^{-j}.$$

Note that

$$\rho(G_j(z), G_\infty(z)) \le 2^{-j-1},$$

$$\rho(G_\infty(z), G_\infty(z')) = \rho(z, z') = 2^{-i} > 2^{-j-1},$$

$$\rho(G_\infty(z'), G_k(z')) \le 2^{-k-1} < 2^{-j-1}$$

and hence

$$\rho(H_A(z), H_A(z')) = \rho(G_j(z), G_k(z')) = 2^{-i} = \rho(z, z')$$

because ρ is non-archimedean. Thus H_A is an isometry. ■

Since isometries are surjective in compact spaces, we have $H_A \in \mathcal{H}(C^2)$ for each $A \in \mathcal{K}(C)$, see [32, p. 181].

Claim 5 $H\colon \mathcal{K}(C) \to \mathcal{H}(C^2)$ *is continuous.*

Proof Let $A_1, A_2 \in \mathcal{K}(C)$ be distinct so there is an $i \in \mathbb{N}$ with $d_{\mathrm{H}}(A_1, A_2) = 2^{-i}$. Consider a $z \in C^2$. If $n_{A_1}(z) = n_{A_2}(z)$, then obviously $H_{A_1}(z) = H_{A_2}(z)$. So we may assume that, for instance, $j = n_{A_1}(z) < n_{A_2}(z) = k$, which means that $\rho(z, A_1 \times \{0\}) = 2^{-j}$ and $\rho(z, A_2 \times \{0\}) = 2^{-k} \leq 2^{-j-1}$. Observe that

$$\begin{aligned} 2^{-j} = \rho(z, A_1 \times \{0\}) &\leq \max\big\{\rho\big(z, A_2 \times \{0\}\big), \rho_{\mathrm{H}}\big(A_1 \times \{0\}, A_2 \times \{0\}\big)\big\} \\ &\leq \max\{2^{-j-1}, d_{\mathrm{H}}(A_1, A_2)\} = \max\{2^{-j-1}, 2^{-i}\}. \end{aligned}$$

Consequently, we have that $j \geq i$. Recall that $\rho(G_n(z), G_\infty(z)) \leq 2^{-n-1}$ for each $n \in \mathbb{N} \cup \{\infty\}$. Thus

$$\begin{aligned} \rho(H_{A_1}(z), H_{A_2}(z)) &= \rho(G_j(z), G_k(z)) \\ &\leq \max\{\rho(G_j(z), G_\infty(z)), \rho(G_\infty(z), G_k(z))\} \\ &\leq \max\{2^{-j-1}, 2^{-k-1}\} = 2^{-j-1} \\ &\leq 2^{-i-1}. \end{aligned}$$

In conclusion, $\rho(H_{A_1}(z), H_{A_2}(z)) \leq d_{\mathrm{H}}(A_1, A_2)$ for each $z \in C^2$, thus H is continuous. ■

Claim 6 $H^{-1}(\mathcal{H}(C^2 \mid B)) = \mathcal{K}(D)$.

Proof Let $A \in \mathcal{K}(C)$ such that $H_A \in \mathcal{H}(C^2 \mid B)$. If $a \in A$ then we have $H_A(a, 0) = (0, a) \in D \times C = B$, so $(a, 0) \in D \times C$. Consequently, $A \in \mathcal{K}(D)$.

Let $A \in \mathcal{K}(D)$. If $a \in A$, then $(a, 0) \in B$ and $H_A(a, 0) = (0, a) \in B$. If $z \in C^2 \setminus (A \times \{0\})$, then $H_A(z) = G_i(z)$ for some $i \in \mathbb{N}$. Since $G_i \in \mathcal{H}(C^2 \mid B)$, we have in this case that $z \in B$ if and only if $H_A(z) \in B$. ■

In conclusion, $(\mathcal{H}(C^2 \mid B), \mathcal{H}(C^2))$ reduces $\mathcal{K}(D)$ and the proof of Theorem 5.2 is complete. ■

6 $\mathbb{R}^n$ and the Hilbert Cube

We will show in this section that the results we obtained for $\mathbb{R}$ do not generalize to $\mathbb{R}^2, \mathbb{R}^3, \ldots$ or the Hilbert cube, because the group of homeomorphisms that leave a countable dense set or the zero-dimensional pseudoboundary invariant is not zero-dimensional in these higher dimensional cases.

We call a space X *almost zero-dimensional* if every point has a neighbourhood basis consisting of sets that can be written as intersections of clopen sets. This concept is due to Oversteegen and Tymchatyn [33]. The definition we use here is different from

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the original one, but its equivalence is established in [17]. Note that almost zero-dimensionality is hereditary. It is proved in [33] that every almost zero-dimensional space is at most one-dimensional, see also [1, 26].

