Proceedings of the Edinburgh Mathematical Society (1993) 37, 47–51 ©

AN EXAMPLE OF A NON-EXPOSED EXTREME FUNCTION IN THE UNIT BALL OF H^1

by JYUNJI INOUE*

(Received 12th May 1992)

Dedicated to professor Mitsuru Nakai on his sixtieth birthday

We construct a non-exposed extreme function f of the unit ball of H^1, the classical Hardy space on the unit disc of the plane, which has the property: $f(z)/(1-q(z))^2 \notin H^1$ for any nonconstant inner function $q(z)$. This function constitutes a counterexample to a conjecture in D. Sarason [7].

1991 *Mathematics subject classification*: Primary 30 D 55, 46 J 15; secondary 46 B 20.

Let D be the open unit disc in the complex plane C, with boundary $T=\partial D$. For $1 \leqq p \leqq \infty$, $L^p = L^p(T)$ denotes the Lebesgue space on T, and $H^p = H^p(D)$ the Hardy space on D. Occasionally, we identify $f \in H^1(D)$ with its boundary function $f(e^{it})$ on T. For $\phi \in L^q(1/p+1/q=1)$, $L=L_\phi$ denotes a bounded linear functional on H^p defined by

$$L(f)=\int_0^{2\pi} f(e^{it})\phi(e^{it})\,dt/2\pi$$

with the norm $\|L\| = \sup\{|L(f)|: f \in H^p, \|f\|_p \leqq 1\}$. If L is nonzero, we put $S_L = \{f \in H^p: L(f) = \|L\|, \|f\|_p \leqq 1\}$. S_L is the solution set of a well-known linear extremal problem in H^p.

When $1 < p \leqq \infty$, the structure of the set S_L is simple, because S_L consists of exactly one point. But when $p=1$, the situation is quite different: S_L does not generally consist of one point. It may be empty, or a singleton, or an infinite set (cf. [1]).

In [2] deLeeuw and Rudin studied the set S_L, and determined the structure of S_L in some restricted cases. An element $f \in H^1$ is called an exposed point of the unit ball of H^1 if $S_L = \{f\}$ for some $L = L_\phi$ with $\phi \in L^\infty$. In [2], a function f in H^1 is called *strong outer* if f is not divisible in H^1 by any function of the form $(a-z)^2$ with $\alpha \in T$. It was proved that, in the restricted case they considered, f is an exposed point if and only if f is a strong outer function of norm 1. It is natural to ask whether this result can be extended to more general cases.

In [3] E. Hayashi presented examples of strong outer functions of norm 1 which are not exposed. His examples are of the form $(1-q(z))^2$ for some nonconstant inner

*This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

function $q(z)$, and, in [7], it is conjectured that $f \in H^1$ of norm 1 is exposed if and only if f is not divisible in H^1 by any function of the form $(1-q(z))^2$ with a nonconstant inner function $q(z)$.

The purpose of this paper is to present a counterexample to the conjecture in [7] stated above. Our example is related to an infinite product of outer functions, which takes non-negative boundary values a. e. on $T = \partial D$. For the problem of characterizing exposed points in H^1, we can consult papers such as [3], [4], [6] and [7].

Construction of a Counterexample. Let

$$F(z) = \prod_{k=1}^{\infty} \frac{(z-a_k)(1-\overline{a_k}z)}{(z-1)(1-z)} (z \in D) \tag{1}$$

where $a_k = e^{i/k^2}$, $k = 1, 2, 3, \ldots$ Note that the right-hand infinite product in (1) converges on each compact set K of $C\backslash\{1\}$, since we have

$$\sup_{z \in K} \sum_{k=1}^{\infty} \left|1 - \frac{z-a_k}{z-1}\right| + \left|1 - \frac{1-\overline{a_k}z}{1-z}\right| \leqq \sup_{z \in K} \frac{1+|z|}{|z-1|} \sum_{k=1}^{\infty} |1-e^{i/k^2}| < \infty.$$

The following properties of $F(z)$ hold:

(i) $F(z)$ is analytic on D and can be extended analytically across $T\backslash\{1\}$.

(ii) $F(e^{it}) \geqq 0$ for each $e^{it} \in T\backslash\{1\}$.

(iii) If $z \in \bar{D}\backslash\{1\}$, $F(z) = 0$ if and only if $z = \alpha_k$, and α_k is a zero of order 2 of F for $k = 1, 2, \ldots$

(iv) $F(z)$ is outer.

