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Extrema of Low Eigenvalues of the Dirichlet–Neumann Laplacian on a Disk

Eveline Legendre

Abstract. We study extrema of the first and the second mixed eigenvalues of the Laplacian on the disk among some families of Dirichlet–Neumann boundary conditions. We show that the minimizer of the second eigenvalue among all mixed boundary conditions lies in a compact 1-parameter family for which an explicit description is given. Moreover, we prove that among all partitions of the boundary with bounded number of parts on which Dirichlet and Neumann conditions are imposed alternately, the first eigenvalue is maximized by the uniformly distributed partition.

1 Introduction

Let Ω be a d-dimensional Euclidean domain with Lipschitz boundary $\partial\Omega$. Let $\partial_D\Omega$ be an open subset of $\partial\Omega$ with a finite number of connected components, and let $\partial_N\Omega$ be the open remainder $\partial\Omega \setminus \overline{\partial_D\Omega}$. Consider the mixed Dirichlet–Neumann eigenvalue problem determined by imposing Dirichlet condition on $\partial_D\Omega$ and Neumann condition on $\partial_N\Omega$ for the eigenfunction of the Laplacian. That is

$$-\Delta u = \lambda(\Omega, \partial_D\Omega)u, \quad u|_{\partial_D\Omega} = 0, \quad \tfrac{\partial u}{\partial n}|_{\partial_N\Omega} = 0,$$

where $\frac{\partial u}{\partial n}$ denotes the derivative of u with respect to the vector field, normal to the boundary of Ω. So there is an infinite, discrete, and positive spectrum which we order and denote $\lambda_1(\Omega, \partial_D\Omega) < \lambda_2(\Omega, \partial_D\Omega) \leq \lambda_3(\Omega, \partial_D\Omega) \leq \cdots$. This boundary value problem is also called Zaremba's problem (see [Za]).

Our interest is in understanding the dependence of these eigenvalues on the geometric properties of the partition of the boundary into Dirichlet and Neumann parts. In order to do so, we study the extrema, minima or maxima, of low eigenvalues among all parts of the boundary $\partial_D\Omega$ with fixed volume. As far as we know, this point of view appeared in Denzler's work on the first mixed eigenvalue (see [D1,D2]).

Mixed Dirichlet–Neumann problems appear naturally in many physical models and the low eigenvalues $(\lambda_1(\Omega, \partial_D\Omega), \lambda_2(\Omega, \partial_D\Omega), \dots)$ are especially important in these models, since they correspond to lower energy. The following interpretation, first suggested by Walter Craig as reported by Denzler [D1], can be particularly enlightening. Interpreting the domain Ω as a room, Neumann parts $\partial_N\Omega$ may be viewed as perfectly insulated walls while Dirichlet parts $\partial_D\Omega$ are non-insulated windows. From this point of view, low eigenvalues (depending on the initial distribution

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of temperature) determine generically the rate of heat diffusion through windows after large time (here we ignore convection). Hence, loosely speaking, extrema of low eigenvalues can be understood as extrema of heat loss through windows after large time.

Extrema of the first mixed eigenvalue have been studied in different cases, but extrema of higher mixed eigenvalues are essentially unknown at this time. This paper focuses on the study of the minimal arrangement of boundary conditions for the second eigenvalue on the disk. More precisely, we point out an explicit, compact, 1-parameter family of boundary conditions containing the minimizer of the second mixed eigenvalue.

Note that it is natural to minimize, or maximize, eigenvalues over the family of Dirichlet parts of given length since $\partial_{D_1}\Omega \subset \partial_{D_2}\Omega$ implies $\lambda_k(\partial_{D_1}\Omega) \leq \lambda_k(\partial_{D_2}\Omega)$. We define the following.

Definition 1.1 (**Boundary family** $\mathcal{F}_\ell(\Omega)$) For $0 < \ell < |\partial\Omega| := \mathrm{Vol}_{d-1}(\partial\Omega)$, we define the set of Dirichlet parts, $\partial_D\Omega \subset \partial\Omega$, with volume ℓ, as

$$\mathcal{F}_\ell(\Omega) := \{\partial_D\Omega \subset \partial\Omega \;:\; |\partial_D\Omega| = \ell \\ \text{and } \partial_D\Omega \text{ has a finite number of connected components}\}.$$

This paper focuses on the case where $\Omega = \mathcal{B}$, the open unit disk in $\mathbb{R}^2$. We denote $\mathcal{F}_\ell = \mathcal{F}_\ell(\mathcal{B})$. In addition, we define the following subfamily of special arrangements of the boundary conditions which is, as we shall see, of particular interest.

Definition 1.2 (**Uniform** n**-partition**) For $n \in \mathbb{N}$, a uniform n-partition of length ℓ, denoted by $\Gamma_{n,\ell} \in \mathcal{F}_\ell$ (or simply Γ_n), is the union of n connected parts of equal length $\frac{\ell}{n}$ uniformly distributed in $\partial\mathcal{B}$.

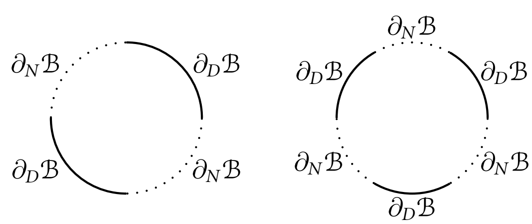

Figure 1: Uniform 2-partition (left) and uniform 3-partition (right).

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1.1 The First Mixed Eigenvalue

The extrema of the first mixed eigenvalue has been studied by Denzler [D1]. In a later paper [D2] he studied the same problem with a particular focus on minimization and disks of any dimension. Denzler showed that the minimal first eigenvalue of the d-dimensional ball is achieved when the Dirichlet part $\partial_D\Omega$ is a spherical cap. In dimension 2, this is just one connected part, that is, the uniform 1-partition. Cox and Uhlig gave another proof of this result in [CU].

Burchard and Denzler investigated the case of the first mixed eigenvalue of 2-dimensional domains, particularly the square [BD]. In this case, it seems that there is no absolute minimizing arrangement at all. In particular, they showed that there exist, for a fixed total length of the Dirichlet part, subfamilies of boundary conditions with different "shapes" and with different minimizers.

It is known that the maximizing arrangement of boundary conditions is not attained and that one should "smear" the boundary condition in order to increase the eigenvalues. In Cox and Uhlig [CU] and Denzler [D1], this result is shown for the first eigenvalue of the mixed Laplacian on a d-dimensional Lipschitz domain. As we shall see in Section 4, this result can be extended to any higher eigenvalues (Theorem 4.1).

Moreover, we prove that on the disk, with the additional constraint that the number of connected components of the Dirichlet part is bounded above by n, the maximum of the first eigenvalue is attained when these parts are uniformly distributed in the border (Theorem 1.3).

Theorem 1.3 *The uniform n-partition of length ℓ is a maximizer for the first mixed eigenvalue on the disk among all parts of boundary of length ℓ with at most n connected components.*

1.2 Minimizing the Second Mixed Eigenvalue of the Disk

Let us state the following conjecture, which will be justified in this paper.

Conjecture 1.4 *The minimizing arrangement of boundary conditions for the second eigenvalue of the disk, among all mixed Dirichlet–Neumann problems of given length, is given by the uniform* 2*-partition or by the uniform* 1*-partition.*

Numerics in Section 3 support this conjecture. More precisely, they indicate that there exists a real number $\ell_0 \in (0, 2\pi)$ for which the minimizer "jumps" from the uniform 2-partition ($\ell < \ell_0$) to the uniform 1-partition ($\ell > \ell_0$). Based on further tests, we think that in the particular case where $\ell = \ell_0$, there are exactly two minima, given by the uniform 2-partition and the uniform 1-partition of length ℓ_0. As we shall see (see caption of Figure 7), ℓ_0 seems to lie between $\frac{5{,}61\pi}{4}$ and $\frac{5{,}67\pi}{4}$.

Notice that the dependence of the minimizer on the length of Dirichlet part is analogous to the case of the first mixed eigenvalue of the square [BD].

Moreover, we prove some partial results supporting Conjecture 1.4. The following theorem reduces the problem of finding a minimum among a smaller family of boundary conditions containing the uniform 2-partition and the uniform 1-partition.

