2
Relativistic strings

This chapter is devoted to an introduction to bosonic strings and their quantisation. There is no attempt made at performing a rigourous or exhaustive derivation of some of the various formulae we will encounter, since that would take us well away from the main goal. That goal is to understand some of how string theory incorporates some of the familiar spacetime physics that we know from low energy field theory, and then rapidly proceed to the point where many of the remarkable properties which make strings so different from field theory are manifest. That will be a good foundation for appreciating just what D-branes really are. The careful reader who needs to know more of the details behind some of what we will introduce is invited to consult texts devoted to the study of string theory.

2.1 Motion of classical point particles

Let us start by reminding ourselves about a description of a point particle. We already touched on it in section 1.1, but we want to take it a bit further now, in preparation for doing the same thing for the analogous formulation for strings. The particle moves in the 'target spacetime' (with coordinates $(t \equiv X^0, X^1, \dots, X^{D-1})$) sweeping out a 'world-line' (see figure 1.1, page 2) parametrised by τ. We want to write an action principle which yields equations of motion for the allowed paths, $X^\mu(\tau)$.

2.1.1 Two actions

The most obvious action is the total path length swept out in spacetime. The infinitesimal path length traversed is:

$$d\ell = (-ds^2)^{1/2} = (-dX^\mu dX^\nu \eta_{\mu\nu})^{1/2} = (-dX^\mu dX_\mu)^{1/2}, \qquad (2.1)$$

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and we have assumed that the particle is massive and hence that $ds^2 < 0$. The massless case will be discussed below. So the action is

$$S_\mathrm{o} = -m \int d\ell = -m \int d\tau (-\dot{X}^\mu \dot{X}_\mu)^{1/2}, \tag{2.2}$$

where a dot denotes differentiation with respect to τ. Let us vary the action:

$$\begin{aligned} \delta S_\mathrm{o} &= m \int d\tau (-\dot{X}^\mu \dot{X}_\mu)^{-1/2} \dot{X}^\nu \delta \dot{X}_\nu = m \int d\tau u^\nu \delta \dot{X}_\nu \\ &= -m \int d\tau \dot{u}^\nu \delta X_\nu, \end{aligned} \tag{2.3}$$

where the last step used integration by parts, and

$$u^\nu \equiv (-\dot{X}^\mu \dot{X}_\mu)^{-1/2} \dot{X}^\nu. \tag{2.4}$$

So for δX arbitrary, we get $\dot{u}^\nu = 0$, which is Newton's Law of motion:

$$\frac{d^2 X^\mu}{d\tau^2} = 0, \tag{2.5}$$

where we have used $d\ell/d\tau = (-\dot{X}^\mu \dot{X}_\mu)^{1/2}$. There is another action from which we can derive the same physics. Consider the action

$$S = \frac{1}{2} \int d\tau \left(\eta^{-1} \dot{X}^\mu \dot{X}_\mu - \eta m^2 \right), \tag{2.6}$$

for some independent function $\eta(\tau)$ defined on the world-line.

N.B. In preparation for the coming treatment of strings, think of the function η as related to the particle's 'world-line metric', $\gamma_{\tau\tau}$, as $\eta(\tau) = [-\gamma_{\tau\tau}(\tau)]^{1/2}$. The function $\gamma(\tau)$ ensures world-line reparametrisation invariance:

$$ds^2 = \gamma_{\tau\tau} d\tau d\tau = \gamma_{\tau'\tau'} d\tau' d\tau'.$$

This is all a bit redundant in $0+1$ dimensions, but the structure will make more sense when we consider the 1+1 dimensions of the string's world-sheet.

If we vary S with respect to η:

$$\delta S = \frac{1}{2} \int d\tau \left[-\eta^{-2} \dot{X}^\mu \dot{X}_\mu - m^2 \right] \delta\eta. \tag{2.7}$$

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So for $\delta\eta$ arbitrary, we get an equation of motion

$$\eta^2 m^2 + \dot{X}^\mu \dot{X}_\mu = 0, \tag{2.8}$$

which we can solve with $\eta = m^{-1}(-\dot{X}^\mu \dot{X}_\mu)^{1/2}$. Upon substituting this into our expression (2.6) defining S, we get:

$$S = -\frac{1}{2}\int d\tau \left\{ m(-\dot{X}^\mu \dot{X}_\mu)^{1/2} + (-\dot{X}^\mu \dot{X}_\mu)^{1/2} m^{-1} m^2 \right\} = S_\mathrm{o}, \tag{2.9}$$

showing that the two actions are equivalent.

Notice, however, that the action S allows for a treatment of the massless, $m = 0$, case, in contrast to S_o. Another attractive feature of S is that it does not use the awkward square root that S_o does in order to compute the path length. The use of the 'auxiliary' parameter η allows us to get away from that.

2.1.2 *Symmetries*

There are two notable symmetries of the action.

