Can. J. Math. Vol. 44 (5), 1992 pp. 1100–1120

A TAUBERIAN THEOREM FOR THE GENERAL EULER-BOREL SUMMABILITY METHOD

LAYING TAM

ABSTRACT. Our main result is a Tauberian theorem for the general Euler-Borel summability method. Examples of the method include the discrete Borel, Euler, Meyer-König, Taylor and Karamata methods. Every function f analytic on the closed unit disk and satisfying some general conditions generates such a method, denoted by (E,f). For instance the function $f(z) = \exp(z-1)$ generates the discrete Borel method. To each such function f corresponds an even positive integer $p = p(f)$. We show that if a sequence (s_n) is summable (E,f) and

$$(*) \qquad s_m - s_n \longrightarrow 0$$

as $n \longrightarrow \infty$, $m > n$, $(m-n)n^{-1/p(f)} \longrightarrow 0$, then (s_n) is convergent. If the Maclaurin coefficients of f are nonnegative, then $p(f) = 2$. In this case we may replace the condition $(*)$ by $\underline{\lim}(s_m - s_n) \geq 0$. This generalizes the Tauberian theorems for Borel summability due to Hardy and Littlewood, and R. Schmidt. We have also found new examples of the method and proved that the exponent $-p(f)$ in the Tauberian condition $(*)$ is the best possible.

1. **Introduction.** The main result of this paper is a Tauberian theorem for the general Euler-Borel summability method. The method is also called the *Sonnenschein method* and was first studied in detail in [1]. We begin with its definition.

Let $f(z)$ be a function analytic at the origin. The general Euler-Borel method generated by $f(z)$, denoted by (E,f), is the sequence-to-sequence transformation defined by the matrix $(a_{nk})_{n\geq 0, k\geq 0}$, where a_{nk} satisfies

$$\big(f(z)\big)^n = \sum_{k=0}^{\infty} a_{nk} z^k.$$

Thus if (s_k) is a sequence of numbers such that $\sum_{k=0}^{\infty} a_{nk}s_k$ is convergent for all large n and $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}s_k$ exists and is equal to S, then we say that (s_k) is summable (E,f) to S. We denote this fact by

$$s_k \longrightarrow S(E,f).$$

Examples of the method (E,f) include the following.

1. The Euler method (E,q), $q > 0$.

$$s_k \longrightarrow S(E,q) \text{ means } (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \longrightarrow S \text{ as } n \longrightarrow \infty.$$

Received by the editors July 3, 1991.
AMS subject classification: Primary: 40E05, 40G05, 40G10.

2. The discrete Borel method B.

$$s_k \longrightarrow S(B) \text{ means } \sum_{k=0}^{\infty} \frac{e^{-n}n^k}{k!} s_k \longrightarrow S \text{ as } n \longrightarrow \infty.$$

3. The Meyer-König method $S_r, 0 < r < 1$.

$$s_k \longrightarrow S(S_r) \text{ means } (1-r)^n \sum_{k=0}^{\infty} \binom{n-1+k}{k} r^k s_k \longrightarrow S \text{ as } n \longrightarrow \infty.$$

4. The Taylor method $T_r, 0 < r < 1$.

$$s_k \longrightarrow S(T_r) \text{ means } (1-r)^n \sum_{k=0}^{\infty} \binom{n-1+k}{k} r^k s_{n+k} \longrightarrow S \text{ as } n \longrightarrow \infty.$$

5. The Karamata method $E(\alpha,\beta)$, where $\alpha < 1, \beta < 1$, and $\alpha+\beta > 0$.

$$s_k \longrightarrow S\big(E(\alpha,\beta)\big) \text{ means } \sum_{k=0}^{\infty} c_{nk} s_k \longrightarrow S \text{ as } n \longrightarrow \infty, \text{ where}$$

$$c_{nk} = \sum_{j=0}^{\min(n,k)} \binom{n}{j} \binom{n-1+k-j}{n-1} (1-\alpha-\beta)^j \alpha^{n-j} \beta^{k-j}.$$

We will refer to the discrete Borel method as the Borel method.

The functions defining the above methods are, respectively,

1. $f(z) = (z+q)/(1+q), q > 0,$
2. $f(z) = \exp(z-1),$
3. $f(z) = (1-r)/(1-rz), 0 < r < 1,$
4. $f(z) = (1-r)z/(1-rz), 0 < r < 1,$ and
5. $f(z) = \big(\alpha+(1-\alpha-\beta)z\big)/(1-\beta z)$, where α and β satisfy the above conditions.

Most of these examples have been studied thoroughly. They are all regular. In other words, they sum a convergent sequence to its limit. Tauberian theorems for them are known. For further references of the method see [14].

Regarding the general Euler-Borel method we have Theorems A, B, and 1 below. Theorem A, due to Bajšanski [1], and Theorem B, due to Bajšanski [1] and Clunie and Vermes [5], are criteria for the regularity of the method. Theorem 1, our main result, is a Tauberian theorem for the method.

THEOREM A. *The following conditions are sufficient for the regularity of* (E,f).

1. $f(z)$ *is analytic for* $|z| < R, R > 1$.
2. $|f(z)| < 1$ *for* $|z| \leq 1, z \neq 1$.
3. $f(1) = 1$.
4. *Let the numbers* p *and* $A \neq 0$ *be defined by*

$$f(z) - z^{\alpha} = Ai^p(z-1)^p + o(1)(z-1)^p, \; z \longrightarrow 1, \; \alpha = f'(1).$$

Then $\Re A \neq 0$.

Assuming that conditions 1–3 of the theorem hold, then the seemingly complicated condition 4 is also necessary for the regularity of the method. See Theorem B.

https://doi.org/10.4153/CJM-1992-067-9 Published online by Cambridge University Press

If a function $f(z)$ satisfies the conditions of Theorem A then we denote the parameters A, α, and p in condition 4 by $A(f)$, $\alpha(f)$, and $p(f)$, respectively.

It is proved in [1] that under the assumptions of Theorem A,

$$\Re A(f) < 0, \ \alpha(f) > 0, \text{ and } p(f) \text{ is an even integer.}$$

We note that for each of our examples of the method (E,f), $f(z)$ satisfies the conditions of Theorem A. Moreover, $A(f)$ is a real number and $p(f) = 2$. The preceding two conditions hold whenever the coefficients a_{nk} are nonnegative (see [1, pp. 134–135]), as in the examples, except the Karamata method.

THEOREM B. *Suppose that*

1. *$f(z)$ is analytic for $|z| < R$, where $R > 1$, and is not a monomial, i.e., $f(z) \neq z^m$, where m is a non-negative integer.*

Then (E,f) is regular if and only if the following conditions are satisfied.

2. *$|f(z)| < 1$ for $|z| < 1$ except at finitely many points ζ.*
3. *$f(1) = 1$.*
4. *If $|f(\zeta)| = 1$, $h_\zeta(z) = f(\zeta z)/f(\zeta)$, $\alpha_\zeta = h'_\alpha(1)$, and p_ζ and the nonzero number A_ζ are defined by*

$$h_\zeta(z) - z^{\alpha_\zeta} = A_\zeta i^{p_\zeta}(z-1)^{p_\zeta} + o(1)(z-1)^{p_\zeta}, \quad z \longrightarrow 1,$$

then $\Re A_\zeta \neq 0$.

The sufficiency of these conditions follows from Theorem A. (See [1]). Their necessity is due to Clunie and Vermes [5].

The above results of Bajšanski and Clunie and Vermes were rediscovered by Newman [10], whose proof is considerably shorter.

In this paper we will consider regular methods (E,f) where f satisfies the conditions of Theorem A. We will not repeat this assumption.

Here is our Tauberian theorem.

THEOREM 1. *(a) If a sequence of complex numbers (s_k) is summable (E,f) and*

$$\lim(s_m - s_n) = 0$$

as $n \longrightarrow \infty$, $m > n$, and $(m-n)n^{-1/p(f)} \longrightarrow 0$, then s_k is convergent.

(b) If all the coefficients a_{nk} of the method (E,f) are nonnegative (so that $p(f) = 2$), a sequence of real numbers (s_k) is summable (E,f), and

$$\underline{\lim}(s_m - s_n) \geq 0$$

as $n \longrightarrow \infty$, $m > n$, and $(m-n)n^{-1/2} \longrightarrow 0$, then (s_k) is convergent.