Theorem 1.5 *The second mixed eigenvalue of the disk, among all Dirichlet parts of length ℓ, admits at least one minimizer. Any such minimizer has at most two connected components, and if it has exactly two connected components, they are of equal length.*

Remark 1.6 Theorem 1.5 implies that there exists a minimizer of the second eigenvalue among all Dirichlet parts of fixed length. That is, the minimal second mixed eigenvalue lies in the following family.

Definition 1.7 **(Boundary family $\mathcal{F}^e_\ell$)** Let $\mathcal{F}^e_\ell$ be the family of Dirichlet parts of length ℓ which have at most two connected components and if they have exactly two connected components, they are of equal length.

Such partitions are necessarily symmetric with respect to a single line of symmetry. The uniform 2-partition is the only one of these which is symmetric with respect to two (perpendicular) axes of symmetry. As we shall see in Section 2.1, for a given ℓ this family can be parameterized by a compact interval. Theorem 1.5 claims that the minimizer of the second eigenvalue among $\mathcal{F}_\ell$ lies in $\mathcal{F}^e_\ell$. Note that $\Gamma_{2,\ell}$ and $\Gamma_{1,\ell} \in \mathcal{F}^e_\ell$. More precisely, we prove in Section 2.1 (Lemma 2.4) that the second mixed eigenvalue of any boundary condition on the disk is greater or equal than one of the second mixed eigenvalue associated with this family.

One of the difficulties in obtaining a more precise result, that is proving Conjecture 1.4, is the lack of knowledge concerning the behavior of nodal lines in the case of mixed boundary conditions. As we shall see, the behavior of the nodal line of an eigenfunction associated with the second eigenvalue, usually called the first nodal line, is related to the minimizer (see Remark 2.5). But in the cases of mixed boundary conditions, we do not even know if the nodal line is closed or if it reaches the border; that is, Payne's Conjecture for mixed cases is not yet proved. We recall that this conjecture claims that first nodal lines, associated with a pure Dirichlet problem on a planar domain, reach the boundary. After partial results obtained by Payne [P] and Lin [L] concerning this conjecture, Melas [Me] proved that it is true for smooth convex domains, and finally Alessandrini [A] managed to remove the smoothness assumption.

In the case of mixed boundary conditions, usual techniques and tools of optimization using derivatives cannot be used directly because mixed eigenfunctions may be nonsmooth on the border of the domain. One can refer to [M, pp. 233–234] for a review of known results about the regularity of these solutions. In particular, we cannot extend directly Melas' [Me] or Alessandrini's proof [A] of Payne's Conjecture for the mixed cases (see Theorem 2.6 for a partial result).

Another difficulty is the rigidity of the problem. We have to work on a fixed domain, with functions that are not explicitly known. The technique usually used to find minima of low eigenvalues is first to perform a rearrangement the eigenfunctions (*e.g.*, a spherical symmetrization) that decreases their Rayleigh quotient, and then to make use of the variational characterization of eigenvalues. However, there are not many rearrangements that can be used here since one has to obtain functions defined on the disk, satisfying some orthogonality relation.

2 Minimizing the Second Mixed Eigenvalue of the Disk

2.1 A Key Lemma

We now assume that the domain is the unit disk, $\mathcal{B} \subset \mathbf{R}^2$, and we denote

$$\lambda_k(\partial_D \mathcal{B}) := \lambda_k(\mathcal{B}, \partial_D \mathcal{B}).$$

The next lemma implies the existence of a minimizer and reduces the problem of its location to the family $\mathcal{F}_\ell^e$. First, we give another description of this family (see Definition 1.7), by taking a parametrization for the disk.

Definition 2.1 $\mathcal{F}_\ell^e = \{\Gamma_\ell(\beta) | \, \beta \in [0, \frac{2\pi-\ell}{4}]\}$, where

$$\Gamma_\ell(\beta) := \left\{ e^{i\theta} \;\middle|\; \begin{array}{l} \theta \in [0, 2\pi] \text{ and} \\ \theta \notin (-\beta, \beta) \cup \left(\pi - \left(\frac{\ell}{2} - \beta\right), \pi + \left(\frac{\ell}{2} - \beta\right)\right) \end{array} \right\}$$

(see Figure 2). If there is no possible confusion we simply denote $\Gamma_\ell(\beta)$ by $\Gamma(\beta)$.

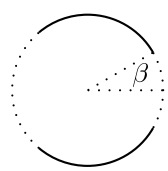

Figure 2: Boundary condition $\Gamma_\ell(\beta)$

Note that the uniform 2-partition is given by $\Gamma_\ell(\frac{2\pi-\ell}{4}) = \Gamma_2$, and the uniform 1-partition is given by $\Gamma_\ell(0) = \Gamma_1$.

Remark 2.2 Recall that the nodal domains of a continuous function f on a domain $\Omega \subset \mathbb{R}^n$ are by definition the connected components of $\Omega \backslash f^{-1}(0)$. The famous Courant's Nodal Theorem (see[C1]) claims that for a given mixed boundary value problem on Ω, any eigenfunction associated with the k-th eigenvalue has at most k nodal domains. In particular, any first eigenfunction (that is, an eigenfunction for the first eigenvalue) has exactly one nodal domain and thus has constant sign. Since any second eigenfunction is L^2-orthogonal to the first eigenfunctions, this means that any second eigenfunction has exactly two nodal domains. In the case of a 2-dimensional domain Ω and a second eigenfunction f, the set $f^{-1}(0)$ is called the nodal line.

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Recall that $\mathcal{B}$ is the *open* unit disk centered at the origin of $\mathbb{R}^2$. For a mixed problem on $\mathcal{B}$, determined by a Dirichlet part $\partial_D\mathcal{B} \in \mathcal{F}_\ell$, and an associated second eigenfunction u_2, the nodal domains are the connected open sets:

$$\mathcal{B}_+(u_2) := \{p \in \mathcal{B} | u_2(p) > 0\} \quad \text{and} \quad \mathcal{B}_-(u_2) := \{p \in \mathcal{B} | u_2(p) < 0\}.$$

Definition 2.3 Given $\partial_D\mathcal{B} \in \mathcal{F}_\ell$ and a corresponding second eigenfunction u_2 with nodal domains $\mathcal{B}_+$ and $\mathcal{B}_-$, we define

$$\beta_{u_2} := \min\{|\partial_N\mathcal{B} \cap \partial\mathcal{B}_+|, |\partial_N\mathcal{B} \cap \partial\mathcal{B}_-|\}/2.$$

Lemma 2.4 *Given $\partial_D\mathcal{B} \in \mathcal{F}_\ell$ and any associated second eigenfunction u_2, we have $\lambda_2(\partial_D\mathcal{B}) \geq \lambda_2(\Gamma_\ell(\beta_{u_2}))$, with equality if and only if $\partial_D\mathcal{B} = \Gamma_\ell(\beta_{u_2})$ up to a rotation, a reflection, and a set of zero measure.*

Remark 2.5 There are two extremal cases worth noting :

(i) If the nodal line of u_2 divides $\partial_D\mathcal{B}$ in two parts of equal length then $\beta(u) = \frac{2\pi-\ell}{4}$. Hence Lemma 2.4 gives that $\lambda_2(\partial_D\mathcal{B}) \geq \lambda_2(\Gamma_2)$, where Γ_2 stands for the uniform 2-partition.

(ii) If the nodal line of u_2 intersects the boundary of $\mathcal{B}$ in at most one point, then $\beta(u) = 0$, and, by Lemma 2.4, $\lambda_2(\partial_D\mathcal{B}) \geq \lambda_2(\Gamma_1)$, where Γ_1 stands for the uniform 1-partition.

These extremal cases point out that it can be useful to know what is the behavior of the nodal lines and in particular if the nodal lines of the second mixed eigenfunctions of the disk are closed. Theorem 2.6 claims that this never happens for a mixed eigenvalue problem associated with some $\Gamma_\ell(\beta) \in \mathcal{F}_\ell^e$.