- Spacetime Lorentz/Poincaré:

$$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + A^\mu,$$

  where Λ is an $SO(1,3)$ Lorentz matrix and A^μ is an arbitrary constant four-vector. This is a trivial global symmetry of S (and also S_o), following from the fact that we wrote them in covariant form.

- world-line reparametrisations:

$$\begin{aligned} \delta X &= \zeta(\tau)\frac{dX(\tau)}{d\tau} \\ \delta\eta &= \frac{d}{d\tau}\left[\zeta(\tau)\eta(\tau)\right], \end{aligned}$$

  for some parameter $\zeta(\tau)$. This is a non-trivial local or 'gauge' symmetry of S. This large extra symmetry on the world-line (and its analogue when we come to study strings) is very useful. We can, for example, use it to pick a nice gauge where we set $\eta = m^{-1}$. This gives a nice simple action, resulting in a simple expression for the conjugate momentum to X^μ:

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = m\dot{X}^\mu. \tag{2.10}$$

  We will use this much later.

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2.2 Classical bosonic strings

Turning to strings, we parametrise the 'world-sheet' which the string sweeps out with coordinates $(\sigma^1, \sigma^2) = (\tau, \sigma)$. The latter is a spatial coordinate, and for now, we take the string to be an open one, with $0 \le \sigma \le \pi$ running from one end to the other. The string's evolution in spacetime is described by the functions $X^\mu(\tau, \sigma)$, $\mu = 0, \ldots, D-1$, giving the shape of the string's world-sheet in target spacetime (see figure 1.4, p. 13).

2.2.1 Two actions

As we already discussed in section 1.3, using the induced metric on the world-sheet which we recall here:

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{2.11}$$

we can measure distances on the world-sheet as an object embedded in spacetime, and hence define an action analogous to the one for the particle: the total area swept out by the world-sheet (equation (1.25)), which we repeat here:

$$S_{\rm o} = -T\int dA = -T\int d\tau d\sigma\,(-\det h_{ab})^{1/2} \equiv \int d\tau d\sigma\ \mathcal{L}(\dot{X}, X'; \sigma, \tau). \tag{2.12}$$

$$\begin{aligned} S_{\rm o} &= -T\int d\tau d\sigma\left[\left(\frac{\partial X^\mu}{\partial\sigma}\frac{\partial X^\mu}{\partial\tau}\right)^2 - \left(\frac{\partial X^\mu}{\partial\sigma}\right)^2\left(\frac{\partial X_\mu}{\partial\tau}\right)^2\right]^{1/2} \\ &= -T\int d\tau d\sigma\left[(X'\cdot\dot{X})^2 - X'^2\dot{X}^2\right]^{1/2}, \end{aligned} \tag{2.13}$$

where X' means $\partial X/\partial\sigma$ and a dot means differentiation with respect to τ. This is the Nambu–Goto action.

Varying the action, we have generally:

$$\begin{aligned} \delta S_{\rm o} &= \int d\tau d\sigma\left\{\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\delta\dot{X}^\mu + \frac{\partial\mathcal{L}}{\partial X'^\mu}\delta X'^\mu\right\} \\ &= \int d\tau d\sigma\left\{-\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} - \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial X'^\mu}\right\}\delta X^\mu \\ &\quad + \int d\tau\left\{\frac{\partial\mathcal{L}}{\partial X'^\mu}\delta X'^\mu\right\}\Bigg|_{\sigma=0}^{\sigma=\pi}. \end{aligned} \tag{2.14}$$

Requiring this to be zero, we get:

$$\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} + \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial X'^\mu} = 0 \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial X'^\mu} = 0 \quad \text{at} \quad \sigma = 0, \pi, \tag{2.15}$$

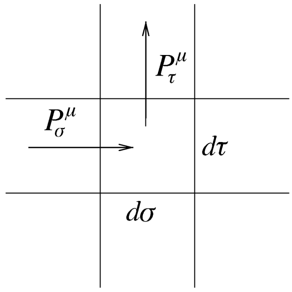

Fig. 2.1. The infinitesimal momenta on the world sheet.

which are statements about the conjugate momenta:

$$\frac{\partial}{\partial\tau}P^\mu_\tau + \frac{\partial}{\partial\sigma}P^\mu_\sigma = 0 \quad \text{and} \quad P^\mu_\sigma = 0 \quad \text{at} \quad \sigma = 0, \pi. \tag{2.16}$$

Here, P^μ_σ is the momentum running along the string (i.e. in the σ direction) while P^μ_τ is the momentum running transverse to it. The total spacetime momentum is given by integrating up the infinitesimal (see figure 2.1):

$$dP^\mu = P^\mu_\tau d\sigma + P^\mu_\sigma d\tau. \tag{2.17}$$

Actually, we can choose any slice of the world-sheet in order to compute this momentum. A most convenient one is a slice $\tau =$ constant, revealing the string in its original paramaterisation: $P^\mu = \int P^\mu_\tau d\sigma$, but any other slice will do.