Theorem 1 clearly implies the following

COROLLARY. *(a) If a sequence of complex numbers (s_k) is summable (E,f), and*

$$s_k - s_{k-1} = O(k^{-1/p(f)}),$$

then (s_k) is convergent.

(b) If all the coefficients a_{nk} of the method (E,f) are nonnegative, a sequence of real numbers (s_k) is summable (E,f), and there is a positive constant M such that for each k,

$$s_k - s_{k-1} \geq -Mk^{-1/2},$$

then (s_k) is convergent.

Theorem 1(b) contains, for instance, the Tauberian theorem for the Borel method of summability [8, Theorem 241] and that for the Meyer-König method [11]. Indeed, as we have remarked earlier, examples 1–5 of the method (E,f) satisfy $p(f) = 2$. This fact and the validity of Theorem 1 explain why these methods have the same Tauberian condition.

The corresponding result for the Karamata method is new. A Tauberian theorem for the Karamata method has been obtained by Fridy and Powell in [6]. Their Tauberian condition, $s_k = O(1)$ and $s_k - s_{k-1} = o(1/k)$, is stronger than ours. Bingham [3] has found our Tauberian condition for two principal special cases ($\alpha = 0$ and $\beta = 1 - \alpha$) of the method. He has also treated the Karamata-Stirling method, which is related to the Karamata method, in [4].

We also note that as $p(f)$ increases, the Tauberian condition in Theorem 1 becomes weaker. Hence if (E,f) and (E,g) are regular methods and $p(f) > p(g)$ then the former is, loosely speaking, closer to convergence than the latter.

So far we have mentioned examples of methods (E,f) with $p(f) = 2$ only. [5] contains the following question: is there a regular method (E,f) with a single maximum of $|f(z)|$ on $|z| = 1$ and with $p(f) > 2$? We will answer the question positively. In Section 2 we will show that for each positive even integer there is a regular method (E,f) with $p(f)$ equal to that integer. (Recall that $p(f)$ is always even.) We will prove Theorem 1 in Section 3. The proof relies on Pitt's Tauberian theorem. In Section 4 we show by an example that the exponent $-1/p(f)$ in Theorem 1 is the best possible.

2. **More examples of the method.** Let

$$f(z) = z^{2k} - \big((z-1)/2\big)^{4k},$$

where k is a positive integer. Then f clearly satisfies conditions 1 and 3 in Theorem A. For $0 < t < 2\pi$ we have

$$\begin{aligned}\left|(e^{it})^{2k} - \big((e^{it}-1)/2\big)^{4k}\right| &= \left|e^{2kit} - e^{2kit}\big((e^{it/2} - e^{-it/2})/2\big)^{4k}\right| \\ &= |1 - \sin^{4k}(t/2)| < 1.\end{aligned}$$

Hence by the maximum modulus principle $f(z)$ satisfies condition 2. Since $f'(1) = 2k$ and

$$f(z) - z^{2k} = -(1/2)^{4k}(i)^{4k}(z-1)^{4k}$$

$f(z)$ satisfies condition 4 as well. By Theorem A, (E,f) is regular. Moreover the last equality shows that $p(f) = 4k$.

Similarly we can prove that if

$$g(z) = z^{2k+1} + \big((z-1)/2\big)^{4k+2}, \quad k \geq 0$$

then (E, g) is regular and $p(g) = 4k + 2$.

Thus for each positive even integer there is a method (E,f) with $p(f)$ equal to that integer.

3. **Proof of Theorem 1.** First we will prove part (a) of the theorem. Thus, we assume that f satisfies the conditions of Theorem A, so that (E,f) is a regular method, (s_k) is summable (E,f) to S, and

$$\lim(s_m - s_n) = 0$$

as $n \longrightarrow \infty$, $m > n$, and $(m-n)n^{-1/p(f)} \longrightarrow 0$.

Without loss of generality we may assume that $S = 0$. For otherwise we just have to consider $t_k = s_k - S$. Hence,

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}s_k = 0.$$

We will denote constants by K, not necessarily the same at each occurrence. But a letter with a subscript, e.g., K_1, denotes always the same constant. For simplicity we will write p, A, and α instead of $p(f)$, $A(f)$, and $\alpha(f)$.

We need seven lemmas. The proof of the first one is similar to that of Lemma ϵ in [13] (or [8, Theorem 239]) and will be omitted.

LEMMA 1. *Let (s_k) be a sequence which satisfies the given Tauberian condition. Then there is a constant $M_1 > 0$ such that if $m > n$, we have*

$$|s_m - s_n| < M_1(m-n)n^{-1/p} + 1.$$

LEMMA 2. *If the hypotheses of Theorem 1 are satisfied then (s_k) is bounded.*

PROOF. Suppose we have shown that the subsequence $(s_{[\alpha n]})_{n\geq 0}$ is bounded. Then we can prove the lemma easily as follows. For every positive integer k, there exists n such that $\alpha n < k \leq \alpha(n+1)$, and we have

$$\begin{aligned} |s_k| &\leq |s_k - s_{[\alpha n]}| + |s_{[\alpha n]}| \\ &\leq |s_k - s_{[\alpha n]}| + O(1) \\ &\leq M_1(k - [\alpha n])[\alpha n]^{-1/p} + 1 + O(1), \quad \text{by Lemma 1,} \\ &\leq M_1\alpha[\alpha n]^{-1/p} + O(1). \end{aligned}$$

Hence $s_k = O(1)$.

Since $\sum_{k=0}^{\infty} a_{nk}s_k = o(1)$, to prove that $s_{[\alpha n]} = O(1)$ it suffices to show that

$$\sum_{k=0}^{\infty} a_{nk}s_k - s_{[\alpha n]} = O(1).$$

Since

$$f(1) = 1 = \big(f(1)\big)^n = \sum_{k=0}^{\infty} a_{nk}, \quad n = 0, 1, \ldots$$

we have

$$\sum_{k=0}^{\infty} a_{nk}s_k - s_{[\alpha n]} = \sum_{k=0}^{\infty} a_{nk}s_k - s_{[\alpha n]} \sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} a_{nk}(s_k - s_{[\alpha n]}).$$

Let $0 < H < 1$. We will prove that the following three sums

$$\sum_{0 \le k < (1-H)\alpha n} a_{nk}(s_k - s_{[\alpha n]}), \quad \sum_{(1-H)\alpha n \le k \le (1+H)\alpha n} a_{nk}(s_k - s_{[\alpha n]}),$$
$$\text{and} \quad \sum_{(1+H)\alpha n < k} a_{nk}(s_k - s_{[\alpha n]})$$

are bounded.

By Lemma 1,

$$|s_k - s_{k-1}| \le M_1 k^{-1/p} + 1$$

for every k. Hence

$$|s_k| \le Kk.$$

Thus we can estimate the first and third of the above sums as follows.

$$\Big|\sum_{0 \le k < (1-H)\alpha n} a_{nk}(s_k - s_{[\alpha n]})\Big| \le K \sum_{0 \le k < (1-H)\alpha n} n|a_{nk}| = KS_1, \text{ and}$$
$$\Big|\sum_{(1+H)\alpha n < k} a_{nk}(s_k - s_{[\alpha n]})\Big| \le K \sum_{(1+H)\alpha n < k} k|a_{nk}| = KS_2,$$

where S_1 and S_2 have the obvious definitions.

By Lemma 1 again, we estimate the second of the above sums.

$$\begin{aligned}
&\Big|\sum_{(1-H)\alpha n \le k \le (1+H)\alpha n} a_{nk}(s_k - s_{[\alpha n]})\Big| \\
&\quad \le \sum_{(1-H)\alpha n \le k \le \alpha n} \big(Kn^{-1/p}(\alpha n - k) + 1\big)|a_{nk}| \\
&\quad\quad + \sum_{\alpha n < k \le (1+H)\alpha n} \big(Kn^{-1/p}(k - [\alpha n]) + 1\big)|a_{nk}| \\
&\quad \le Kn^{-1/p} \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k)|a_{nk}| \\
&\quad\quad + Kn^{-1/p} \sum_{\alpha n < k \le (1+H)\alpha n} (k - [\alpha n])|a_{nk}| + \sum_{k=0}^{\infty} |a_{nk}| \\
&\quad = KS_3 + KS_4 + \sum_{k=0}^{\infty} |a_{nk}|, \text{ say.}
\end{aligned}$$

Since f satisfies the conditions of Theorem A, (E,f) is regular. By the Toeplitz-Schur theorem $\sum_{k=0}^{\infty}|a_{nk}| = O(1)$, $n \longrightarrow \infty$. Hence to complete the proof it suffices to show that S_1, S_2, S_3, and S_4 are bounded. In fact we will see that S_1 and S_2 tend to zero as $n \longrightarrow \infty$.