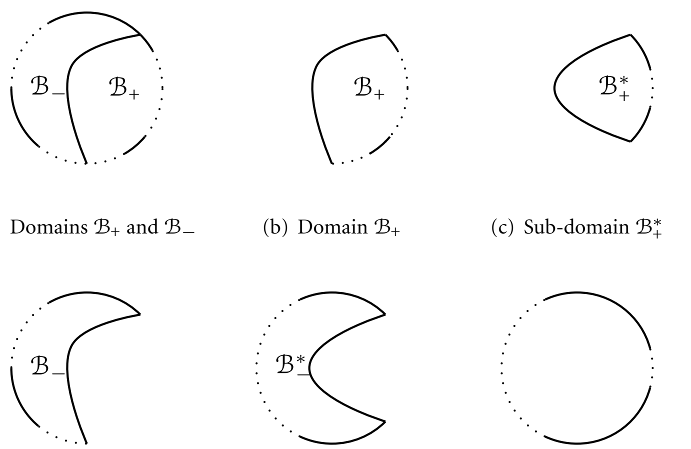
Proof of Lemma 2.4 Denote the restrictions of u_2 to $\mathcal{B}_+$ and $\mathcal{B}_-$ by u_+ and u_-, respectively. Taking $-u_2$ if necessary, we can suppose that

$$2\beta_{u_2} = |\partial_N\mathcal{B} \cap \partial\mathcal{B}_+| \leq |\partial_N\mathcal{B} \cap \partial\mathcal{B}_-|.$$

We denote the nodal line

$$\mathcal{N} := \{(x,y) \in \mathcal{B} | u_2(x,y) = 0\} = \mathcal{B} \setminus (\mathcal{B}_+ \cup \mathcal{B}_-).$$

The function u_+ is a first eigenfunction on the domain B_+ with Dirichlet boundary condition on $\mathcal{N} \cup \partial_D\mathcal{B} \cap \partial\mathcal{B}_+$ and Neumann boundary condition on the remainder. Now use spherical symmetrization, as described by Polyà and Szegö in [PS], to rearrange the function u_+ in the angular direction with respect to the positive x-axis and centered at the origin. The result is a smooth function u_+^* defined on the domain $\mathcal{B}_+^*$. We recall that the spherical symmetrization with respect to the origin and the positive x-axis is defined by the property that for any circle S_r of radius $r \leq 1$ centered at the origin, we have $|S_r \cap \mathcal{B}_+| = |S_r \cap \mathcal{B}_+^*|$ and $S_r \cap \mathcal{B}_+^*$ is connected, symmetric with respect to the x-axis and passes through the positive x-axis (see Figures 3(b) and 3(c)). Moreover, the function u_+^* is symmetric with respect to the x-axis. Its restriction to S_1 is a strictly positive function for $\theta \in (-\beta_{u_2}, \beta_{u_2})$ and vanishes elsewhere on the boundary of $\mathcal{B}_+^*$.



(a) Domains $\mathcal{B}_+$ and $\mathcal{B}_-$ (b) Domain $\mathcal{B}_+$ (c) Sub-domain $\mathcal{B}_+^*$

(d) Domain $\mathcal{B}_-$ (e) Sub-domain $\mathcal{B}_-^*$ (f) Domain $\mathcal{B}$ with Dirichlet boundary conditions $\Gamma(\beta_{u_2})$

Figure 3: Domains Used in the Proof of Lemma 2.4

Similarly, u_- is a first eigenfunction on the domain B_- with Dirichlet boundary condition on $\mathcal{N} \cup \partial_D\mathcal{B} \cap \partial\mathcal{B}_-$ and Neumann boundary condition on the remainder. We proceed in a similar way to rearrange u_-, but the spherical symmetrization should be done with respect to the negative x-axis. Hence we get the function u_-^* defined and smooth on $\mathcal{B}_-^*$, strictly negative on an arc in S_1 of length $|\partial_N\mathcal{B} \cap \mathcal{B}_-| = 2\pi - \ell - 2\beta_{u_2}$ and vanishing elsewhere on $\partial\mathcal{B}_-$ (see Figures 3(d) and 3(e)).

Since we have $|S_r \cap \mathcal{B}_+| = |S_r \cap \mathcal{B}_+^*|$ and $|S_r \cap \mathcal{B}_-| = \left|S_r \cap \mathcal{B}_-^*\right|$ for any circle, S_r, the interior of $\overline{\mathcal{B}_+^*} \cup \overline{\mathcal{B}_-^*}$ is a disk of radius one. Notice also that $\{p \in S_1 | u_+(p) = 0 \text{ or } u_-(p) = 0\} = \Gamma(\beta_{u_2})$.

By a result due to Sperner [Sp], the spherical symmetrization process ensures that

$$
\begin{aligned}
\lambda_2(\partial_D\mathcal{B}) = \lambda_1(\mathcal{B}_+, \mathcal{N} \cup (\partial_D\mathcal{B} \cap \partial\mathcal{B}_+)) &= \frac{\int_{\mathcal{B}_+} |\nabla u_+|^2}{\int_{\mathcal{B}_+} |u_+|^2} \\
&\geq \frac{\int_{\mathcal{B}_+^*} |\nabla u_+^*|^2}{\int_{\mathcal{B}_+^*} |u_+^*|^2} \geq \lambda_1(\mathcal{B}_+^*, \mathcal{N}^* \cup (\Gamma(\beta_{u_2}) \cap \partial\mathcal{B}_+^*)).
\end{aligned}
\tag{2.1}
$$

Similarly for domain $\mathcal{B}^*_-$, we have

$$\lambda_2(\partial_D\mathcal{B}) = \lambda_1(\mathcal{B}_-, \mathcal{N}\cup(\partial_D\mathcal{B}\cap\partial\mathcal{B}_-)) = \frac{\int_{\mathcal{B}_-}|\nabla u_-|^2}{\int_{\mathcal{B}_-}|u_-|^2} \tag{2.2}$$
$$\geq \frac{\int_{\mathcal{B}^*_-}|\nabla u^*_-|^2}{\int_{\mathcal{B}^*_-}|u^*_-|^2} \geq \lambda_1(\mathcal{B}^*_-, \mathcal{N}^*\cup(\Gamma(\beta_{u_2})\cap\partial\mathcal{B}^*_-)).$$

But the Theorem of Domain Monotonicity of Eigenvalues, for Dirichlet data (see *e.g.*, [C1]) implies that

$$\lambda_2(\Gamma(\beta_{u_2})) \leq \max\big\{\lambda_1(\mathcal{B}^*_+, \mathcal{N}^*\cup(\Gamma(\beta_{u_2})\cap\partial\mathcal{B}^*_+), \tag{2.3}$$
$$\lambda_1(\mathcal{B}^*_-, \mathcal{N}^*\cup(\Gamma(\beta_{u_2})\cap\partial\mathcal{B}^*_-))\big\}.$$

Hence, inequalities (2.3), (2.1), and (2.2) complete the proof of the fact that $\lambda_2(\Gamma(\beta_{u_2})) \leq \lambda_2(\partial_D\mathcal{B})$.

The case of equality The equality $\lambda_2(\Gamma(\beta_{u_2})) = \lambda_2(\partial_D\mathcal{B})$ implies equality in (2.3). Denote $\partial_D\mathcal{B}^*_\pm := \mathcal{N}^*\cup(\Gamma(\beta_{u_2})\cap\partial\mathcal{B}^*_\pm)$ and $\partial_N\mathcal{B}^*_\pm$ the respective reminders.