Similarly, one can define the angular momentum:

$$M^{\mu\nu} = \int (P^\mu_\tau X^\nu - P^\nu_\tau X^\mu) d\sigma. \tag{2.18}$$

It is a simple exercise to work out the momenta for our particular Lagrangian:

$$\begin{aligned} P^\mu_\tau &= T\,\frac{\dot{X}^\mu X'^2 - X'^\mu(\dot{X}\cdot X')}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}} \\ P^\mu_\sigma &= T\,\frac{X'^\mu \dot{X}^2 - \dot{X}^\mu(\dot{X}\cdot X')}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2}}. \end{aligned} \tag{2.19}$$

It is interesting to compute the square of P^μ_σ from this expression, and one finds that

$$P^2_\sigma \equiv P^\mu_\sigma P_{\mu\sigma} = -2T^2\dot{X}^2. \tag{2.20}$$

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This is our first (perhaps) non-intuitive classical result. We noticed that P_σ vanishes at the endpoints, in order to prevent momentum from flowing off the ends of the string. The equation we just derived implies that $\dot{X}^2 = 0$ at the endpoints, which is to say that they move at the speed of light.

Just like we did in the point particle case, we can introduce an equivalent action which does not have the square root form that the current one has. Once again, we do it by introducing a independent metric, $\gamma_{ab}(\sigma,\tau)$, on the world-sheet, and write the 'Polyakov' action:

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma(-\gamma)^{1/2}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \\ &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\,(-\gamma)^{1/2}\gamma^{ab}h_{ab}. \end{aligned} \tag{2.21}$$

If we vary γ, we get

$$\delta S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\left\{-\frac{1}{2}(-\gamma)^{1/2}\delta\gamma\gamma^{ab}h_{ab} + (-\gamma)^{1/2}\delta\gamma^{ab}h_{ab}\right\}. \tag{2.22}$$

Using the fact that $\delta\gamma = \gamma\gamma^{ab}\delta\gamma_{ab} = -\gamma\gamma_{ab}\delta\gamma^{ab}$, (which we already used in higher dimensions, see equation (1.13)) we get

$$\delta S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,(-\gamma)^{1/2}\delta\gamma^{ab}\left\{h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd}\right\}. \tag{2.23}$$

Therefore we have

$$h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} = 0, \tag{2.24}$$

from which we can derive

$$\gamma^{ab}h_{ab} = 2(-h)^{1/2}(-\gamma)^{-1/2}, \tag{2.25}$$

and so substituting into S, we recover (just as in the point-particle case) that it reduces to the Nambu–Goto action, $S_{\rm o}$.

2.2.2 Symmetries

Let us again study the symmetries of the action.

- Spacetime Lorentz/Poincaré:

  $$X^\mu \to X'^\mu = \Lambda^\mu{}_\nu X^\nu + A^\mu,$$

  where Λ is an $SO(1,3)$ Lorentz matrix and A^μ is an arbitrary constant four-vector. Just as before this is a trivial global symmetry of S (and also $S_{\rm o}$), following from the fact that we wrote them in covariant form.

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- world-sheet reparametrisations:

$$\begin{aligned}\delta X^\mu &= \zeta^a \partial_a X^\mu \\ \delta\gamma^{ab} &= \zeta^c \partial_c \gamma^{ab} - \partial_c \zeta^a \gamma^{cb} - \partial_c \zeta^b \gamma^{ac},\end{aligned} \tag{2.26}$$

  for two parameters $\zeta^a(\tau, \sigma)$. This is a non-trivial local or 'gauge' symmetry of S. This is a large extra symmetry on the world-sheet of which we will make great use.

- Weyl invariance:

$$\gamma_{ab} \to \gamma'_{ab} = e^{2\omega}\gamma_{ab}, \tag{2.27}$$

  specified by a function $\omega(\tau, \sigma)$. This ability to do local rescalings of the metric results from the fact that we did not have to choose an overall scale when we chose γ^{ab} to rewrite S_o in terms of S. This can be seen especially if we rewrite the relation (2.25) as $(-h)^{-1/2}h_{ab} = (-\gamma)^{-1/2}\gamma_{ab}$.

N.B. We note here for future use that there are just as many parameters needed to specify the local symmetries (three) as there are independent components of the world-sheet metric. This is very useful, as we shall see.

2.2.3 String equations of motion

We can get equations of motion for the string by varying our action (2.21) with respect to the X^μ:

$$\begin{aligned}\delta S = \frac{1}{2\pi\alpha'}\int d^2\sigma\, \partial_a \left\{(-\gamma)^{1/2}\gamma^{ab}\partial_b X_\mu\right\}\delta X^\mu \\ - \frac{1}{2\pi\alpha'}\int d\tau\, (-\gamma)^{1/2}\partial_\sigma X_\mu \delta X^\mu\Big|_{\sigma=0}^{\sigma=\pi},\end{aligned} \tag{2.28}$$

which results in the equations of motion:

$$\partial_a\left((-\gamma)^{1/2}\gamma^{ab}\partial_b X^\mu\right) \equiv (-\gamma)^{1/2}\nabla^2 X^\mu = 0, \tag{2.29}$$

with *either*:

$$\left.\begin{aligned}X'^\mu(\tau, 0) = 0 \\ X'^\mu(\tau, \pi) = 0\end{aligned}\right\} \quad \begin{matrix}\text{Open string} \\ \text{(Neumann b.c.s)}\end{matrix} \tag{2.30}$$