Proposition 6.1 *If A is a zero-dimensional and dense subset of a locally compact space X, then $\mathcal{H}(X \mid A)$ is almost zero-dimensional.*

Proof If X is not compact, then $\mathcal{H}(X \mid A) \subset \mathcal{H}(\alpha X \mid A)$. Thus it suffices to prove the proposition for compact X, which means that we can use the compact-open topology. Note that in order to prove almost zero-dimensionality, it suffices to construct neighbourhood *sub*bases. Since $\mathcal{H}(X \mid A)$ is a topological group we only need to construct a neighbourhood subbasis $\mathcal{S}$ for the identity 1_X. Let $\mathcal{S}$ consist of all sets $[K, F] \cap \mathcal{H}(X \mid A)$ where K and F are closed subsets of X such that $K = \overline{\operatorname{int} K}$ and $K \subset \operatorname{int} F$. It is easily verified that $\mathcal{S}$ is a neighbourhood subbasis at 1_X.

Let $[K, F] \cap \mathcal{H}(X \mid A)$ be an arbitrary element of $\mathcal{S}$ and let $\beta \in \mathcal{H}(X \mid A) \setminus [K, F]$. Then $\{x \in K : \beta(x) \notin F\}$ is an open nonempty subset of K. Since $K = \overline{\operatorname{int} K}$ and A is dense in X, we have that $A \cap K$ is dense in K and hence there is an $a \in A \cap K$ such that $\beta(a) \notin F$. Since A is zero-dimensional, we can find an open subset C of X such that $\beta(a) \in C$, $C \cap F = \varnothing$, and $C \cap A$ is clopen in A. Consider the open subset $\tilde{C} = \{h \in \mathcal{H}(X \mid A) : h(a) \in C\}$ of $\mathcal{H}(X \mid A)$. Note that $\tilde{C}$ is disjoint from $[K, F]$ and that it contains β. If $h \in \mathcal{H}(X \mid A) \setminus \tilde{C}$ then $h(a) \in A \setminus C = A \setminus \overline{C}$. So the complement of $\tilde{C}$ equals the obviously open set $\{h \in \mathcal{H}(X \mid A) : h(a) \notin \overline{C}\}$. Thus $\tilde{C}$ is clopen in $\mathcal{H}(X \mid A)$ which concludes the proof. ■

We will now consider the most famous example of a space that is both one-dimensional and almost zero-dimensional: Erdős space [20]. For reasons of convenience we will use a variation on the original space that is constructed in the Banach space ℓ^1 rather than in ℓ^2. Recall that the space ℓ^1 consists of all absolutely summable sequences of real numbers and is equipped with the norm

$$\|z\| = \sum_{i=1}^{\infty} |z_i| \quad \text{for } z = (z_1, z_2, \dots) \in \ell^1.$$

Recall also that the norm topology on ℓ^1 is generated by the coordinate projections and the norm function, *i.e.*, it is the weakest topology that makes all these functions into $\mathbb{R}$ continuous. Define

$$E = \{z \in \ell^1 : z_i \in \mathbb{Q} \text{ and } z_i \geq 0 \text{ for all } i \in \mathbb{N}\}.$$

The proof that E is one-dimensional is completely analogous to the proof in [20], see also [13].

Theorem 6.2 *If A is a subset of $\mathbb{R}$ that is a vector space over the field $\mathbb{Q}$, then $\mathcal{H}^+(\mathbb{R}^2 \mid A^2)$ contains a copy of the Erdős space E and hence* $\dim \mathcal{H}^+(\mathbb{R}^2 \mid A^2) > 0$.

Proof If $A = \{0\}$ or $A = \mathbb{R}$, then the statement is obviously true so we may assume that there exist an $a \neq 0$ and a b such that $a \in A$ and $b \in \mathbb{R} \setminus A$. By re-scaling we can arrange that $a = 1$ and hence that $\mathbb{Q} \subset A$. Select two sequences $p_0 < p_1 < \cdots$ and $q_0 > q_1 > \cdots$ of rational numbers that both converge to b. For every $z \in E$ we define the function $\varphi_z : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_z(x) = \begin{cases} 0 & \text{if } x \leq p_0 \text{ or } x \geq q_0, \\ \displaystyle\sum_{i=1}^{n} z_i + \frac{z_{n+1}}{p_{n+1} - p_n}(x - p_n) & \text{if } p_n \leq x \leq p_{n+1} \text{ for some } n, \\ \displaystyle\sum_{i=1}^{n} z_i + \frac{z_{n+1}}{q_{n+1} - q_n}(x - q_n) & \text{if } q_{n+1} \leq x \leq q_n \text{ for some } n, \\ \displaystyle\sum_{i=1}^{\infty} z_i & \text{if } x = b. \end{cases}$$

for each $x \in \mathbb{R}$. Note that $\varphi_z(p_n) = \varphi_z(q_n) = \sum_{i=1}^{n} z_i$ for $z \in E$ and $n \in \{0, 1, \dots\}$ and that φ_z simply connects these points with linear segments. It is clear that φ_z is well defined and continuous and that

$$|\varphi_z(x) - \varphi_{z'}(x)| \leq \|z - z'\|$$

for every $z, z' \in E$ and $x \in \mathbb{R}$. Furthermore, since A is a vector space over $\mathbb{Q}$, we have that $\varphi_z(x) \in A$ whenever $x \in A$.