We first prove that, in this case, we must have

$$\lambda_1(\mathcal{B}^*_+, \partial_D\mathcal{B}^*_+) = \lambda_1(\mathcal{B}^*_-, \partial_D\mathcal{B}^*_-).$$

Indeed, otherwise we can suppose without loss of generality that

$$\lambda_2(\Gamma(\beta_{u_2})) = \lambda_1(\mathcal{B}^*_+, \partial_D\mathcal{B}^*_+) > \lambda_1(\mathcal{B}^*_-, \partial_D\mathcal{B}^*_-).$$

Thus, by continuity of the first mixed eigenvalue with respect to (Hausdorff)-continuous variations of the Dirichlet part of the boundary (see Šverak's theorem [Sv] or [HO2, p. 111]), there exists a domain $D\subset\mathcal{B}$ such that $\mathcal{B}^*_+\subset D$ and $\mathcal{B}\backslash\overline{D}\subset\mathcal{B}_-$ in such a way that

$$\lambda_1(D, \partial D\backslash\partial_N\mathcal{B}^*_+) = \lambda_1(\mathcal{B}\backslash\overline{D}, \partial(\mathcal{B}\backslash\overline{D})\backslash\partial_N\mathcal{B}^*_-).$$

On another hand, the Theorem of Domain Monotonicity of Eigenvalues (for Dirichlet Data) implies that

$$\begin{aligned}\lambda_2(\Gamma(\beta_{u_2})) &= \lambda_1(\mathcal{B}^*_+, \partial_D\mathcal{B}^*_+) > \lambda_1(D, \partial D\backslash\partial_N\mathcal{B}^*_+)\\ &= \lambda_1(\mathcal{B}\backslash\overline{D}, \partial(\mathcal{B}\backslash\overline{D})\backslash\partial_N\mathcal{B}^*_-)\\ &\geq \lambda_2(\Gamma(\beta_{u_2})).\end{aligned}$$

This is a contradiction.

This proves that the equality $\lambda_2(\Gamma(\beta_{u_2})) = \lambda_2(\partial_D\mathcal{B})$ implies equality in *both* (2.1) and (2.2). But this can occur if and only if $\partial_N\mathcal{B}_\pm$ equals $(\partial_N\mathcal{B}_\pm)^*$ (and $\mathcal{B}_\pm$ equals $\mathcal{B}^*_\pm$) up to rotation, reflection and set of zero measure. Indeed, this can be deduced directly (in view of the definition of $u_\pm$) from the proof given by Denzler in [D2] concerning the case of equality. Therefore, we have $\partial_D\mathcal{B} = \Gamma(\beta_{u_2})$ up to a rotation, a reflection, and a set of zero measure. ■

2.2 Corollaries and Extensions of Lemma 2.4

Theorem 1.5 is a straightforward corollary of Lemma 2.4.

Proof of Theorem 1.5 One can show that the map $\beta \mapsto \lambda_2(\Gamma_\ell(\beta))$ is continuous via standard methods such as monotonicity of mixed eigenvalues and continuity of the first mixed eigenvalue. Since β lies in a compact interval, there exists a minimum in this family, and Lemma 2.4 gives that for each $\partial_D\mathcal{B} \in \mathcal{F}_\ell$, there exists $\Gamma_\ell(\beta) \in \mathcal{F}_\ell^e$ such that $\lambda_2(\partial_D\mathcal{B}) \geq \lambda_2(\Gamma_\ell(\beta))$ with equality if and only if $\partial_D\mathcal{B} = \Gamma_\ell(\beta)$ up to a set of zero measure and an isometry. The inequality implies that the minimum among this family is a global minimum, and the case of equality implies that any minimizer lies in this family. ■

Extension to higher eigenvalues The proof of Lemma 2.4 uses rearrangements. A natural question is whether or not the proof can be adapted to higher eigenvalues. Since we use the fact that the number of nodal domains of a second eigenfunction is exactly two, the adaptation does not seem straightforward.

Extension to higher dimensions Lemma 2.4 can be partially extended to higher dimensions in the following way. Let $\Gamma_{\ell_N}(\beta) \subset S^{n-1}$ be the union of two opposite spherical caps of volume β and $\ell_N - \beta$ (where $\beta < \ell_N/2$ and $\ell_N \in [0, \mathrm{Vol}(S^{n-1})]$). Consider a mixed eigenvalue problem on the ball $\mathcal{B}^n$ in $\mathbb{R}^n$ with Dirichlet boundary condition on $\partial_D\mathcal{B}^n \subset S^{n-1}$ and Neumann condition on the open remainder $\partial_N\mathcal{B}^n = S^{n-1}\backslash\overline{\partial_D\mathcal{B}^n}$ with $|\mathrm{Vol}(\partial_N\mathcal{B}^n)| = \ell_N$. A second eigenfunction, u_2, associated with this problem has exactly two nodal domains, $\mathcal{B}^n_+$ and $\mathcal{B}^n_-$, by Courant's Nodal Theorem (see Remark 2.2). As above, let

$$\beta_{u_2} := \min\left\{|\partial_N\mathcal{B}^n \cap \partial\mathcal{B}^n_+|, |\partial_N\mathcal{B}^n \cap \partial\mathcal{B}^n_-|\right\}/2.$$

The first part of the proof of Lemma 2.4 works with spherical symmetrization in this framework as well, but the proof of the statement about the equality case makes use of Šverak's theorem which holds for 2–dimensional domains. Hence, given $\partial_D\mathcal{B}^n \in \mathcal{F}_\ell(\mathcal{B}^n)$ and any associated second eigenfunction u_2, we have

$$\lambda_2(\mathcal{B}^n, \partial_D\mathcal{B}^n) \geq \lambda_2(\mathcal{B}^n, S^{n-1}\backslash\overline{\Gamma_{\ell_N}(\beta_{u_2})}).$$

Nodal line and the family $\mathcal{F}_\ell^e$ By Courant's Nodal Theorem, for a given mixed boundary problem on a 2-dimensional domain Ω, one can define the *first nodal lines* to be the curves in Ω that appear as the nodal line of a second eigenfunction. The number of different first nodal lines is given by the multiplicity of the second eigenvalue.

The next theorem claims that for any mixed problem on the disk associated with an element of $\mathcal{F}_\ell^e$, associated first nodal lines cannot be closed. This theorem is a particular case of a result stated in [Gr] which can be considered as a mixed counterpart to Lin's result [L] about Payne's Conjecture in the case of a pure Dirichlet problem for symmetric, smooth, and convex domains. We give another proof of the particular case which is of interest in our setting, based on the idea which appears in Payne's proof [P].

Theorem 2.6 *Consider a mixed boundary value problem on a disk such that the Dirichlet and Neumann parts of the boundary Γ_D are symmetric under a reflection with respect to a line passing through the origin. Let u be an eigenfunction corresponding to the second eigenvalue λ_2. Then the nodal line of u, $\mathcal{N}(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}$, cannot be closed in the open disk.*

Proof Let Γ_D be symmetric with respect to the x-axis. We denote

$$\Gamma_D^+ := \{(x,y) \in \Gamma_D \mid y > 0\} \quad \text{and} \quad \Gamma_D^- := \{(x,y) \in \Gamma_D \mid y < 0\}$$

and we let u be a second mixed eigenfunction which has a closed nodal line $\mathcal{N}(u)$ (which thus does not meet the boundary). In particular, the x-axis is not contained in the nodal line and so u cannot be anti-symmetric. Thus u may be chosen symmetric with respect to the x-axis by considering the function $u(x,y) + u(x,-y)$ if necessary. The function $v := \frac{\partial u}{\partial y}$ is well defined and anti-symmetric on $\mathcal{B} \cup \Gamma_D$. Define $\mathcal{B}_*^+ := \{(x,y) \in \mathcal{B} \mid v(x,y) < 0 \text{ and } y > 0\}$, and $\mathcal{B}_* := \{(x,y) \in \mathcal{B} \mid (x, \pm y) \in \mathcal{B}_*^+\}$ is the union of $\mathcal{B}_*^+$ and its reflection. The restriction of the function v on $\mathcal{B}_*$ is such that:

- $-\Delta v = \lambda_2(\Gamma_D) v$ (since the Laplacian commutes with the y-derivative),
- v is L^2-orthogonal to the first eigenfunction of any mixed problem on $\mathcal{B}_*$ (because it is anti-symmetric),
- $v = 0$ on $\partial\mathcal{B}_* \cap \mathcal{B}$.

Hence v is a test function for the second mixed eigenvalue $\lambda_2(\mathcal{B}_*, \partial\mathcal{B}_* \cap \mathcal{B})$ and

$$\lambda_2(\Gamma_D) \geq \lambda_2(\mathcal{B}_*, \partial\mathcal{B}_* \cap \mathcal{B}). \tag{2.4}$$

In order to use monotonicity of mixed eigenvalue, we need to prove that $\mathcal{B}_*$ is nonempty and does not contain an open subset (other than the empty one) of Γ_D in its boundary.