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or:

$$\left.\begin{aligned} X'^{\mu}(\tau,0) &= X'^{\mu}(\tau,\pi) \\ X^{\mu}(\tau,0) &= X^{\mu}(\tau,\pi) \\ \gamma_{ab}(\tau,0) &= \gamma_{ab}(\tau,\pi) \end{aligned}\right\} \quad \begin{array}{c}\text{Closed string}\\ \text{(periodic b.c.s)}\end{array} \tag{2.31}$$

We shall study the equation of motion (2.29) and the accompanying boundary conditions a lot later. We are going to look at the standard Neumann boundary conditions mostly, and then consider the case of Dirichlet conditions later, when we uncover D-branes, using T-duality. Notice that we have taken the liberty of introducing closed strings by imposing periodicity (see also insert 2.1 (p. 32)).

2.2.4 Further aspects of the two dimensional perspective

The action (2.21) may be thought of as a two dimensional model of D bosonic fields $X^\mu(\tau,\sigma)$. This two dimensional theory has reparameterisation invariance, as it is constructed using the metric $\gamma_{ab}(\tau,\sigma)$ in a covariant way. It is natural to ask whether there are other terms which we might want to add to the theory which have similar properties.

With some experience from General Relativity two other terms spring effortlessly to mind. One is the Einstein–Hilbert action (supplemented with a boundary term):

$$\chi = \frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\,(-\gamma)^{1/2}R + \frac{1}{2\pi}\int_{\partial\mathcal{M}} ds K, \tag{2.32}$$

where R is the two dimensional Ricci scalar on the world-sheet $\mathcal{M}$ and K is the trace of the extrinsic curvature tensor on the boundary $\partial\mathcal{M}$. This latter quantity may be less familiar to some, and we will use it a lot in diverse dimensions much later in this book. (There is a discussion of it in insert 10.2 (p. 229), and we will not worry about it in detail here lest we get sidetracked.)

The other term is:

$$\Theta = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\,(-\gamma)^{1/2}, \tag{2.33}$$

which is the cosmological term. What is the role of these terms here? Well, under a Weyl transformation (2.27), it can be seen that $(-\gamma)^{1/2} \to e^{2\omega}(-\gamma)^{1/2}$ and $R \to e^{-2\omega}(R - 2\nabla^2\omega)$, and so χ is invariant, (because R changes by a total derivative which is cancelled by the variation of K) but Θ is not.

So we will include χ, but not Θ in what follows. Let us anticipate something that we will do later, which is to work with Euclidean signature to help make sense of the topological statements to follow: γ_{ab} with signature $(-+)$ has been replaced by g_{ab} with signature $(++)$. Now, since as

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Insert 2.1. T is for tension

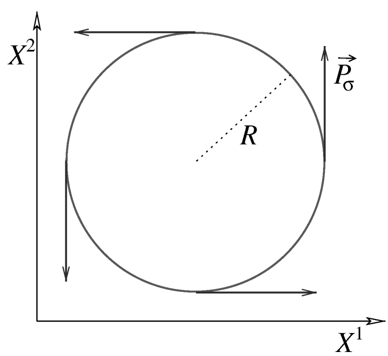
As a first non-trivial example (and to learn that T, a mass per unit length, really is the string's tension) let us consider a closed string lying in the (X^1, X^2) plane.

$$\begin{aligned} X^0 &= 2R\tau; \\ X^1 &= R\sin 2\sigma \\ X^2 &= R\cos 2\sigma. \end{aligned}$$

We have made it by arranging that the $\sigma = 0, \pi$ ends meet, that momentum flows across that join. An examination of the equations of motion shows that this configuration is not a solution, and there are terms which do not vanish corresponding to the fact that the string does not want to stay at rest: since the string has tension, it is likely to want to shrink its length away if put into this shape. So let us think of this as a snapshot of such a situation, ignoring the non-vanishing terms which involve time derivative. It is worth taking the time to use this to show that one gets

$$P^\mu_\tau = T\,(2R, 0, 0), \quad P^\mu_\sigma = T\,(0, -2R\cos 2\sigma, 2R\sin 2\sigma),$$

which is interesting, as a sketch shows.



There is momentum flowing around the string (which is lying in a circle of radius R). The total momentum is

$$P^\mu = \int_0^\pi d\sigma\, P^\mu_\tau.$$

The only non-zero component is the mass-energy: $M = 2\pi RT =$ length$\times T$.