The rest of the proof of Lemma 2 is similar to that of Théorème 1 in [1].

Let $g(z) = z^{-\alpha}f(z)$ and $\psi(r,t) = \log|g(re^{it})|$. Then (see [1, pp. 137–138]) there are positive numbers ε, δ, N_0, and K such that N_0 is so large that $f(z)$ is analytic on $|z| < 1 + N_0^{-1/p}$,

(1) $$|f(re^{it})| \leq 1-\delta \text{ for } |t| \geq \varepsilon \text{ and } |r-1| \leq N_0^{-1/p}, \text{ and}$$

(2) $$\int_{-\varepsilon}^{\varepsilon} |g(re^{it})|^n\,dt = \int_{-\varepsilon}^{\varepsilon} e^{n\psi(r,t)}\,dt \leq Kn^{-1/p},$$

where $r = 1 \pm n^{-1/p}$, and $n > N_0$.

Let C_n be the circle centered at the origin with radius $1 - n^{-1/p}$. By Cauchy's integral formula we have, with $r = 1 - n^{-1/p}$ and $n > N_0$,

$$\begin{aligned} S_1 &= (n/2\pi) \sum_{0\leq k<(1-H)\alpha n} \left| \int_{C_n} \big(f(z)\big)^n z^{-k-1}\,dz \right| \\ &\leq (n/2\pi) \sum_{0\leq k<(1-H)\alpha n} \int_{\varepsilon}^{2\pi-\varepsilon} |f(re^{it})|^n r^{-k}\,dt \\ &\quad + n \sum_{0\leq k<(1-H)\alpha n} \int_{-\varepsilon}^{\varepsilon} |f(re^{it})|^n r^{-k}\,dt \\ &\leq n(1-\delta)^n \sum_{0\leq k<(1-H)\alpha n} r^{-k} + n \sum_{0\leq k<(1-H)\alpha n} r^{\alpha n-k} \int_{-\varepsilon}^{\varepsilon} |g(re^{it})|^n\,dt, \end{aligned}$$

by (1) and the definition of g.

Since $r = 1 - n^{-1/p}$ the first term on the right is less than or equal to

$$\begin{aligned} & n(1-\delta)^n(1-n^{-1/p})^{-(1-H)\alpha n} \sum_{0\leq k<(1-H)\alpha n} (1-n^{-1/p})^{(1-H)\alpha n-k} \\ &\qquad \leq n(1-\delta)^n(1-n^{-1/p})^{-(1-H)\alpha n} \sum_{k=0}^{\infty} (1-n^{-1/p})^k \\ &\qquad \leq n(1-\delta)^n(1-n^{-1/p})^{-(1-H)\alpha n} \frac{1}{1-(1-n^{-1/p})} \\ &\qquad \leq n^{1+1/p}(1-\delta)^n(1-n^{-1/p})^{-(1-H)\alpha n} \\ &\qquad = o(1) \text{ as } n \longrightarrow \infty. \end{aligned}$$

To estimate the second term we apply inequality (2). We have

$$\begin{aligned} n \sum_{0\leq k<(1-H)\alpha n} r^{\alpha n-k} \int_{-\varepsilon}^{\varepsilon} |g(re^{it})|^n\,dt &\leq n \sum_{0\leq k<(1-H)\alpha n} r^{\alpha n-k} K n^{-1/p} \\ &\leq Kn^{1-1/p} \sum_{0\leq k<(1-H)\alpha n} (1-n^{-1/p})^{\alpha n-k} \\ &\leq Kn(1-n^{-1/p})^{H\alpha n} \\ &= o(1) \text{ as } n \longrightarrow \infty. \end{aligned}$$

Hence $S_1 = o(1)$ as $n \longrightarrow \infty$.

The proof that $S_2 = o(1)$ as $n \longrightarrow \infty$ is similar. Here we represent a_{nk} as a Cauchy integral over a circle centered at the origin with radius $1 + n^{-1/p}$, where $n > N_0$. We omit the details.

Next we show that S_3 is bounded. Again we represent a_{nk} as a Cauchy integral over the circle C_n. We have, with $r = 1 - n^{-1/p}$ and $n > N_0$,

$$\begin{aligned} S_3 &= n^{-1/p} \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k) \left| (1/2\pi) \int_{C_n} \big(f(z)\big)^n z^{-k-1}\, dz \right| \\ &\le (n^{-1/p}/2\pi) \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k) \int_{\varepsilon}^{2\pi-\varepsilon} |f(re^{it})|^n r^{-k}\, dt \\ &\quad + n^{-1/p} \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k) \int_{-\varepsilon}^{\varepsilon} |f(re^{it})|^n r^{-k}\, dt \\ &\le n^{-1/p}(1-\delta)^n \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k) r^{-k} \\ &\quad + n^{-1/p} \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k) r^{\alpha n - k} \int_{-\varepsilon}^{\varepsilon} |g(re^{it})|^n\, dt, \\ &\quad \text{by (1) and the definition of } g, \\ &\le n^{-1/p}(1-\delta)^n \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k) r^{-k} \\ &\quad + n^{-1/p} \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k) r^{\alpha n - k} K n^{-1/p}, \qquad \text{by (2)}, \\ &\le n^{-1/p}(1-\delta)^n \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k)(1 - n^{-1/p})^{-k} \\ &\quad + K n^{-2/p} \sum_{(1-H)\alpha n \le k \le \alpha n} (\alpha n - k)(1 - n^{-1/p})^{\alpha n - k}. \end{aligned}$$

It is easy to see that the first term on the right $= o(1)$ as $n \longrightarrow \infty$, the dominant factor being $(1-\delta)^n$. Since

$$\sum_{k=0}^{\infty} k a^{k-1} = (1-a)^{-2}$$

if $|a| < 1$, the second term is less than

$$K n^{-2/p} \Big(1 - (1 - n^{-1/p})\Big)^{-2} = K.$$

Thus S_3 is bounded. Similarly, representing a_{nk} as a Cauchy integral over the circle centered at the origin with radius $1 + n^{-1/p}$ we can prove that S_4 is bounded. This completes the proof of Lemma 2.

Lemma 2 plays a crucial role in the proof. Assuming that (s_k) is bounded and $S_k \longrightarrow 0(E,f)$ we can prove Lemmas 3–6, as we will see. This remark will be important in the proof of Theorem 1(b).

LEMMA 3. *$\lim_{n\to\infty} \sum_{\log n < |\alpha n - k| < n^{1/p}\log n} n^{-1/p} s_k \phi\big((\alpha n - k) n^{-1/p}\big) = 0$, where*

$$\phi(x) = \int_{-\infty}^{\infty} \exp(At^p + ixt)\, dt.$$

https://doi.org/10.4153/CJM-1992-067-9 Published online by Cambridge University Press

PROOF. Girard's paper [7], which is based on his doctoral thesis, contains the essential ingredients required for the proof of Lemma 3. Since

$$a_{nk} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^n(e^{it})e^{-ikt}\,dt$$

and, by Lemma 2, (s_k) is bounded, it follows from the proof in [7, pp. 362–364] that for each $\varepsilon > 0$, we have

$$\sum_{k=0}^{\infty} a_{nk}s_k = \frac{1}{2\pi}\sum_{\log n<|\alpha n-k|<n^{1/p}\log n} s_k\int_{-\varepsilon}^{\varepsilon} f^n(e^{it})e^{-ikt}\,dt + o(1) \text{ as } n \longrightarrow \infty.$$

We choose ε so small that the following three conditions hold.