Courant's Nodal Theorem (see Remark 2.2) implies that u has only two nodal domains and by hypothesis the first nodal line $\mathcal{N}(u)$ is closed, so:

- $\mathcal{N}(u)$ divides $\mathcal{B}$ in 2 disjoint, connected components: the inside part, whose boundary is only $\mathcal{N}(u)$, and the outside part, whose boundary contains also the boundary of the disk,
- we can assume that u is positive on the inside part and negative on the outside one,
- $\mathcal{N}(u) \cap \partial\mathcal{B} = \varnothing$, so u is smooth around $\mathcal{N}(u)$ and $\mathcal{N}(u)$ is a C^2-immersed circle without intersection (see [SY, pp.123–124]).

Denote by $\frac{\partial}{\partial \eta}$ the outward unit vector field, normal to the boundary $\mathcal{N}(u)$ of the inside part and by $\frac{\partial}{\partial n}$ the outward unit vector field, normal to the boundary $\partial\mathcal{B}$ of the disk. Then $\frac{\partial u}{\partial \eta} < 0$ on $\mathcal{N}(u)$ and $\frac{\partial u}{\partial n} > 0$ on Γ_D (Hopf Boundary Lemma).

Hence the set $\mathcal{B}_*$ is not empty since there has to exist a point of $\mathcal{N}(u)$, in the upper half plane, where $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \eta}$ are collinear and have the same orientation since $\mathcal{N}(u)$ is a closed path by hypothesis.

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Denote by $\frac{\partial}{\partial\theta}$ a unit vector field, tangent to S^1 (and thus to Γ_D). There exist a and $b \in C^\infty(\Gamma_D)$ such that $\frac{\partial}{\partial y} = a\frac{\partial}{\partial n} + b\frac{\partial}{\partial\theta}$ on Γ_D. Since u is constant on Γ_D, for any $P \in \Gamma_D$

$$\nu(P) = \frac{\partial u}{\partial y}(P) = a(P)\frac{\partial u}{\partial n}(P) + b(P)\frac{\partial u}{\partial\theta}(P) = a(P)\frac{\partial u}{\partial n}(P).$$

Observe moreover that $a = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial n}\rangle > 0$ on Γ_D^+ and $a = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial n}\rangle < 0$ on Γ_D^- and as stated before $\frac{\partial u}{\partial n} > 0$ on Γ_D. Hence

$$\nu(P) = \begin{cases} > 0 & \text{for } P \in \Gamma_D^+, \\ < 0 & \text{for } P \in \Gamma_D^-. \end{cases}$$

Thus $\partial\mathcal{B}_* \cap \Gamma_D = \varnothing$ or is only a finite set (the intersection of the x-axis and Γ_D). In particular, $\mathcal{B}_*$ is not $\mathcal{B}$ itself.

So we can use monotonicity (with Dirichlet data [C1]), and we get

$$\lambda_2(\mathcal{B}_*, \partial\mathcal{B}_* \cap \mathcal{B}) > \lambda_2(\Gamma_D),$$

which contradicts inequality (2.4). ■

3 Numerics

Theorem 1.5 states that among all partitions of the boundary of given length, the minimizer of the second eigenvalue of the disk belongs to the family $\mathcal{F}_\ell^e$ (see Definition 1.7 and Figure 2). Hence, we now focus on this particular family.

3.1 Numerical Results

Figure 4 shows the graph of the first three eigenvalues over the family $\mathcal{F}_\pi^e$, computed for $\beta = \{\frac{i\pi}{512} \mid i = 0, 1, \dots, 128\}$. The graph indicates that, in this family, the minimum of the second eigenvalue is reached for $\beta = \frac{2\pi-\ell}{4} = \frac{\pi}{4}$. Since the corresponding partition is $\Gamma_\ell(\frac{2\pi-\ell}{4}) = \Gamma_{2,\ell}$, the uniform 2-partition of length ℓ, this numerical test supports Conjecture 1.4.

Let us denote by β_c the point in the family where the two curves representing λ_2 and λ_3 on Figure 4 seems to intersect. It is the only element of the family for which the multiplicity of λ_2 could be two. Moreover, this particular point corresponds to an important change in the position of the corresponding nodal line. In fact, for $0 \leq \beta < \beta_c$, simulations show that the nodal line is horizontal with respect to the orientation given in Figure 2 and then, the corresponding eigenfunction is anti-symmetric. For $\beta_c < \beta \leq \frac{\pi}{2}$, the nodal line is vertical according to numerics and the second eigenfunction should be symmetric with respect to the line of symmetry of the Family $\mathcal{F}_\pi^e$.

Figure 5 shows level lines of only two tests, namely for $\beta = \frac{40{,}25\pi}{512} < \beta_c$ (case `b`) and $\beta = \frac{40{,}5\pi}{512} > \beta_c$ (case `a`). The nodal line is represented by the dotted line. This suggests that second eigenfunctions are anti-symmetric for $\beta < \beta_c$ and symmetric

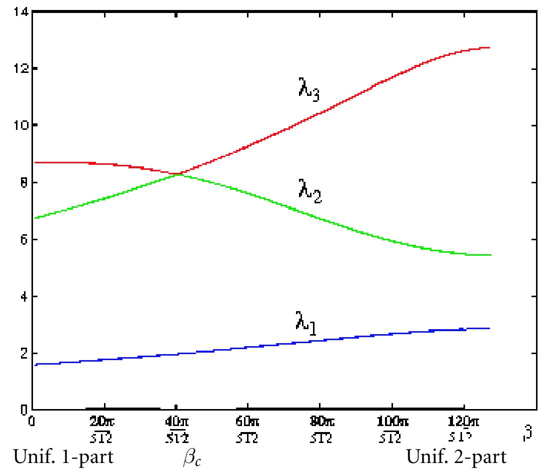

Figure 4: The first three eigenvalues associated with the family $\mathcal{F}_\ell^e$ for $\ell = \pi$ and $0 \le \beta \le \pi/4$.

for $\beta > \beta_c$. Then, Lemma 2.4, the first case of Remark 2.5, and previous numerical evidence indicate that the uniform 2-partition is the minimizer of the second eigenvalue on the subset of $\mathcal{F}_\pi^e$ consisting of Dirichlet parts $\Gamma_\pi(\beta) \in \mathcal{F}_\pi^e$ with $\beta < \beta_c$. So, it seems to indicate that the main difficulty in establishing a proof of Conjecture 1 is to understand what are the minimizers of the complement of this set of boundary value problem, that is for $\beta \ge \beta_c$.

Figure 6 shows the graph of the second eigenvalue among $\mathcal{F}_\ell^e$ for different values of ℓ, $\ell_n = \frac{n\pi}{4}$ for $n = 1, \dots, 7$. On each curve $2\beta \mapsto \lambda_2(\Gamma_{\ell_n}(\beta))$, $\beta \in [0, \frac{2\pi - \ell_n}{4}]$, the second eigenvalue corresponding to the uniform 1-partition is given at $\beta = 0$ and the one corresponding to the uniform 2-partition at $\beta = \frac{2\pi - \ell_n}{4}$ (the "last" point of each curve). So numerics make us conjecture the existence of a real number $\ell_0 \in (0, 2\pi)$ for which the minimizer "jumps" from the uniform 2-partition ($\ell < \ell_0$) to the uniform 1-partition ($\ell > \ell_0$). Further tests seem to validate this idea and let us precise that $\frac{5{,}6111\pi}{4} < \ell_0 < \frac{5{,}6667\pi}{4}$. For example, Figure 7 shows more closely the evolution from one situation to the other.

3.2 Discussion on Numerical Results

We relate these numerical observations to the cases of half-disks and squares.