(i)~(iii) follows from the standard properties of infinite products. To see (iv), we consider $\log F(z)$ on $\bar{D}\backslash\{1\}$ such that $\operatorname{Im}[\log F(-1)] = 0$, where $\operatorname{Im} z$ means the imaginary part of the complex number z. Since $\operatorname{Im}[\log F(e^{it})]$ is a monotone decreasing step function on $(0, 2\pi)$ with jumps -2π at $t = 1/k^2$ $(k = 1, 2, \ldots)$ by the properties (i), (ii) and (iii) of $F(z)$ above, one sees that $\operatorname{Im}[\log F]$ is a real harmonic function belonging to the class $L \log^+ L$. Therefore the harmonic conjugates of $\operatorname{Im}[\log F]$ are in

$$h^1 = \left\{ u\text{: harmonic on } D,\ \sup_{0 \leqq r < 1} \int_0^{2\pi} |u(re^{it})|\, dt < \infty \right\}$$

(cf. [5]), which implies that $F(z)$ is outer.

Next, choose $\varepsilon_k > 0$ so that

$$\frac{1}{k^2} > \frac{1}{k^2} - \varepsilon_k > \frac{1}{(k+1)^2} + \varepsilon_{k+1}, |F(e^{it})| \leqq 1 \left(t \in \left(\frac{1}{k^2} - \varepsilon_k, \frac{1}{k^2} + \varepsilon_k \right) \right) k = 1, 2, \ldots;$$

and put

$$\Omega=\bigcup_{k=1}^{\infty}\left(\frac{1}{k^2}-\varepsilon_k,\frac{1}{k^2}+\varepsilon_k\right).$$

If we define a function $g(e^{it})\in L^1(T)$ by

$$g(e^{it})=\begin{cases}\min\left\{\dfrac{1}{|F(e^{it})|},1\right\}: t\in(0,2\pi)\backslash\Omega\\ \dfrac{1}{\varepsilon_k k^4}: t\in\left(\dfrac{1}{k^2}-\varepsilon_k,\dfrac{1}{k^2}+\varepsilon_k\right), \quad k=1,2,3,\ldots,\end{cases}$$

$\log g(e^{it})$ belongs to $L^1(T)$. Indeed, we can see this from:

$$\int_0^{2\pi}|\log g(e^{it})|\,dt$$

$$\leqq \int_{(0,2\pi)\backslash\Omega}|\log|F(e^{it})||\,dt+\sum_{k=1}^{\infty}\int_{1/k^2-\varepsilon_k}^{1/k^2+\varepsilon_k}\left|\log\frac{1}{\varepsilon_k k^4}\right|dt$$

$$\leqq \|\log|F|\|_1-2\varepsilon_1\log\varepsilon_1+\sum_{k=2}^{\infty}2\varepsilon_k\left(\log\frac{1}{\varepsilon_k}+\left|\log\frac{1}{k^2}\right|\right)$$

$$\leqq \|\log|F|\|_1-2\varepsilon_1\log\varepsilon_1+\sum_{k=2}^{\infty}\frac{4}{k^2}\left|\log\frac{1}{k^2}\right|<\infty,$$

since $1/k^2>\varepsilon_k$ by definition and $x\log 1/x$ is an increasing function of x on $(0,1/e)$.

Using $g(e^{it})$, we define an outer function $f(z)\in H^1(D)$ by

$$f(z)=\lambda\exp\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log g(e^{it})\,dt/2\pi\ (z\in D),$$

where λ is a positive constant to assure $\|f\|_1=1$. Then $f(z)$ has the following properties:

(a) For each δ $(2>\delta>0)$, $\inf\{|f(z)|: z\in D, |z-1|\geqq\delta\}>0$,

(b) $f(z)/(1-q(z))^2\notin H^1(D)$ for each non-constant inner function $q(z)$,

(c) $f(z)$ is not an exposed point of the unit ball of $H^1(D)$.

Proof of (a). We choose a continuous function $h(e^{it})$ on $T\backslash\{1\}$ such that $0<h(e^{it})\leqq \lambda g(e^{it})$ $(t\in(0,2\pi))$ with $\log h\in L^1(T)$, and let H the harmonic extension of $\log h$ to $\bar{D}\backslash\{1\}$.