(A) For $|t| \le \varepsilon$,

$$f(e^{it})e^{-i\alpha t} = 1 + Ai^p e^{-i\alpha t}(e^{it}-1)^p + O(t^{p+1}) = \exp\big(At^p + G(t)\big),$$

where $|G(t)| \le K_1|t|^{p+1}$ for some constant K_1.

Such a function G exists because of condition 4 of Theorem A and the fact that if $|t|$ is small enough, then

$$\begin{aligned} Ai^p e^{-i\alpha t}(e^{it}-1)^p &= Ai^p\big(1+O(t)\big)\big((it)^p + O(t^{p+1})\big)\\ &= Ai^{2p}t^p + O(t^{p+1})\\ &= At^p + O(t^{p+1}), \end{aligned}$$

since p is an even integer.

(B) $\Re A + K_1\varepsilon < 0$.

(C) $|\exp(x) - 1| \le 2|x|$ if $|x| \le K_1\varepsilon^{p+1}$.

For simplicity let $T(n) = \{k : \log n < |\alpha n - k| < n^{1/p}\log n\}$. We note that the number of elements in $T(n)$ is less than $2n^{1/p}\log n$.

By condition A we have

$$\begin{aligned} \sum_{k\in T(n)} s_k\int_{-\varepsilon}^{\varepsilon} f^n(e^{it})e^{-ikt}\,dt &= \sum_{k\in T(n)} s_k\int_{-\varepsilon}^{\varepsilon}\big(f(e^{it})e^{-i\alpha t}\big)^n e^{i(\alpha n-k)t}\,dt\\ &= \sum_{k\in T(n)} s_k\int_{-\varepsilon}^{\varepsilon}(e^{At^p+G(t)})^n e^{i(\alpha n-k)t}\,dt\\ &= \sum_{k\in T(n)} s_k\int_{-\varepsilon}^{\varepsilon}\exp\big(nAt^p + nG(t) + i(\alpha n-k)t\big)\,dt. \end{aligned}$$

Making the substitution $v = n^{1/p}t$ in the preceding integrals we have

$$\begin{aligned} &\sum_{k\in T(n)} s_k\int_{-\varepsilon}^{\varepsilon} f^n(e^{it})e^{-ikt}\,dt\\ &\quad= \sum_{k\in T(n)} s_k n^{-1/p}\int_{-\varepsilon n^{1/p}}^{\varepsilon n^{-1/p}}\exp\big(Av^p + nG(n^{-1/p}v) + i(\alpha n-k)n^{-1/p}v\big)\,dv\\ &\quad= \sum_{k\in T(n)} s_k n^{-1/p}\int_{-\varepsilon n^{1/p}}^{\varepsilon n^{1/p}}\exp\big(Av^p + i(\alpha n-k)n^{-1/p}v\big)\,dv + U, \end{aligned}$$

where $U =$

$$\sum_{k\in T(n)} s_k n^{-1/p} \int_{-\varepsilon n^{1/p}}^{\varepsilon n^{1/p}} \left\{\exp\left(Av^p + i(\alpha n - k)n^{-1/p}v\right)\right\}\left\{\exp\left(nG(n^{-1/p}v)\right) - 1\right\} dv.$$

We write

$$U = \sum_{k\in T(n)} s_k n^{-1/p}\left(\int_{-\varepsilon n^{1/p(p+1)}}^{\varepsilon n^{1/p(p+1)}} + \int_{\varepsilon n^{1/p(p+1)}}^{\varepsilon n^{1/p}} + \int_{-\varepsilon n^{1/p}}^{-\varepsilon n^{1/p(p+1)}}\right)\cdots dv$$
$$= U_1 + U_2 + U_3.$$

First we show that $U_1 = o(1)$ as $n \longrightarrow \infty$.

If $|v| \leq n^{1/p}\varepsilon$ then $|n^{-1/p}v| \leq \varepsilon$. Hence by condition A we have

$$|nG(n^{-1/p}v)| \leq nK_1|n^{-1/p}v|^{p+1} = K_1 n^{-1/p}|v|^{p+1}. \tag{3}$$

If in addition we have $|v| \leq \varepsilon n^{1/p(p+1)}$ then $|v|^{p+1} \leq n^{1/p}\varepsilon^{p+1}$. Hence by (3),

$$|nG(n^{-1/p}v)| \leq K_1 n^{-1/p} n^{1/p}\varepsilon^{p+1} = K_1\varepsilon^{p+1}.$$

It follows from this inequality and condition C that

$$|U_1| \leq \sum_{k\in T(n)} |s_k| n^{-1/p} \int_{-\varepsilon n^{1/p(p+1)}}^{\varepsilon n^{1/p(p+1)}} \{\exp(\Re A v^p)\} 2|nG(n^{-1/p}v)|\, dv.$$

Hence by (3) and the fact that (s_k) is bounded,

$$|U_1| \leq Kn^{-1/p} \sum_{k\in T(n)} \int_{-\varepsilon n^{1/p(p+1)}}^{\varepsilon n^{1/p(p+1)}} \{\exp(\Re A v^p)\} K_1 n^{-1/p}|v|^{p+1}\, dv$$
$$\leq Kn^{-2/p} \sum_{k\in T(n)} \int_{-\infty}^{\infty} \{\exp(\Re A v^p)\}|v|^{p+1}\, dv$$
$$\leq Kn^{-2/p} \sum_{k\in T(n)} 1$$
$$\leq Kn^{-2/p}(n^{1/p}\log n),$$

since the number of elements in $T(n)$ is less than $2n^{1/p}\log n$.

This shows that $U_1 \longrightarrow 0$ as $n \longrightarrow \infty$.

Next we prove that $U_2 \longrightarrow 0$ as $n \longrightarrow \infty$.

For $|v| \leq \varepsilon n^{1/p}$, or $|n^{-1/p}v| \leq \varepsilon$, we have, by condition A,

$$n|G(n^{-1/p}v)| \leq nK_1(n^{-1/p}|v|)^{p+1} \leq K_1(n^{-1/p}|v|)|v|^p \leq K_1\varepsilon|v|^p.$$

Therefore if $|v| \leq \varepsilon n^{1/p}$ then

$$\left|\exp\left(nG(n^{-1/p}v)\right) - 1\right| \leq \exp(K_1\varepsilon|v|^p).$$

Hence

$$|U_2| \leq Kn^{-1/p} \sum_{k\in T(n)} \int_{-\varepsilon n^{1/p(p+1)}}^{\varepsilon n^{1/p}} \exp(\Re A|v|^p)\exp(K_1\varepsilon|v|^p)\, dv.$$

Since $\Re A + K_1\varepsilon < 0$ (by condition B) we have,

$$\begin{aligned}
|U_2| &\leq Kn^{-1/p}\int_{\varepsilon n^{1/p(p+1)}}^{\infty} \exp\big((\Re A + K_1\varepsilon)|v|^p\big)\,dv \sum_{k\in T(n)} 1 \\
&\leq Kn^{-1/p}n^{1/p}\log n \int_{\varepsilon n^{1/p(p+1)}}^{\infty} \exp\big((\Re A + K_1\varepsilon)|v|^p\big)\,dv, \\
&\quad \text{since the number of elements in } T(n) \text{ is less than } 2n^{1/p}\log n, \\
&\leq K\log n \int_{\varepsilon n^{1/p(p+1)}}^{\infty} \exp\big((\Re A + K_1\varepsilon)v\big)\,dv, \ \text{if } \varepsilon n^{1/p(p+1)} > 1, \\
&\leq K\log n \frac{1}{|\Re A + K_1\varepsilon|} \exp\big((\Re A + K_1\varepsilon)n^{1/p(p+1)}\varepsilon\big) \\
&= o(1) \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Similarly we can show that $U_3 \longrightarrow 0$ as $n \longrightarrow \infty$.

Hence we have

$$\begin{aligned}
\sum_{k=0}^{\infty} a_{nk}s_k &= \frac{1}{2\pi}\sum_{k\in T(n)} s_k \int_{-\varepsilon}^{\varepsilon} f^n(e^{it})e^{-ikt}\,dt + o(1) \\
&= \frac{1}{2\pi}\sum_{k\in T(n)} s_k n^{-1/p}\int_{-\varepsilon n^{1/p}}^{\varepsilon n^{1/p}} \exp\big(Av^p + i(\alpha n - k)n^{-1/p}v\big)\,dv \\
&\quad + o(1),
\end{aligned}$$

as $n \longrightarrow \infty$.