Related boundary problems on the half-disk The fact that, for each ℓ, the curve described by the second eigenvalue $\beta \mapsto \lambda_2(\Gamma_\ell(\beta))$ is the minimum between two (continuous) curves comes from the symmetry of these boundary value problems (see Figures 4 or 6). Fix some $\Gamma_\ell(\beta) \in \mathcal{F}_\ell^e$; we can choose an L^2-basis of eigenfunctions which are symmetric or antisymmetric with respect to the axis of symmetry of the elements of $\mathcal{F}_\ell^e$ (that is, the axis $\{y = 0\}$ according to the parametrization given in Definition 2.1 and Figure 2). For such a basis, the restriction of any second eigen-

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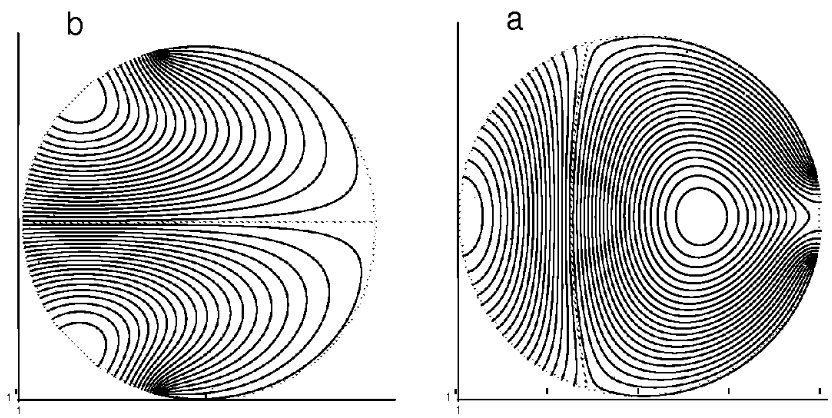

Figure 5: Level lines of a second eigenfunction for β near β_c, namely for $\beta = \frac{40{,}25\pi}{512} < \beta_c$ (case `b`) and $\beta = \frac{40{,}5\pi}{512} > \beta_c$ (case `a`).

function to the upper half disk,

$$\mathcal{D} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \quad \text{and} \quad y > 0\},$$

agrees with an eigenfunction associated with one of these mixed eigenvalues (we denote by $\tilde{\Gamma}_\ell(\beta)$ the restriction of $\Gamma_\ell(\beta)$ on $\partial\mathcal{D}$):

(i) the first mixed eigenvalue of $\mathcal{D}$ with Dirichlet part given by $\tilde{\Gamma}_\ell(\beta)$ and the diameter, that is $\lambda_1(\mathcal{D}, \tilde{\Gamma}_\ell(\beta) \cup (\partial\mathcal{D} \cap \{y = 0\}))$,

(ii) the second mixed eigenvalue of $\mathcal{D}$ with Dirichlet part given by $\tilde{\Gamma}_\ell(\beta)$, that is $\lambda_2(\mathcal{D}, \tilde{\Gamma}_\ell(\beta))$,

which are both continuous with respect to β. Case (i) corresponds to case `b` in Figure 5, that is $\beta < \beta_c$, and case (ii) corresponds to case `a` in Figure 5, that is $\beta > \beta_c$.

The first case can be easily handled via Lemma 2.4, which implies that

$$\lambda_1(\mathcal{D}, \tilde{\Gamma}_\ell(\beta) \cup \{y = 0\}) \geq \lambda_2(\Gamma_{2,\ell}).$$

So, in order to prove Conjecture 1.4, it is sufficient to show that

$$\lambda_2(\mathcal{D}, \tilde{\Gamma}_\ell(\beta)) \geq \min\{\lambda_2(\Gamma_{1,\ell}),\ \lambda_2(\Gamma_{2,\ell})\}$$

for any $\ell \in [0, 2\pi]$ and $\beta \in [0, (2\pi - \ell)/4]$. Notice that in Figure 4 the map $\beta \mapsto \lambda_2(\mathcal{D}, \tilde{\Gamma}_\pi(\beta))$ is the decreasing curve corresponding to the third eigenvalue for $\beta < \beta_c$ and to the second one for $\beta > \beta_c$.

Note that for some $\ell > \pi$ (ℓ_n with $n > 4$), numerics of Figures 6 and 7 indicate that the second mixed eigenvalue $\lambda_2(\Gamma(\ell))$ is given by $\lambda_2(\mathcal{D}, \tilde{\Gamma}_\ell(\beta))$.

Related observation on the square In the paper [BD], Burchard and Denzler point out that the shape of the Dirichlet part (not only its length) minimizing the first

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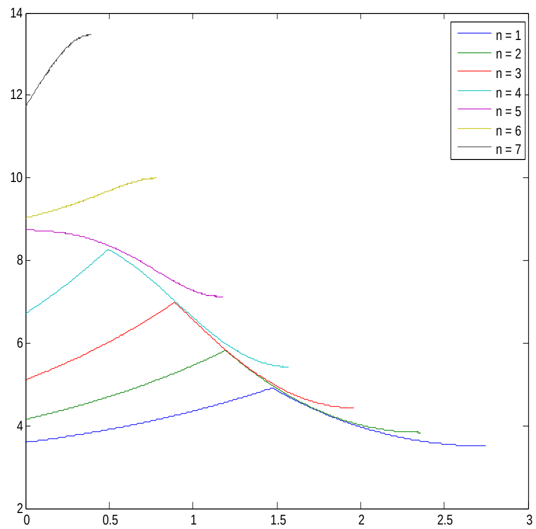
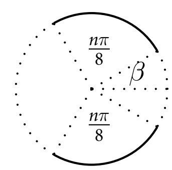




$\Gamma_{\ell_n}(\beta)$ with $\ell_n := \frac{n\pi}{4}$

Figure 6: $\lambda_2(\Gamma_{\ell_n}(\beta))$ for $0 \le \beta \le \frac{2\pi - \ell_n}{4}$ and $n = 1, \dots, 7$.

mixed eigenvalue of the square, among boundary problems (with fixed length of Dirichlet parts) depends of the family where the minimizer is taken. That is, for example, for a given $\ell > 0$, the first mixed eigenvalue of the square with Dirichlet part of length ℓ, entirely contained in one side of the square (so ℓ is smaller than one side), decreases when the Dirichlet part moves to a connected subset centered at the center of the segment where it lies. On the other hand, among all Dirichlet parts of fixed length that lie on only two sides of the square, the minimizer of the first eigenvalue is connected and contains one corner of the square [BD, Theorem 3.2].

The nontrivial relation between the shape of the minimizer taken on some family of boundary conditions and this family also appears in the case of the second mixed eigenvalue of the disk. The analogy between the first mixed eigenvalue of the square and the second of the disk may be deepened by noting that in our problem on the disk, for some $\ell < \pi$, numerics of Figures 6 and 7 indicate that the second mixed eigenvalue $\lambda_2(\Gamma(\ell))$ is given by the uniform 2-partition, and, for some higher ℓ the minimizer is given by the uniform 1-partition, that is when the Dirichlet's part reaches a corner of the half disk $\mathcal{D}$.

4 Maximization

We now study the existence and geometric properties of an arrangement of boundary conditions maximizing some mixed eigenvalue.

In [CU] and [D1], it is shown that the first eigenvalue of the mixed Laplacian has no maximizing arrangement on d-dimensional Lipschitz domain and that the first Dirichlet eigenvalue is the supremum. The following theorem is a natural extension

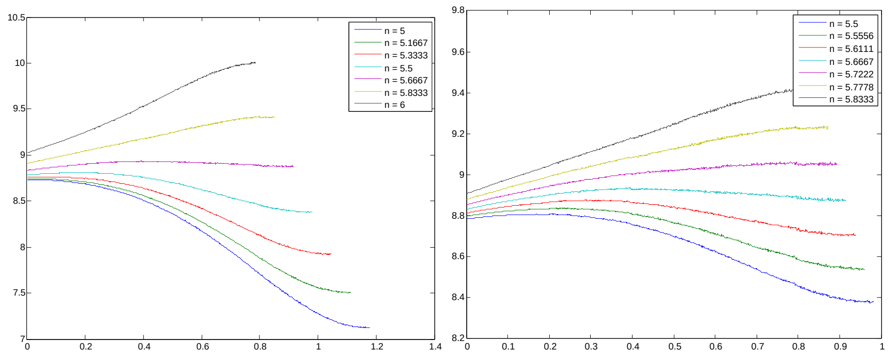


Figure 7: Graphs of $2\beta \mapsto \lambda_2(\Gamma_{\ell_n}(\beta))$ for $0 \leq \beta \leq \frac{2\pi-\ell_n}{4}$ and n between 5 and 6 (left) and between 5.5 and 5.8333 (right).

of this result for any eigenvalue, and our proof is a straightforward adaptation of Denzler's.