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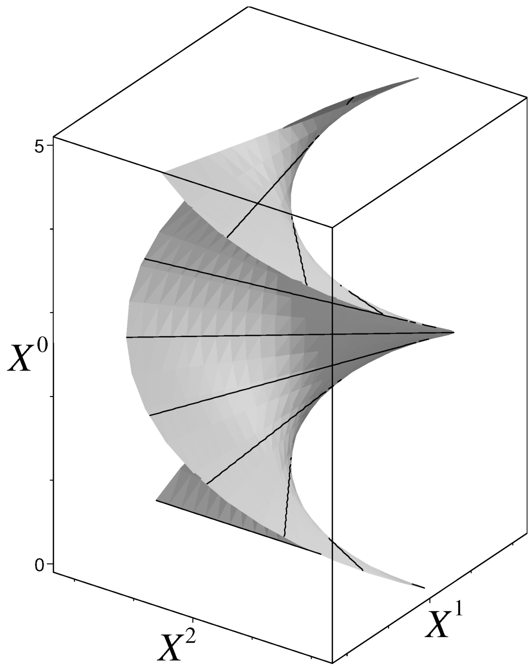
Insert 2.2. A rotating open string

As a second non-trivial example consider the following open string rotating at a constant angular velocity in the (X^1, X^2) plane. Such a configuration is:

$$X^0 = \tau; \quad X^1 = A\left(\sigma - \frac{\pi}{2}\right)\cos\omega\tau, \quad X^2 = A\left(\sigma - \frac{\pi}{2}\right)\sin\omega\tau,$$

where it should be checked that the equations of motion fix $A = \frac{2}{\pi\omega}$. This is what it looks like (the spinning string is shown in frozen snapshots).



It is again a worthwhile exercise to compute P^μ, and also $M^{\mu\nu}$. With $J \equiv M^{12}$ and $M \equiv P^0$, some algebra shows that

$$\frac{|J|}{M^2} = \frac{1}{2\pi T} = \alpha'.$$

This parameter, α', is the slope of the celebrated 'Regge' trajectories: the straight line plots of J vs. M^2 seen in nuclear physics in the 1960s. There remains the determination of the intercept of this straight line graph with the J-axis. It turns out to be one for the bosonic string as we shall see.

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we said earlier, the full string action resembles two dimensional gravity coupled to D bosonic 'matter' fields X^μ, and the equations of motion are, of course,

$$R_{ab} - \frac{1}{2}\gamma_{ab}R = T_{ab}. \tag{2.34}$$

The left hand side vanishes identically in two dimensions, and so there are no dynamics associated to (2.32). The quantity χ depends only on the topology of the world-sheet (it is the Euler number) and so will only matter when comparing world sheets of different topology. This will arise when we compare results from different orders of string perturbation theory and when we consider interactions.

We can see this in the following. Let us add our new term to the action, and consider the string action to be:

$$S = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\, g^{1/2} g^{ab}\partial_a X^\mu \partial_b X_\mu + \lambda\left\{\frac{1}{4\pi}\int_{\mathcal{M}} d^2\sigma\, g^{1/2}R + \frac{1}{2\pi}\int_{\partial\mathcal{M}} ds K\right\}, \tag{2.35}$$

where λ is – for now – and arbitrary parameter that we have not fixed to any particular value.

> N.B. It will turn out that λ is not a free parameter. In the full string theory, it has dynamical meaning, and will be equivalent to the expectation value of one of the massless fields – the 'dilaton' – described by the string.

So what will λ do? Recall that it couples to Euler number, so in the full path integral defining the string theory:

$$\mathcal{Z} = \int \mathcal{D}X\mathcal{D}g\; e^{-S}, \tag{2.36}$$

resulting amplitudes will be weighted by a factor $e^{-\lambda\chi}$, where $\chi = 2-2h-b-c$. Here, h, b, c are the numbers of handles, boundaries and crosscaps, respectively, on the world sheet. Consider figure 2.2. An emission and reabsorption of an open string results in a change $\delta\chi = -1$, while for a closed string it is $\delta\chi = -2$. Therefore, relative to the tree level open string diagram (disc topology), the amplitudes are weighted by e^{λ} and $e^{2\lambda}$, respectively. The quantity $g_s \equiv e^{\lambda}$ therefore will be called the closed string coupling. Note that it is the square of the open string coupling, which justifies the labelling we gave of the two three-string diagrams in figure 1.3.

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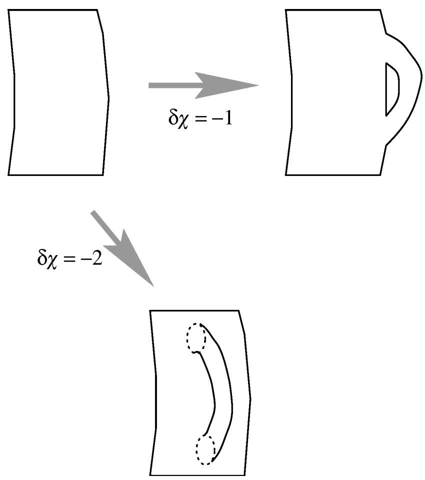

Fig. 2.2. World-sheet topology change due to emission and reabsorption of open and closed strings.