We now show that we may extend the limits of the last integrals to infinity. For every real number r_{nk}, we have

$$\begin{aligned}
\Big|\phi(r_{nk}) &- \int_{-\varepsilon n^{1/p}}^{\varepsilon n^{1/p}} \exp(Av^p + ir_{nk})\,dv\Big| \\
&= \Big|\int_{-\infty}^{\infty} \exp(Av^p + ir_{nk}v)\,dv - \int_{-\varepsilon n^{1/p}}^{\varepsilon n^{1/p}} \exp(Av^p + ir_{nk}v)\,dv\Big| \\
&\leq \Big(\int_{-\infty}^{-\varepsilon n^{1/p}} + \int_{\varepsilon n^{1/p}}^{\infty}\Big) \exp(\Re Av^p)\,dv \\
&\leq 2\int_{\varepsilon n^{1/p}}^{\infty} \exp(\Re Av^p)\,dv, \ \text{since } p \text{ is even}, \\
&\leq 2\int_{\varepsilon n^{1/p}}^{\infty} \exp(\Re Av)\,dv, \ \text{if } \varepsilon n^{1/p} > 1, \\
&= \frac{2\exp(\Re A\varepsilon n^{1/p})}{|\Re A|}.
\end{aligned}$$

Hence if $\varepsilon n^{1/p} > 1$ then

$$\Big|\sum_{k\in T(n)} s_k n^{-1/p}\Big\{\phi\big((\alpha n-k)n^{-1/p}\big) - \int_{-\varepsilon n^{1/p}}^{\varepsilon n^{1/p}} \exp\big(Av^p + i(\alpha n-k)n^{-1/p}v\big)\,dv\Big\}\Big|$$

$$\le Kn^{-1/p}\sum_{k\in T(n)} \exp(\Re A\varepsilon n^{1/p})$$

$$\le Kn^{-1/p}n^{1/p}(\log n)\exp(\Re A\varepsilon n^{1/p}),$$

since the number of elements of $T(n)$ is less than $2n^{1/p}\log n$,

$$\le K(\log n)\exp(\Re A\varepsilon n^{1/p})$$

$$= o(1) \text{ as } n \longrightarrow \infty.$$

It follows that

$$\sum_{k=0}^{\infty} a_{nk}s_k = \frac{1}{2\pi}\sum_{k\in T(n)} s_k n^{-1/p}\phi\big((\alpha n-k)n^{-1/p}\big) + o(1), \tag{4}$$

as $n \longrightarrow \infty$. By the hypothesis that $\sum_{k=0}^{\infty} a_{nk}s_k \longrightarrow 0$, we have

$$\lim_{n\to\infty}\sum_{k\in T(n)} s_k n^{-1/p}\phi\big((\alpha n-k)n^{-1/p}\big) = 0.$$

The proof of Lemma 3 is complete.

LEMMA 4. *$\lim_{n\to\infty} n^{-1/p}\int_0^\infty s(t)\phi\big((\alpha n-t)n^{-1/p}\big)\,dt = 0$, where $s(t) = s_{[t]}$, where in turn $[t]$ denotes the greatest integer in t.*

PROOF. We begin with a few remarks about the function ϕ. Since ϕ is the inverse Fourier transform of the function $\exp(At^p)$, which is rapidly decreasing, it itself is rapidly decreasing. Hence ϕ is bounded, infinitely differentiable and integrable on the real line. Also, for every nonegative integers m and n, $x^m\phi^{(n)}(x)$ is bounded, where as usual $\phi^{(0)} = \phi$ and $\phi^{(n)}$ is the nth derivative of ϕ if $n \ge 1$. It follows that the integral in Lemma 4 exists. Finally we note that ϕ is an even funtion, since p is an even integer.

We will prove (5)–(9) below, which imply the lemma.

$$\lim_{n\to\infty} n^{-1/p}\int_{\alpha n-\log n}^{\alpha n+\log n} s(t)\phi\big((\alpha n-t)n^{-1/p}\big)\,dt = 0. \tag{5}$$

$$\lim_{n\to\infty} n^{-1/p}\int_{\alpha n+n^{1/p}\log n}^{\infty} s(t)\phi\big((\alpha n-t)n^{-1/p}\big)\,dt = 0. \tag{6}$$

$$\lim_{n\to\infty} n^{-1/p}\int_{0}^{\alpha n-n^{1/p}\log n} s(t)\phi\big((\alpha n-t)n^{-1/p}\big)\,dt = 0. \tag{7}$$

$$\lim_{n\to\infty} n^{-1/p}\Big\{\int_{U(n)} s(t)\phi\big((\alpha n-t)n^{-1/p}\big)\,dt - \sum_{k\in U(n)} s_k\phi\big((\alpha n-k)n^{-1/p}\big)\Big\} = 0, \tag{8}$$

where $U(n)$ denotes the open interval $(\alpha n - n^{1/p}\log n, \alpha n - \log n)$.

$$\text{(9)}\quad \lim_{n\to\infty} n^{-1/p}\Big\{\int_{V(n)} s(t)\phi\big((\alpha n - t)n^{-1/p}\big)\,dt - \sum_{k\in V(n)} s_k\phi\big((\alpha n - k)n^{-1/p}\big)\Big\} = 0,$$

where $V(n)$ denotes the open interval $(\alpha n + \log n, \alpha n + n^{1/p}\log n)$.

Since $s(t)$ and ϕ are bounded we have

$$\Big|n^{-1/p}\int_{\alpha n-\log n}^{\alpha n+\log n} s(t)\phi\big((\alpha n - t)n^{-1/p}\big)\,dt\Big| \le Kn^{-1/p}\log n,$$

and (5) follows.

To prove (6) we note that since ϕ is an even function and is integrable,

$$\begin{aligned}\Big|n^{-1/p}\int_{\alpha n+n^{1/p}\log n}^{\infty} s(t)\phi\big((\alpha n - t)n^{-1/p}\big)\,dt\Big| \\ &\le Kn^{-1/p}\int_{\alpha n+n^{1/p}\log n}^{\infty}\Big|\phi\big((\alpha n - t)n^{-1/p}\big)\Big|\,dt \\ &\le K\int_{\log n}^{\infty}|\phi(v)|\,dv \\ &= o(1)\end{aligned}$$

as $n \longrightarrow \infty$.

Similarly we can prove (7).

Since

$$\begin{aligned} n^{-1/p}\Big\{\int_{U(n)} s(t)\phi\big((\alpha n - t)n^{-1/p}\big)\,dt \\ &- \sum_{k\in U(n)} s_k\phi\big((\alpha n - k)n^{-1/p}\big)\Big\} \\ &= n^{-1/p}\sum_{k\in U(n)}\Big\{\int_k^{k+1} s(t)\phi\big((\alpha n - t)n^{-1/p}\big)\,dt \\ &\qquad - \int_k^{k+1} s(t)\phi\big((\alpha n - k)n^{-1/p}\big)\,dt\Big\} \\ &+ n^{-1/p}\int_{\alpha n - n^{1/p}\log n}^{[\alpha n - n^{1/p}\log n]+1} s(t)\phi\big((\alpha n - t)n^{-1/p}\big)\,dt \\ &+ n^{-1/p}\int_{[\alpha n-\log n]}^{\alpha n - \log n} s(t)\phi\big((\alpha n - t)n^{-1/p}\big)\,dt,\end{aligned}$$

since the last two terms of the preceding line $= o(1)$ as $n \longrightarrow \infty$, since analogous statements hold for the interval $V(n)$, and since $s(t) = s_{[t]}$ is bounded, to prove (8) and (9) it suffices to prove that

$$n^{-1/p}\sum_{k\in T(n)}\int_k^{k+1}\Big|\phi\big((\alpha n - t)n^{-1/p}\big) - \phi\big((\alpha n - k)n^{-1/p}\big)\Big|\,dt = o(1),$$

where $T(n)$ is defined in the proof of Lemma 3.