Theorem 4.1 *Let Ω be a bounded Lipschitz domain with boundary $\partial\Omega$, and fix $0 < A < |\partial\Omega|$. Then*

$$\sup\{\lambda_k(\Omega, \partial_D\Omega) \mid |\partial_D\Omega| = A\} = \lambda_k(\Omega, \partial\Omega),$$

and a maximizing sequence of parts can be given in such a way that their characteristic functions converge weakly to a constant in $L^2(\partial\Omega)$.

Proof of Theorem 4.1 Following [D1], we construct a sequence $(\partial_D\Omega)_n$ of Dirichlet parts such that the corresponding k-th eigenfunctions $\phi_{k,n}$ strongly converge to some limit ϕ_k. By strong convergence we know that the orthogonality relations are preserved and that there is strong convergence $|\phi_{k,n}| \to |\phi_k|$. Hence, the conclusion follows from the argument used by Denzler [D1]. ■

Hence, in order to increase any mixed eigenvalue, one should *smear* the boundary condition as much as possible. This concludes the question of maximization of mixed eigenvalue without other constraints than fixed volume. Nevertheless, we prove below that the uniform n-partition maximizes the first eigenvalue under the constraint that the number of connected components is bounded above by n. It seems to be a natural refinement of Theorem 4.1 on the disk, for $k = 1$. In our notation and for the disk, such a *smearing* takes the form of the uniform n-partition, and we would expect the first eigenvalue to increase with n. However, as indicated by Conjecture 1.4, the results of Section 2.2, and the numerics of Section 3, this is not true for higher eigenvalues since the uniform 2-partition seems to minimize the second mixed eigenvalue among boundary problems with Dirichlet part of small length.

We finally give the proof of Theorem 1.3. First, we establish an intermediate lemma which gives the basic rearrangement repeatedly used in the proof of the theorem. For $\xi \in [-\pi, \pi]$ and $\gamma \in [0, \pi]$, we denote by $S(\xi, \gamma)$ the closed sector of the disk $\mathcal{B}$ of angle 2γ, centered on the axis determined by ξ. The corresponding open (as a subset of $\partial\mathcal{B}$), connected arc will be denoted by $A(\xi,\gamma)$. Actually, we will use $A(\xi,\gamma)$ to denote this arc either as the set of *points of the boundary* or the set of *angles*. Notice that, if $\gamma = 0$, $S(\xi,\gamma)$ is a line segment and $A(\xi,\gamma)$ is empty.

Definition 4.2 Let $f \in H^{1,2}(S(\xi,\gamma)) \cap C^0(S(\xi,\gamma))$ for some $\xi \in [-\pi,\pi]$ and $\gamma \in [0,\pi]$. For an angle $\epsilon > 0$, we say that f is ϵ-partially symmetric around ξ' if $S(\xi',\epsilon) \subset S(\xi,\gamma)$ and if, for any $\theta \in [-\epsilon,\epsilon]$, $f(\xi'+\theta,r) = f(\xi'-\theta,r)$.

Lemma 4.3 *Let $f \in H^{1,2}(S(\xi,\gamma)) \cap C^0(S(\xi,\gamma))$ for some $\xi \in [-\pi,\pi]$ and $\gamma \in [0,\pi]$. If f is ϵ-partially symmetric around $\xi' \in A(\xi,\gamma)$, then for any $\alpha \in [-\epsilon,\epsilon]$ the function given by*

$$f_{\xi',\alpha}(\theta,r) := \begin{cases} f(\theta+\alpha,r) & \theta \in [\xi-\gamma-\alpha,\xi'] \\ f(\theta-\alpha,r) & \theta \in [\xi',\xi+\gamma+\alpha] \end{cases}$$

is well defined on $S(\xi^*,\gamma+\alpha)$, where $\xi^* = \xi$ if $\xi' = \xi$ and $\xi^* = \xi + \operatorname{sign}(\xi-\xi')\alpha$ otherwise. Moreover, its restriction to $S(\xi^*,\gamma)$ is an element of $H^{1,2}(S(\xi^*,\gamma)) \cap C^0(S(\xi^*,\gamma))$ which satisfies

$$\int_{S(\xi^*,\gamma+\alpha)} |\nabla f_{\xi,\alpha}|^2 = \int_{S(\xi,\gamma)} |\nabla f|^2 + \operatorname{sign}(\alpha)\int_{S(\xi',|\alpha|)} |\nabla f|^2 \quad \textit{and}$$

$$\int_{S(\xi^*,\gamma+\alpha)} f_{\xi,\alpha}^2 = \int_{S(\xi,\gamma)} f^2 + \operatorname{sign}(\alpha)\int_{S(\xi',|\alpha|)} f^2.$$

We omit the proof, which is elementary.

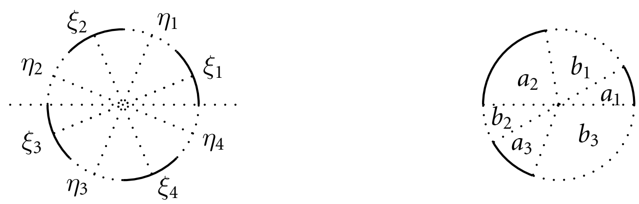
Notice that if f satisfies the hypothesis of Lemma 4.3 and vanishes on $A(\xi',\gamma')$ for some $\gamma' < \epsilon$, then, for any $\alpha \in [-\gamma',\gamma']$, $f_{\xi',\alpha}$ vanishes on $A(\xi',\gamma'+\alpha)$.

Proof of Theorem 1.3 We give the main ideas underlying the proof of the general case and, for $n = 2$ and 3, a detailed construction of the test functions which play the central role in the proof. In the general case, one can use the same construction. In particular, we introduce our notation in view of the general case.

Any subset $\partial_D\mathcal{B}$ of the boundary of the unit disk with at most n connected components and total length ℓ_D is determined, up to an isometry, by $2n$ ordered numbers $a_1, b_1, \dots, a_n, b_n$. The ordered real numbers $a_1,\dots,a_n \in [0,\ell_D]$ are the lengths of the n connected components (some of them possibly empty) of $\partial_D\mathcal{B}$. They satisfy $\sum_{i=1}^n a_i = \ell_D$. Similarly, $b_1,\dots,b_n \in [0,\ell_N]$ are the lengths of the connected components of the remainder $\partial_N\mathcal{B}$ and $\sum_{i=1}^n b_i = \ell_N := 2\pi - \ell_D$. In view of this, we adopt the following notation : $\partial_D\mathcal{B} = \Gamma_{a_1b_1,\dots,a_nb_n}$.

For example, the uniform n-partition is $\Gamma_n = \Gamma_{\frac{\ell_D}{n}\frac{\ell_N}{n}\cdots\frac{\ell_D}{n}\frac{\ell_N}{n}}$. If we denote by ξ_i the center of its i-th arc, we can also describe Γ_n as $\bigcup_{i=1}^n A(\xi_i,\frac{\ell_D}{2n})$. Similarly, $\Gamma_n^N = \bigcup_{i=1}^n A(\eta_i,\frac{\ell_N}{2n})$, with η_i the center of the i-th connected component of the remainder.

Let u denote a first mixed eigenfunction associated with Γ_n and u_i (respectively v_i), the restriction of u to the Dirichlet sector $S(\xi_i,\frac{\ell_D}{2n})$ (respectively $S(\eta_i,\frac{\ell_N}{2n})$). Since



(a) Example $\Gamma_4 = \Gamma_{\frac{2\pi}{4},\frac{2\pi}{4},\frac{2\pi}{4},\frac{2\pi}{4},\frac{2\pi}{4},\frac{2\pi}{4}}$

(b) Example $\Gamma_{a_1,b_1,a_2,b_2,a_3,b_3}$

u is invariant by any isometry of the disk which preserves the uniform n-partition, we know that u is $\frac{\ell_D}{2n}$-partially symmetric around ξ_i and $\frac{\ell_N}{2n}$-partially symmetric around η_i for $i = 1, \dots, n$. Moreover, for any two integers i and $j \in \{1, \dots, n\}$, there exists a rotation ρ of the disk which preserves Γ_n and maps ξ_i on ξ_j, so that $u_j \circ \rho = u_i$ (and similarly for Neumann sectors).