2.2.5 The stress tensor

Let us also note that we can define a two dimensional energy-momentum tensor:

$$T^{ab}(\tau,\sigma) \equiv -\frac{2\pi}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma_{ab}} = -\frac{1}{\alpha'}\left\{\partial^a X_\mu \partial^b X^\mu - \frac{1}{2}\gamma^{ab}\gamma_{cd}\partial^c X_\mu \partial^d X^\mu\right\}. \tag{2.37}$$

Notice that

$$T^a_a \equiv \gamma_{ab}T^{ab} = 0. \tag{2.38}$$

This is a consequence of Weyl symmetry. Reparametrisation invariance, $\delta_\gamma S' = 0$, translates here into (see discussion after equation (2.34))

$$T^{ab} = 0. \tag{2.39}$$

These are the classical properties of the theory we have uncovered so far. Later on, we shall attempt to ensure that they are true in the quantum theory also, with interesting results.

2.2.6 Gauge fixing

Now recall that we have three local or 'gauge' symmetries of the action:

$$\begin{aligned} \text{2D reparametrisations}: &\ \ \sigma,\tau \to \tilde\sigma(\sigma,\tau), \tilde\tau(\sigma,\tau), \\ \text{Weyl}: &\ \ \gamma_{ab} \to \exp(2\omega(\sigma,\tau))\gamma_{ab}. \end{aligned} \tag{2.40}$$

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The two dimensional metric γ_{ab} is also specified by three independent functions, as it is a symmetric 2×2 matrix. We may therefore use the gauge symmetries (see equations (2.26) and (2.27)) to choose γ_{ab} to be a particular form:

$$\gamma_{ab} = \eta_{ab} e^{\phi} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e^{\phi}, \tag{2.41}$$

i.e. the metric of two dimensional Minkowski, times a positive function known as a *conformal factor*. In this 'conformal' gauge, our X^μ equations of motion (2.29) become:

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu(\tau, \sigma) = 0, \tag{2.42}$$

the two dimensional wave equation. (In fact, the reader should check that the conformal factor cancels out entirely of the action in equation (2.21).) As the wave equation is $\partial_{\sigma^+} \partial_{\sigma^-} X^\mu = 0$, we see that the full solution to the equation of motion can be written in the form:

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-), \tag{2.43}$$

where $\sigma^\pm \equiv \tau \pm \sigma$.

N.B. Write $\sigma^\pm = \tau \pm \sigma$. This gives metric $ds^2 = -d\tau^2 + d\sigma^2 \to -d\sigma^+ d\sigma^-$. So we have $\eta_{-+} = \eta_{+-} = -1/2$, $\eta^{-+} = \eta^{+-} = -2$ and $\eta_{++} = \eta_{--} = \eta^{++} = \eta^{--} = 0$. Also, $\partial_\tau = \partial_+ + \partial_-$ and $\partial_\sigma = \partial_+ - \partial_-$.

Our constraints on the stress tensor become:

$$\begin{aligned} T_{\tau\sigma} &= T_{\sigma\tau} \equiv \frac{1}{\alpha'} \dot{X}^\mu X'_\mu = 0 \\ T_{\sigma\sigma} &= T_{\tau\tau} = \frac{1}{2\alpha'} \left(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu \right) = 0, \end{aligned} \tag{2.44}$$

or

$$\begin{aligned} T_{++} &= \frac{1}{2}(T_{\tau\tau} + T_{\tau\sigma}) = \frac{1}{\alpha'} \partial_+ X^\mu \partial_+ X_\mu \equiv \frac{1}{\alpha'} \dot{X}_L^2 = 0 \\ T_{--} &= \frac{1}{2}(T_{\tau\tau} - T_{\tau\sigma}) = \frac{1}{\alpha'} \partial_- X^\mu \partial_- X_\mu \equiv \frac{1}{\alpha'} \dot{X}_R^2 = 0, \end{aligned} \tag{2.45}$$

and T_{-+} and T_{+-} are identically zero.

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2.2.7 The mode decomposition

Our equations of motion (2.43), with our boundary conditions (2.30) and (2.31) have the simple solutions:

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i(2\alpha')^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma, \tag{2.46}$$

for the open string and

$$\begin{aligned} X^\mu(\tau,\sigma) &= X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+) \\ X_R^\mu(\sigma^-) &= \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-} \\ X_L^\mu(\sigma^+) &= \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+}, \end{aligned} \tag{2.47}$$

for the closed string, where, to ensure a real solution we impose $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ and $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$. Note that x^μ and p^μ are the centre of mass position and momentum, respectively. In each case, we can identify p^μ with the zero mode of the expansion:

$$\begin{aligned} \text{open string:} \qquad \alpha_0^\mu &= (2\alpha')^{1/2} p^\mu; \\ \text{closed string:} \qquad \alpha_0^\mu &= \left(\frac{\alpha'}{2}\right)^{1/2} p^\mu. \end{aligned} \tag{2.48}$$

N.B. Notice that the mode expansion for the closed string (2.47) is simply that of a pair of independent left and right moving travelling waves going around the string in opposite directions. The open string expansion (2.46) on the other hand, has a standing wave for its solution, representing the left and right moving sector reflected into one another by the Neumann boundary condition (2.30).