Since $\log|f(z)|$ and $H(z)$ are the Poisson integrals of $\log\lambda g$ and $\log h$ respectively, we get for each δ $(0<\delta<2)$

$$\inf\{|f(z)|: z\in D, |z-1|\geqq\delta\}\geqq\inf\{\exp H(z): z\in\bar{D}, |z-1|\geqq\delta\}>0.$$

Proof of (b). First, we show that

$$\frac{f(z)}{(1-az)^2}\notin H^1$$

for each $a\in T$. By (a), we may assume $a=1$. Then

$$\int_0^\pi |f(e^{it})|\frac{1}{|1-e^{it}|^2}dt\geqq\int_\Omega \lambda g(e^{it})\frac{1}{|1-e^{it}|^2}dt$$

$$=\sum_{k=1}^\infty \lambda\int_{1/k^2-\varepsilon_k}^{1/k^2+\varepsilon_k}\frac{1}{\varepsilon_k\cdot k^4}\cdot\frac{1}{4\sin^2 t/2}dt$$

$$\geqq\lambda\sum_{k=1}^\infty\frac{1}{\varepsilon_k\cdot k^4}\cdot\frac{1}{4\sin^2 k^{-2}}\cdot 2\varepsilon_k=\lambda\sum_{k=1}^\infty\frac{1}{2}\left(\frac{k^{-2}}{\sin k^{-2}}\right)^2=\infty,$$

and hence $f(z)/(1-z)^2\notin H^1$.

Next, we consider the general case. Suppose that $f(z)/(1-q(z))^2\in H^1$ for an inner function $q(z)$. Then $(-q(z))/(1-q(z))^2$ is non-negative on a. e. on T, and belongs locally to H^1 at every point of $T\backslash\{1\}$ by (a). Therefore, $(-q(z))/(1-q(z))^2$ can be extended analytically beyond every point of $T\backslash\{1\}$ by the Schwarz reflection principle, and hence the singular support of $q(z)$ can exist only at $z=1$. If $q(z)\neq z$ and $\neq$constant, $1-q(z)$ takes zeros at some points of $T\backslash\{1\}$, and hence $f(z)/(1-q(z))^2\notin H^1(D)$, contradicting the assumption above. Therefore, if we recall that $f(z)/(1-z)^2\notin H^1$ which we proved above, the only possibility is $q(z)=$constant, which implies that (b) holds.

Proof of (c). Since

$$|f(e^{it})F(e^{it})|\leqq\begin{cases}\lambda: t\in\Omega\\ \dfrac{\lambda}{\varepsilon_k\cdot k^4}: t\in\left(\dfrac{1}{k^2}-\varepsilon_k,\dfrac{1}{k^2}+\varepsilon_k\right) k=1,2,\dots,\end{cases}$$

we get

$$\int_0^{2\pi}|f(e^{it})\cdot F(e^{it})|\,dt\leqq 2\pi\lambda+\sum_{k=1}^\infty\frac{\lambda}{\varepsilon_k\cdot k^4}\cdot 2\varepsilon_k<\infty$$

Thus, $f(z)F(z)/\|f(z)F(z)\|_1$ is in the boundary of the unit ball of H^1 and has the same argument as $f(z)$ at almost every point of T, and hence $f(z)$ is not an exposed point of the unit ball of H^1.

The author would like to express his thanks to the referee for careful reading and valuable suggestions.

REFERENCES

1. P. L. Duren, *Theory of H^p spaces* (Academic Press Inc., New York and London 1970).

2. K. deLeeuw and W. Rudin, Extreme points and extremum problems in H_1, *Pacific J. Math.* **8** (1958), 467–485.

3. E. Hayashi, Solution sets of extremal problems in H^1, *Proc. Amer. Math. Soc.* **93** (1985), 690–696.

4. H. Helson, Large analytic functions II, in *Analysis and Partial Differential Equations* (ed. Cora Sadosky, Marcel Dekker, 1990), 217–220.

5. P. Koosis, *Introduction of H^1* spaces (Cambridge University Press, 1980).

6. T. Nakazi, Sum of two inner functions and exposed points in H^1, *Proc. Edinburgh Math. Soc.* **35** (1992), 349–357.

7. D. Sarason, Exposed points in H^1, I, *Oper. Theory Adv. Appl.* **41** (1989), 485–496.

Department of Mathematics
Hokkaido University
Sapporo 060
Japan