Let $\partial_D \mathcal{B} = \Gamma_{a_1 b_1 \dots a_n b_n}$ be a subset of $\partial\mathcal{B}$ with at most n connected components and total length ℓ_D, as described above and let $\alpha_i := \frac{1}{2}\left(a_i - \frac{\ell_D}{n}\right)$ and $\beta_i := \frac{1}{2}\left(b_i - \frac{\ell_N}{n}\right)$ quantify the differences between $\partial_D\mathcal{B}$ and Γ_n, in terms of Dirichlet and Neumann parts. In particular, we have $\sum_{i=1}^n \alpha_i = 0$ and $\sum_{i=1}^n \beta_i = 0$.

For $n = 2$, $\partial_D\mathcal{B} = \Gamma_{a_1 b_1 a_2 b_2}$, where $a_2 = \ell_D - a_1$ and $b_2 = \ell_N - b_1$, so that $\alpha_1 = -\alpha_2$ and $\beta_1 = -\beta_2$. The functions $u_i^* = (u_i)_{\xi_i,\alpha_i}$ and $v_i^* = (v_i)_{\eta_i,\beta_i}$ for $i = 1$ and 2 are respectively defined on sectors $S(\xi_i, \frac{\ell_D}{4} + \alpha_i = \frac{a_i}{2})$ and $S(\eta_i, \frac{\ell_N}{4} + \beta_i = \frac{b_i}{2})$. Since $2\sum_i(\frac{\ell_D}{4} + \alpha_i + \frac{\ell_N}{4} + \beta_i) = 2\pi$, these functions can be "glued" together (in an adequate order). This leads to a continuous function u^*, defined on the disk, which is a test function for the mixed boundary problem $\partial_D\mathcal{B} = \Gamma_{a_1 b_1 a_2 b_2}$. Lemma 4.3 and the invariance of u imply that the global Lebesgue norm is preserved:

$$\int_{\mathcal{B}} (u^*)^2 = \sum_{i=1}^{2} \int_{S(\xi_i,\frac{a_i}{2})} (u_i^*)^2 + \int_{S(\eta_i,\frac{b_i}{2})} (v_i^*)^2 = \sum_{i=1}^{2} \int_{S(\xi_i,\frac{\ell_D}{4})} u_i^2 + \int_{S(\eta_i,\frac{\ell_N}{4})} v_i^2$$
$$+ \sum_{i=1}^{2} \operatorname{sign}(\alpha_i) \int_{S(\xi_i,|\alpha_i|)} u_i^2 + \operatorname{sign}(\beta_i) \int_{S(\eta_i,|\beta_i|)} u_i^2 = \int_{\mathcal{B}} u^2.$$

A similar computation shows that the global Sobolev norm is also preserved along this process. Hence, the standard variational characterization of eigenvalues allows us to conclude that $\lambda_1(\Gamma_2) \geq \lambda_1(\partial_D\mathcal{B})$.

In the general case, the main ideas are the same. We consider a subset of $\partial\mathcal{B}$, $\partial_D\mathcal{B} = \Gamma_{a_1 b_1 \cdots a_n b_n}$, with at most n connected components and total length ℓ_D. We define $\alpha_1, \dots, \alpha_n \in [-\frac{\ell_D}{2n}, (n-1)\frac{\ell_D}{2}]$ and $\beta_1, \dots, \beta_n \in [-\frac{\ell_N}{2n}, (n-1)\frac{\ell_N}{2}]$ as above.

(i) We build n test functions $u_1^*, \dots, u_n^*$ respectively defined on sectors $S_1, \dots, S_n$ with angles $a_1, \dots, a_n$, by using a finite number of times the rearrangement described by Lemma 4.3, respectively on $u_1, \dots, u_n$, such that

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- the global L^2 and $H^{1,2}$-norms are preserved, that is,

$$\sum_i \|u_i^*\|_{L^2(S_i)} = \|u_|\|_{L^2(\cup S(\xi_i, \frac{\ell_D}{2n}))} \quad \text{and}$$

$$\sum_i \|u_i^*\|_{H^{1,2}(S_i)} = \|u_|\|_{H^{1,2}(\cup S(\xi_i, \frac{\ell_D}{2n}))}.$$

- these rearrangements are "minimal" in the following sense: if $\alpha_i \le 0$, then $u_i^* = (u_i)_{\xi_i,\alpha_i}$, and there is no rearrangement $f_{\xi,\alpha}$ with $|\alpha| > |\alpha_k|$, α_k being the smallest strictly negative α_i.

Via the same process, we build n test functions $v_1^*, \dots, v_n^*$ corresponding to the connected components of the Neumann part.

(ii) We then "glue", in an adequate way, the $2n$ resulting functions, u_1^*, $v_1^*, \dots, v_n^*$, u_n^*, in order to obtain a new function u^* which is defined on the whole disk. This function is a test function for the mixed boundary problem associated with $\partial_D\mathcal{B}$.

(iii) By construction, the global Sobolev and Lebesgue norms are preserved. Using this fact and the standard variational characterization of eigenvalues, we conclude that $\lambda_1(\Gamma_n) \ge \lambda_1(\partial_D\mathcal{B})$.

Only the first step is ambiguous. Writing this construction down explicitly is laborious, but one can prove, by induction on the number of non zero α_i's, that such $2n$ functions can be obtained. The fact that the rearrangement has to be minimal is crucial for the inductive step (for the norms to be preserved).

The base case of the induction is similar to the case $n = 2$. We now describe the case $n = 3$ which already contains the main difficulties. Moreover, we shall see that the minimality condition naturally appears (asymmetry of the roles of α_2 and α_3 in the construction of α_1^*, subcase (ii) below).

When $n = 3$, $\partial_D\mathcal{B}$ is an open subset of $\partial\mathcal{B}$ with at most 3 connected components so that $\partial_D\mathcal{B}$ can be described as $\Gamma_{a_1b_1a_2b_2a_3b_3}$. As above, $\alpha_i := \frac{1}{2}\left(a_i - \frac{\ell_D}{3}\right)$ and $\beta_i := \frac{1}{2}\left(b_i - \frac{\ell_N}{3}\right)$, so that $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\beta_1 + \beta_2 + \beta_3 = 0$.

There are three (exclusive) cases worth noting (six cases if we also consider the Neumann sectors).

(i) One of the α_i's is zero. Then the other two are opposite elements of $[-\frac{\ell_D}{6}, \frac{\ell_D}{6}]$. We define $u_i^* := (u_i)_{\xi_i,\alpha_i}$, for $i = 1, 2$, and 3.

(ii) None of them is zero and (exactly) one is positive; the other two are greater or equal than $-\frac{\ell_D}{6}$ (and negative).
Assuming for simplicity that $\alpha_1 > 0$ and $\alpha_2 \le \alpha_3 < 0$, we define $u_i^* = (u_i)_{\xi_i,\alpha_i}$ for $i = 2$ and 3, and $u_1^* = ((u_1)_{\xi_1,-\alpha_2})_{\xi_1-\alpha_2,-\alpha_3}$.

(iii) None of them is zero and (exactly) one is negative, the other two are lesser or equal than $\frac{\ell_D}{6}$ (and positive). Then, assuming that $\alpha_1 < 0$, α_2 and $\alpha_3 > 0$, we define $u_i^* = (u_i)_{\xi_i,\alpha_i}$ for $i = 1$ and 2, and $u_3^* = ((u_3)_{\xi_3,\alpha_2+\alpha_3})_{\xi_3-(\alpha_2+\alpha_3),-\alpha_2}$.

These cases cover all possibilities for $n = 3$. For each of them our construction preserves the norms and satisfies the required minimality condition. Hence this concludes step (i) for the case $n = 3$. ∎

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Université de Montréal, Montréal, QC H3C 3J7
e-mail: legendre@dms.umontreal.ca