2.2.8 Conformal invariance as a residual symmetry

Actually, we have not gauged away all of the local symmetry by choosing the gauge (2.41). We can do a left–right decoupled change of variables:

$$\sigma^+ \to f(\sigma^+) = \sigma'^+;\ \sigma^- \to g(\sigma^-) = \sigma'^-. \tag{2.49}$$

Then, as

$$\gamma'_{ab} = \frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}\gamma_{cd}, \tag{2.50}$$

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we have

$$\gamma'_{+-} = \left(\frac{\partial f(\sigma^+)}{\partial \sigma^+} \frac{\partial g(\sigma^-)}{\partial \sigma^-} \right)^{-1} \gamma_{+-}. \tag{2.51}$$

However, we can undo this with a Weyl transformation of the form

$$\gamma'_{+-} = \exp(2\omega_{\mathrm{L}}(\sigma^+) + 2\omega_{\mathrm{R}}(\sigma^-))\gamma_{+-}, \tag{2.52}$$

if $\exp(-2\omega_{\mathrm{L}}(\sigma^+)) = \partial_+ f(\sigma^+)$ and $\exp(-2\omega_{\mathrm{R}}(\sigma^-)) = \partial_- g(\sigma^-)$. So we still have a residual 'conformal' symmetry. As f and g are independent arbitrary functions on the left and right, we have an infinite number of conserved quantities on the left and right. This is because the conservation equation $\nabla_a T^{ab} = 0$, together with the result $T_{+-} = T_{-+} = 0$, turns into:

$$\partial_- T_{++} = 0 \quad \text{and} \quad \partial_+ T_{--} = 0, \tag{2.53}$$

but since $\partial_- f = 0 = \partial_+ g$, we have

$$\partial_-(f(\sigma^+)T_{++}) = 0 \quad \text{and} \quad \partial_+(g(\sigma^-)T_{--}) = 0, \tag{2.54}$$

resulting in an infinite number of conserved quantities. The fact that we have this infinite dimensional conformal symmetry is the basis of some of the most powerful tools in the subject, for computing in perturbative string theory. We will return to it not too far ahead.

2.2.9 Some Hamiltonian dynamics

Our Lagrangian density is

$$\mathcal{L} = -\frac{1}{4\pi\alpha'} (\partial_\sigma X^\mu \partial_\sigma X_\mu - \partial_\tau X^\mu \partial_\tau X_\mu), \tag{2.55}$$

from which we can derive that the conjugate momentum to X^μ is

$$\Pi^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\tau X^\mu)} = \frac{1}{2\pi\alpha'} \dot{X}^\mu. \tag{2.56}$$

So we have the equal time Poisson brackets:

$$[X^\mu(\sigma), \Pi^\nu(\sigma')]_{\mathrm{P.B.}} = \eta^{\mu\nu}\delta(\sigma - \sigma'), \tag{2.57}$$

$$[\Pi^\mu(\sigma), \Pi^\nu(\sigma')]_{\mathrm{P.B.}} = 0, \tag{2.58}$$

with the following results on the oscillator modes:

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n]_{\mathrm{P.B.}} &= [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n]_{\mathrm{P.B.}} = im\delta_{m+n}\eta^{\mu\nu} \\ [p^\mu, x^\nu]_{\mathrm{P.B.}} &= \eta^{\mu\nu}; \quad [\alpha^\mu_m, \tilde{\alpha}^\nu_n]_{\mathrm{P.B.}} = 0. \end{aligned} \tag{2.59}$$

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We can form the Hamiltonian density

$$\mathcal{H} = \dot{X}^\mu \Pi_\mu - \mathcal{L} = \frac{1}{4\pi\alpha'}\left(\partial_\sigma X^\mu \partial_\sigma X_\mu + \partial_\tau X^\mu \partial_\tau X_\mu\right), \tag{2.60}$$

from which we can construct the Hamiltonian H by integrating along the length of the string. This results in:

$$\begin{aligned} H &= \int_0^\pi d\sigma\, \mathcal{H}(\sigma) = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n \qquad \text{(open)} \\ H &= \int_0^{2\pi} d\sigma\, \mathcal{H}(\sigma) = \frac{1}{2}\sum_{-\infty}^{\infty} (\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) \qquad \text{(closed)}. \end{aligned} \tag{2.61}$$