By the mean value theorem, if $t \in [k, k+1]$ then

$$\Big|\phi\big((\alpha n - t)n^{-1/p}\big) - \phi\big((\alpha n - k)n^{-1/p}\big)\Big| \le n^{-1/p}|\phi'(\xi)|$$

for some ξ satisfying $(\alpha n - t)n^{-1/p} \leq \xi \leq (\alpha n - k)n^{-1/p}$.

Since ϕ is rapidly decreasing, ϕ' is bounded. Hence if $t \in [k, k+1]$ then

$$\left|\phi\big((\alpha n - t)n^{-1/p}\big) - \phi\big((\alpha n - k)n^{-1/p}\big)\right| \leq Kn^{-1/p}.$$

Thus

$$\begin{aligned} n^{-1/p} \sum_{k \in T(n)} \int_k^{k+1} &\left|\phi\big((\alpha n - t)n^{-1/p}\big) - \phi\big((\alpha n - k)n^{-1/p}\big)\right| dt \\ &\leq n^{-1/p} \sum_{k \in T(n)} Kn^{-1/p} \\ &\leq Kn^{-2/p} \sum_{k \in T(n)} 1 \\ &\leq Kn^{-2/p} 2n^{1/p} \log n \\ &= o(1) \text{ as } n \longrightarrow \infty. \end{aligned}$$

This completes the proof of (8) and (9).

LEMMA 5. *$\lim_{x\to\infty} x^{-1/p} \int_0^\infty s(t)\phi\big((\alpha/x)^{1/p}(t-x)\big)\,dt = 0$, where x is a continuous variable.*

PROOF. Let $n = [x/\alpha]$. Since ϕ is an even function it follows from Lemma 4 that

$$\lim_{x\to\infty} x^{-1/p} \int_0^\infty s(t)\phi\big((t - \alpha n)n^{-1/p}\big)\,dt = 0.$$

Thus it suffices to show, since $s(t)$ is bounded, that

$$I = x^{-1/p} \int_0^\infty \left|\phi\big((t-\alpha n)n^{-1/p}\big) - \phi\big((\alpha/x)^{1/p}(t-x)\big)\right| dt \longrightarrow 0$$

as $x \longrightarrow \infty$.

Making the change variable $u = (\alpha/x)^{1/p}(t-x)$ we have

$$\begin{aligned} I &= \alpha^{1/p} \int_{(\alpha/x)^{1/p}(-x)}^{\infty} \left|\phi\big(\beta(x)u + \gamma(x)\big) - \phi(u)\right| du \\ &\leq \alpha^{1/p} \int_{-\infty}^{\infty} \left|\phi\big(\beta(x)u + \gamma(x)\big) - \phi(u)\right| du, \end{aligned}$$

where $\beta(x) = (x/\alpha n)^{1/p}$ and $\gamma(x) = (x - \alpha n)n^{-1/p}$. Since $n = [x/\alpha]$, we have $\beta(x) \longrightarrow 1$ and $\gamma(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

Since ϕ is continuous $\phi\big(\beta(x)u + \gamma(x)\big) - \phi(u) \longrightarrow 0$ for each u as $x \longrightarrow \infty$. On the other hand there exists a positive constant K such that $|\phi(u)| \leq K/(u^2+1)$, ϕ being rapidly decreasing. Also $\big(\beta(x)u + \gamma(x)\big)^2 \geq u^2/4$ when x is large enough and $|u| \geq 1$, since $\beta(x) \longrightarrow 1$ and $\gamma(x) \longrightarrow 0$. Hence for large x and $|u| \geq 1$

$$\left|\phi\big(\beta(x)u + \gamma(x)\big)\right| \leq \frac{K}{(\beta(x)u + \gamma(x))^2 + 1} \leq \frac{K}{\frac{1}{4}u^2 + 1}.$$

By Lebesgue's dominated convergence theorem $I \longrightarrow 0$ as $x \longrightarrow \infty$. This proves Lemma 5.

https://doi.org/10.4153/CJM-1992-067-9 Published online by Cambridge University Press

LEMMA 6. $\lim_{x\to\infty}\int_0^\infty s(\alpha^{1-q}t^q)\phi\big(q(x-t)\big)\,dt = 0$, *where q satisfies* $1/p+1/q=1$.

PROOF. The proof is similar to that in [8, pp. 313–314]. By Lemma 5 we have

$$\lim_{x\to\infty} x^{-1/p}\int_0^\infty s(t)\phi(\alpha^{1/p}x^{-1/p}t-\alpha^{1/p}x^{1-1/p})\,dt = 0.$$

Let q satisfy $1/p+1/q=1$, let $x=\alpha^{1-q}y^q$ and make the substitution $t=\alpha^{1-q}u^q$ in the preceding integral. We have

$$\begin{aligned} x^{-1/p} &= \alpha^{(1-q)^2/q}y^{1-q},\\ \alpha^{1/p}x^{-1/p}t &= y^{1-q}u^q,\\ \alpha^{1/p}x^{1-1/p} &= y, \text{ and}\\ dt &= q\alpha^{1-q}u^{q-1}\,du. \end{aligned}$$

It follows that

$$\lim_{y\to\infty}\alpha^{(1-q)^2/q}y^{1-q}\int_0^\infty s(\alpha^{1-q}u^q)\phi(y^{1-q}u^q-y)q\alpha^{1-q}u^{q-1}\,du = 0, \text{ or}$$
$$\lim_{y\to\infty}\int_0^\infty \big(s(\alpha^{1-q}u^q)\big)(\frac{u}{y})^{q-1}\phi\big(y^{1-q}(u^q-y^q)\big)\,du = 0.$$

Since $s(\alpha^{1-q}u^q)$ is bounded Lemma 6 follows from the fact that

$$J=\int_0^\infty\Big|\phi\big(q(y-u)\big)-(\frac{u}{y})^{q-1}\phi\big(y^{1-q}(u^q-y^q)\big)\Big|\,du\longrightarrow 0 \text{ as } y\longrightarrow\infty,$$

which we will now prove.

Since we will let $y\longrightarrow\infty$ we may assume that $y>1$. Let $u=y+w$. Since ϕ is an even function we have

$$J=\int_{-y}^\infty\Big|\phi(qw)-\big((y+w)/y\big)^{q-1}\phi\Big(y^{1-q}\big((y+w)^q-y^q\big)\Big)\Big|\,dw.$$

Let

$$\lambda(x)=\begin{cases}\dfrac{(x+1)^q-1}{x} & \text{if } x\ge -1, x\ne 0,\\ q & \text{if } x=0.\end{cases}$$

It is easy to see that $\lambda(x)$ is continuous and has a positive minimum m, say, on $[-1,\infty)$. Furthermore we have

$$y^{1-q}\big((y+w)^q-y^q\big)=w\lambda(w/y).$$

Hence

$$J=\int_{-y}^\infty\Big|\phi(qw)-\big((y+w)/y\big)^{q-1}\phi\big(w\lambda(w/y)\big)\Big|\,dw.$$

Let a satisfy $0<3a<1$.

We write

$$J=J_1+J_2,$$

https://doi.org/10.4153/CJM-1992-067-9 Published online by Cambridge University Press

where

$$J_1 = \int_{|w|\geq y^a, w\geq -y} \left|\phi(qw) - \big((y+w)/y\big)^{q-1} \phi\big(w\lambda(w/y)\big)\right| dw, \text{ and}$$
$$J_2 = \int_{|w|<y^a} \left|\phi(qw) - \big((y+w)/y\big)^{q-1} \phi\big(w\lambda(w/y)\big)\right| dw.$$

Notice that since $y > 1$ and $0 < a < 1$, the restriction $w \geq -y$ is satisfied in J_2.