For each $z \in E$ we define the homeomorphism $H_z \in \mathcal{H}^+(\mathbb{R}^2)$ by $H_z(x, y) = (x, y + \varphi_z(x))$. Note that $H_z(x, y) - H_{z'}(x, y) = (0, \varphi_z(x) - \varphi_{z'}(x))$. So $H : E \to \mathcal{H}^+(\mathbb{R}^2)$ is continuous, since it is continuous even if we use the topology of uniform convergence on $\mathcal{H}^+(\mathbb{R}^2)$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection $\pi(x, y) = y$. Observe that $\pi \circ H_z(p_n, 0) - \pi \circ H_z(p_{n-1}, 0) = z_n$ for each $n \in \mathbb{N}$ and that $\pi \circ H_z(b, 0) = \|z\|$. This means that H is a one-to-one map and that if we use H to pull the topology of point-wise convergence back to E, we get a topology that is at least as strong as the norm topology by the remark above. In conclusion, H is an imbedding. If $x, y \in A$ and $z \in E$, then $\varphi_z(x) \in A$, so $H_z(x, y) = (x, y + \varphi_z(x)) \in A^2$. If, on the other hand, $H_z(x, y) \in A^2$, then $x \in A$ and thus $y \in A - \varphi_z(x) = A$. So H is an imbedding of E in $\mathcal{H}^+(\mathbb{R}^2 \mid A^2)$. ■

Observe that if we choose $A = \mathbb{Q}$ in Theorem 6.2, then our construction gives us a closed imbedding of E in $\mathcal{H}^+(\mathbb{R}^2 \mid \mathbb{Q}^2)$. Note also that every H_z is extendible to an element of $\mathcal{H}^+([-\infty, \infty]^2)$ that is the identity on the boundary which means that the construction will work for homeomorphism groups of arbitrary topological manifolds of dimension at least 2, see [12, §4] for details. In particular we have:

Corollary 6.3 *If $n \in \{2, 3, \dots, \infty\}$ and if A is either a countable dense subset or the zero-dimensional pseudoboundary of M^n, then $\mathcal{H}(M^n \mid A)$ is both one-dimensional and almost zero-dimensional.*

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Proof We first consider $\mathbb{R}^n$ for $n \geq 2$. As the countable dense set we use $\mathbb{Q}^n$. Let B be the pseudoboundary as in Theorem 4.2 and note that Lemma 2.4 shows that B^n is a zero-dimensional pseudoboundary of $\mathbb{R}^n$. Theorem 6.2 obviously applies to $\mathcal{H}(\mathbb{R}^2 \mid \mathbb{Q}^2)$ and $\mathcal{H}(\mathbb{R}^2 \mid B^2)$. Observe that by extending elements of $\mathcal{H}(\mathbb{R}^2)$ with the identity for coordinates other than the first two, we obtain closed imbeddings of $\mathcal{H}(\mathbb{R}^2 \mid \mathbb{Q}^2)$ and $\mathcal{H}(\mathbb{R}^2 \mid B^2)$ in $\mathcal{H}(\mathbb{R}^n \mid \mathbb{Q}^n)$, respectively $\mathcal{H}(\mathbb{R}^n \mid B^n)$.

Consider now the Hilbert cube $Q = [-\infty, \infty]^{\mathbb{N}}$. We use as a countable dense subset

$$D = \{z \in \mathbb{Q}^{\mathbb{N}} : \text{there is an } i \in \mathbb{N} \text{ with } z_j = 0 \text{ for all } j > i\}.$$

Application of Theorem 2.3 and Lemma 2.4 gives that

$$B_0^\infty = \{z \in B^{\mathbb{N}} : \text{there is an } i \in \mathbb{N} \text{ with } z_j = 0 \text{ for all } j > i\}$$

is a zero-dimensional pseudoboundary in Q. Observe that the homeomorphisms H_z that were constructed in the proof of Theorem 6.2 are all extendible to elements of $\mathcal{H}([-\infty, \infty]^2)$. Again, by using the identity on the other coordinates we get imbeddings of E in both $\mathcal{H}(Q \mid D)$ and $\mathcal{H}(Q \mid B_0^\infty)$.

Almost zero-dimensionality follows of course from Proposition 6.1. ■

The fact that for topological manifolds M of dimension at least two the group of homeomorphisms that map a countable dense subset D onto itself is at least one-dimensional appears as Theorem 3.3 in Brechner [7]. However, there is a problem with the proof of that theorem. Specifically, Dijkstra [12] presents a counterexample to [7, Lemma 2.4]. Coming attraction: In a forthcoming paper [16] the authors prove that $\mathcal{H}(M \mid D)$ is in fact homeomorphic to E. The Erdős space imbedding that is presented here plays a crucial part in that proof.

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