First we show that $J_1 \longrightarrow 0$ as $y \longrightarrow \infty$. Thus we assume that $|w| \geq y^a$. Since $y > 1$, if $w > 0$ then we have

$$w > y^a > 1.$$

Hence if y is large enough then

$$w < y(w-1), \text{ or } \frac{y+w}{y} < w. \tag{10}$$

On the other hand since $\lambda(x)$ has a positive minimum m on $[-1, \infty)$, if $w > 0$ then

$$w\lambda(w/y) \geq wm.$$

Since ϕ is rapidly decreasing there is a constant $K > 0$ such that

$$\left|\phi\big(w\lambda(w/y)\big)\right| \leq K|w\lambda(w/y)|^{-3} \leq Km^{-3}w^{-3}, \quad w > 0. \tag{11}$$

Since $1/p + 1/q = 1$ and $p \geq 2$ we have $q \leq 2$. So

$$q - 4 \leq -2. \tag{12}$$

Finally the fact that ϕ is an even function, (10), (11), and (12) yield, for large y,

$$\begin{aligned} J_1 &\leq \int_{|w|\geq y^a} |\phi(qw)|\, dw + \int_{|w|\geq y^a} \big((y+w)/y\big)^{q-1} \left|\phi\big(w\lambda(w/y)\big)\right| dw \\ &\leq \int_{|w|\geq y^a} |\phi(qw)|\, dw + \int_{|w|\geq y^a} w^{q-1} K m^{-3} w^{-3}\, dw \\ &\leq \int_{|w|\geq y^a} |\phi(qw)|\, dw + Km^{-3} \int_{|w|\geq y^a} w^{-2}\, dw. \end{aligned}$$

Hence $J_1 \longrightarrow 0$ as $y \longrightarrow \infty$.

Next we consider J_2. Here we assume that $|w| < y^a$. So $w/y = O(y^{a-1})$. Hence,

$$\big((y+w)/y\big)^{q-1} = (1 + w/y)^{q-1} = 1 + O(y^{a-1}).$$

If x is small then $\lambda(x) = q + O(x)$. Hence

$$w\lambda(w/y) = w\big(q + O(w/y)\big) = w\big(q + O(y^{q-1})\big) = wq + O(y^{2a-1}).$$

Thus

$$\begin{aligned} J_2 &= \int_{|w|<y^a} \left|\phi(qw) - \big(1 + O(y^{a-1})\big)\phi(wq + O(y^{2a-1}))\right| dw \\ &\leq \int_{|w|<y^a} \left|\phi(qw) - \phi(wq + O(y^{2a-1}))\right| dw \\ &\quad + O(y^{a-1}) \int_{|w|<y^a} \left|\phi(wq + O(y^{2a-1}))\right| dw. \end{aligned}$$

https://doi.org/10.4153/CJM-1992-067-9 Published online by Cambridge University Press

Since ϕ' is bounded, the mean value theorem implies that the first term on the right is less than

$$Ky^{2a-1}\int_{|w|<y^a} 1\,dw = 2Ky^{3a-1},$$

which tends to zero as $y \longrightarrow \infty$, $3a < 1$.

Since ϕ is bounded the second term is $O(y^{2a-1})$, which also tends to zero.

This proves Lemma 6.

LEMMA 7. *$s(\alpha^{1-q}t^q)$ is a slowly oscillating function.*

PROOF. By definition (see [8, p. 286]), we have to prove that

$$\lim\{s(\alpha^{1-q}u^q) - s(\alpha^{1-q}v^q)\} = 0,$$

as $v \longrightarrow \infty$, $u > v$, $u - v \longrightarrow 0$.

For simplicity let $y = \alpha^{1-q}u^q$ and $z = \alpha^{1-q}v^q$. Then

$$\begin{aligned}(y-z)z^{-1/p} &= (\alpha^{1-q}u^q - \alpha^{1-q}v^q)(\alpha^{1-q}v^q)^{-1/p} \\ &= \alpha^{-1/p}v\Big((\frac{u}{v})^q - 1\Big).\end{aligned}$$

Since $p \geq 2$, $q \leq 2$. Since $u > v$, $u/v > 1$. Hence

$$\begin{aligned}(y-z)z^{-1/p} &\leq \alpha^{-1/p}v\Big((\frac{u}{v})^2 - 1\Big) \\ &= K\frac{u^2 - v^2}{v} \\ &= K(u-v)(\frac{u}{v}+1) \\ &= o(1)\end{aligned}$$

when $v \longrightarrow \infty$, $u > v$, $u - v \longrightarrow 0$. By the Tauberian condition, $s_{[y]} - s_{[z]} \longrightarrow 0$. The lemma is proved.

PROOF OF THEOREM 1(A). We have to prove that $s_n \longrightarrow 0$. By Lemma 2 and Lemma 7, $s(\alpha^{1-q}u^q)$ is a bounded and slowly oscillating function. It will follow from Lemma 6 and Pitt's Tauberian theorem [8, Theorem 221] that $s(\alpha^{1-q}u^q) \longrightarrow 0$ as $u \longrightarrow \infty$ provided that the Fourier transforms of $\phi(qu)$ has no zeros. But ϕ is the inverse Fourier transform of the function $\exp(At^p)$ and is rapidly decreasing. Hence by the Fourier inversion theorem we have

$$\int_{-\infty}^{\infty}\phi(qu)e^{-ixu}\,du = \int_{-\infty}^{\infty}\phi(t)e^{-i(x/q)t}\frac{1}{q}\,dt = \frac{1}{q}\exp\big(A(x/q)^p\big).$$

This function indeed has no zeros. Let $u = (n\alpha^{q-1})^{1/q}$. Then we have

$$s(\alpha^{1-q}u^q) = s(n) = s_n \longrightarrow 0$$

as $n \longrightarrow \infty$. We have proved Theorem 1(a).

PROOF OF THEOREM 1(B). We will sketch the proof. We replace Lemma 1 by [8, Theorem 239] or [13, Lemma ϵ], according to which there exist positive numbers a and b such that for $q \geq p \geq 1$,

$$s_q - s_p \geq -a(\sqrt{q} - \sqrt{p}) - b. \tag{13}$$

Using (13) we can prove that

$$\sum_{k=0}^{M} a_{nk} \longrightarrow 0$$

as $M \longrightarrow \infty$, $n \longrightarrow \infty$, and $\sqrt{\alpha n} - \sqrt{M} \longrightarrow \infty$, and

$$\sum_{k=N}^{\infty} a_{nk}(\sqrt{k} - \sqrt{N}) \longrightarrow 0$$

as $N \longrightarrow \infty$, $n \longrightarrow \infty$, and $\sqrt{N} - \sqrt{\alpha n} \longrightarrow \infty$.

The proofs of these statements are similar to that of Lemma 2. With these facts on hand we next modify the proof of Theorem 238 in [8] to conclude that $(s_{[\alpha n]})_{n \geq 1}$ is bounded. Let k be an arbitrary positive integer. There exists n such that $\alpha n < k \leq \alpha(n+1)$. By (13) we have

$$\begin{aligned} s_k - s_{[\alpha n]} &\geq -a(\sqrt{k} - \sqrt{[\alpha n]}) - b \\ &= -a\Big(\frac{k - [\alpha n]}{\sqrt{k} + \sqrt{[\alpha n]}}\Big) - b \\ &\geq \frac{-a\alpha}{\sqrt{k} + \sqrt{[\alpha n]}} - b. \end{aligned}$$

Since $(s_{[\alpha n]})$ is bounded, this inequality implies that (s_k) is bounded from below. On the other hand we have

$$s_{[\alpha(n+1)]} - s_k \geq -a(\sqrt{[\alpha(n+1)]} - \sqrt{k}) - b,$$

and we can similarly prove that (s_k) is bounded from above.

Since (s_k) is bounded Lemmas 3–6 hold, as we have remarked earlier. Instead of Lemma 7 we show that $s(\alpha^{1-q}t^q)$ is a (real valued) slowly decreasing function. We then apply Pitt's Tauberian theorem to complete the proof.

4. **The Tauberian condition.** Can we replace the Tauberian condition in Theorem 1(a) by the weaker condition $\lim(s_m - s_n) = 0$ as $n \longrightarrow \infty$, $m > n$, and $(m - n)n^{c-1/p(f)} \longrightarrow 0$, where $0 < c < 1/p(f)$, or more generally, by $\lim(s_m - s_n) = 0$ as $n \longrightarrow \infty$, $m > n$, and $(m-n)r_n n^{-1/p(f)} \longrightarrow 0$, where (r_n) is an increasing sequence which tends to ∞, such that $r_n n^{-1/p(f)}$ is decreasing and tends to 0? If so then we could weaken the O-condition in the corollary of the theorem. Theorem 2 shows that this cannot be done. Thus the exponent $-1/p(f)$ in Theorem 1 is the best possible.

The proof of the theorem is a modification of an example due to Kwee [9].

For the best possible nature of Tauberian conditions in this and related context, see also [2] and [12].

https://doi.org/10.4153/CJM-1992-067-9 Published online by Cambridge University Press

THEOREM 2. *Let p_0 be a positive even integer. Let (r_k) be a sequence of increasing positive numbers tending to ∞. Then there is a bounded divergent sequence (s_k) which is summable by every regular method (E,f) with $p(f) = p_0$ and which satisfies*

$$s_k - s_{k-1} = O(r_k k^{-1/p_0}).$$

PROOF. Let (m_k) and (q_k) be two sequences of positive integers with the following properties.

1. $2k \le r_{q_k}, k = 1, 2, \ldots$.
2. $\frac{1}{2}(q_k)^{1/p_0} \le km_k \le (q_k)^{1/p_0}, k = 1, 2, \ldots$.
3. $2(q_k + 2m_k) < q_{k+1}, k = 1, 2, \ldots$.

Property 2 implies that

4. $m_k \le q_k, k = 1, 2, \ldots$.

By property 3 (q_k) is an increasing sequence.

Now we define (s_k). If k is outside intervals of the form $(q_k, q_k + 2m_k)$ then let $s_k = 0$. For each positive integer k, let

$$s_{q_k+w} = \begin{cases} w/m_k, & 1 \le w \le m_k, \\ (2m_k - w)/m_k, & m_k < w < 2m_k. \end{cases}$$

Hence $0 \le s_k \le 1$ for each k. Since $s_{q_k} = 0$, and $s_{q_k+m_k} = 1$ for each k, (s_k) is divergent. Next we prove that $s_k - s_{k-1} = O(r_k k^{-1/p_0})$.

Clearly,

$$|s_j - s_{j-1}| \le 1/m_k \text{ if } q_k \le j \le q_k + 2m_k.$$

By properties 2 and 1 we have

$$1/m_k \le (2k)(q_k)^{-1/p_0} \le r_{q_k}(q_k)^{-1/p_0}.$$

Since (r_k) is increasing, for $q_k \le j \le q_k + 2m_k$ we have

$$r_j(j)^{-1/p_0} \ge r_{q_k}(j)^{-1/p_0} \ge r_{q_k}(q_k + 2m_k)^{-1/p_0} \ge r_{q_k}(3q_k)^{-1/p_0}.$$

The last inequality follows from property 4.

Hence

$$r_{q_k}(q_k)^{-1/p_0} \le 3^{1/p_0} r_j(j)^{-1/p_0}, \text{ if } q_k \le j \le q_k + 2m_k.$$

It follows that for $q_k \le j \le q_k + 2m_k$ we have

$$|s_j - s_{j-1}| \le 1/m_k \le r_{q_k}(q_k)^{-1/p_0} \le 3^{1/p_0} r_j(j)^{-1/p_0}.$$

On the other hand if j is outside intervals of the form $[q_k, q_k + 2m_k]$ then $s_j - s_{j-1} = 0$. Thus for every j,

$$|s_j - s_{j-1}| \le 3^{1/p_0} r_j(j)^{-1/p_0} = O\big(r_j(j)^{-1/p_0}\big).$$

https://doi.org/10.4153/CJM-1992-067-9 Published online by Cambridge University Press

We still have to show that (s_k) is (E,f) summable to 0 if $p(f) = p_0$. Let (E,f) be such a method with matrix (a_{nk}). Since (s_k) is bounded equation (4) in the proof of Lemma 3 holds, i.e.,

$$\sum_{k=0}^{\infty} a_{nk}s_k = \frac{1}{2\pi} \sum_{k\in T(n)} s_k n^{-1/p_0}\phi\Big((\alpha n - k)n^{-1/p_0}\Big) + o(1), \text{ as } n \longrightarrow \infty,$$

where $T(n) = \{k : \log n < |\alpha n - k| < n^{1/p} \log n\}$, $\alpha = \alpha(f)$, and $\phi(x) = \int_{-\infty}^{\infty} \exp\Big(A(f)t^{p_0} + ixt\Big)\, dt$.

We will prove that

$$\lim_{n\longrightarrow\infty} \sum_{k\in T(n)} s_k n^{-1/p_0}\phi\Big((\alpha n - k)n^{-1/p_0}\Big) = 0.$$

Then it follows that (s_k) is summable (E,f) to 0.

If n is large enough then

$$\alpha n + n^{1/p_0} \log n < 2(\alpha n - n^{1/p_0} \log n).$$

Since $2(q_k + 2m_k) < q_{k+1}$ (by property 3), $T(n)$ overlaps with at most one of the intervals $[q_k, q_k + 2m_k]$. If it does not overlap with any of these intervals then the above sum is 0. If it does for some k then q_k or $q_k + 2m_k$ lies between $\alpha n - n^{1/p_0} \log n$ and $\alpha n + n^{1/p_0} \log n$. Hence $q_k = O(n)$. So

$$n^{-1/p_0} = O(q_k^{-1/p_0}).$$

Also, in this case the number of non-zero terms in the sum is at most $2m_k$, the length of the interval $[q_k, q_k + 2m_k]$. Since $0 \leq s_k \leq 1$ and ϕ is bounded, the sum is

$$\begin{aligned} O(m_k n^{-1/p_0}) &= O(m_k q_k^{-1/p_0}) \\ &= O(k^{-1}), \text{ by property 2}, \\ &= o(1). \end{aligned}$$

This completes the proof of Theorem 2.

ACKNOWLEDGEMENT. This paper contains some of the results in my doctoral thesis written under the supervision of Professor Bogdan Baishanski at the Ohio State University. I would like to thank Professor Baishanski for suggesting the problem of finding a Tauberian theorem for the general Euler-Borel method and for pointing out, among other things, the importance of the function ϕ in the proof of Theorem 1. Without the many discussions with him this paper would not have been completed. I am indebted to the referee for her/his comments and Dr. Man Keung Siu for introducing me to my typist.

References

1. B. Bajšanski, *Sur une classe générale de procédés de sommations du type d' Euler-Borel*, Acad. Serbe Sci. Publi. Insti. Math. **10**(1956), 131–152.

2. N. H. Bingham, *On Euler and Borel summability*, J. London Math. Soc. (2) **29**(1984), 141–146.

3. ______, *Tauberian theorems for summability methods of random-walk type*, J. London Math. Soc. (2) **30**(1984), 281–287.

4. ______, *Tauberian theorems for Jakimovski and Karamata-Stirling methods*, Mathematika **35**(1988), 216–224.

5. J. Clunie and P. Vermes, *Regular Sonnenschein type summability methods*, Acad. Roy. Belg. Bull. Cl. Sci. (5) **45**(1959), 930–954.

6. J. A. Fridy and R. E. Powell, *Tauberian theorems for matrices generated by analytic functions*, Pacific J. Math. 1 **92**(1981), 79–85.

7. D. M. Girard, *The asymptotic behavior of norms of powers of absolutely convergent Fourier series*, Pacific J. Math. **37**(1971), 357–381.

8. G. H. Hardy, *Divergent series*, Oxford University Press, 1949.

9. B. Kwee, *An improvement on a theorem of Hardy and Littlewood*, J. London Math. Soc. (2) 1 **28**(1983), 93–102.

10. D. Newman, *Homomorphisms of ℓ_+*, Amer. J. Math. **91**(1969), 37–46.

11. Y. Sitaraman, *On Tauberian theorems for the S_α-method of summability*, Math. Z **95**(1967), 34–49.

12. G. Tenenbaum, *Sur la procédé de sommation de Borel et la répartition du nombre des facteurs premiers des entiers*, Enseignement Math. **26**(1980), 225–245.

13. T. Vijayaraghavan, *A theorem concerning the summability of series by Borel's method*, Proc. London Math. Soc. (2) **27**(1928), 316–326.

14. K. Zeller and W. Beekman, *Theorie der Limitierungsverfahren*, Springer Verlag, 1970.

The Ohio State University
Columbus, Ohio

Current address:
Flat 1625
16 Lai Tak Tsuen Road
Tsuen Wing Lau
Hong Kong