(We have used the notation $\alpha_n \cdot \alpha_n \equiv \alpha_n^\mu \alpha_{n\mu}$.) The constraints $T_{++} = 0 = T_{--}$ on our energy-momentum tensor can be expressed usefully in this language. We impose them mode by mode in a Fourier expansion, defining:

$$L_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma} T_{--} d\sigma = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n, \tag{2.62}$$

and similarly for $\bar{L}_m$, using T_{++}. Using the Poisson brackets (2.59), these can be shown to satisfy the 'Virasoro' algebra:

$$\begin{aligned} &[L_m, L_n]_{\text{P.B.}} = i(m-n)L_{m+n}; \quad [\bar{L}_m, \bar{L}_n]_{\text{P.B.}} = i(m-n)\bar{L}_{m+n}; \\ &[\bar{L}_m, L_n]_{\text{P.B.}} = 0. \end{aligned} \tag{2.63}$$

Notice that there is a nice relation between the zero modes of our expansion and the Hamiltonian:

$$H = L_0 \qquad \text{(open)}; \quad H = L_0 + \bar{L}_0 \qquad \text{(closed)}. \tag{2.64}$$

So to impose our constraints, we can do it mode by mode and ask that $L_m = 0$ and $\bar{L}_m = 0$, for all m. Looking at the zeroth constraint results in something interesting. Note that

$$\begin{aligned} L_0 &= \frac{1}{2}\alpha_0^2 + 2\times\frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \\ &= \alpha' p^\mu p_\mu + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \\ &= -\alpha' M^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n. \end{aligned} \tag{2.65}$$

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Requiring L_0 to be zero – diffeomorphism invariance – results in a (spacetime) mass relation:

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n \qquad \text{(open)}, \tag{2.66}$$

where we have used the zero mode relation (2.48) for the open string. A similar exercise produces the mass relation for the closed string:

$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty} (\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) \qquad \text{(closed)}. \tag{2.67}$$

These formulae (2.66) and (2.67) give us the result for the mass of a state in terms of how many oscillators are excited on the string. The masses are set by the string tension $T = (2\pi\alpha')^{-1}$, as they should be. Let us not dwell for too long on these formulae however, as they are significantly modified when we quantise the theory, since we have to understand the infinite constant which we ignored.

2.3 Quantised bosonic strings

For our purposes, the simplest route to quantisation will be to promote everything we met previously to operator statements, replacing Poisson Brackets by commutators in the usual fashion: $[\ \ ,\ \ ]_{\text{P.B.}} \to -i[\ \ ,\ \ ]$. This gives:

$$\begin{aligned}
[X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')] &= i\eta^{\mu\nu}\delta(\sigma-\sigma'); \quad [\Pi^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')] = 0\\
[\alpha^\mu_m,\alpha^\nu_n] &= [\tilde{\alpha}^\mu_m,\tilde{\alpha}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\\
[x^\nu,p^\mu] &= i\eta^{\mu\nu}; \quad [\alpha^\mu_m,\tilde{\alpha}^\nu_n] = 0.
\end{aligned} \tag{2.68}$$

N.B. One of the first things that we ought to notice here is that $\sqrt{m}\alpha^\mu_{\pm m}$ are like creation and annihilation operators for the harmonic oscillator. There are actually D independent families of them – one for each spacetime dimension – labelled by μ.

In the usual fashion, we will define our Fock space such that $|0;k\rangle$ is an eigenstate of p^μ with centre of mass momentum k^μ. This state is annihilated by α^ν_m.

What about our operators, the L_m? Well, with the usual 'normal ordering' prescription that all annihilators are to the right, the L_m are all

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fine when promoted to operators, except the Hamiltonian, L_0. It needs more careful definition, since α_n^μ and α_{-n}^μ *do not commute*. Indeed, as an operator, we have that

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n + \text{constant}, \tag{2.69}$$

where the apparently infinite constant is composed of the infinite sum $(1/2)\sum_{n=1}^{\infty} n$ for each of the D families of oscillators. As is of course to be anticipated, this infinite constant can be regulated to give a finite answer, corresponding to the total zero point energy of all of the harmonic oscillators in the system.

2.3.1 The constraints and physical states

For now, let us not worry about the value of the constant, and simply impose our constraints on a state $|\phi\rangle$ as*:

$$\begin{aligned} (L_0 - a)|\phi\rangle = 0; &\qquad L_m|\phi\rangle = 0 \quad \text{for } m > 0, \\ (\bar{L}_0 - a)|\phi\rangle = 0; &\qquad \bar{L}_m|\phi\rangle = 0 \quad \text{for } m > 0, \end{aligned} \tag{2.70}$$

where our regulated constant is set by a, which is to be computed. There is a reason why we have not also imposed this constraint for the L_{-m}s. This is because the Virasoro algebra (2.63) in the quantum case is:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n}; \quad [\bar{L}_m, L_n] = 0; \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n}. \end{aligned} \tag{2.71}$$

There is a central term in the algebra, which produces a non-zero constant when $m = n$. Therefore, imposing both L_m and L_{-m} would produce an inconsistency. Note now that the first of our constraints (2.70) produces a modification to the mass formulae:

$$\begin{aligned} M^2 &= \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n - a\right) \qquad (\text{open}) \\ M^2 &= \frac{2}{\alpha'}\left(\sum_{n=1}^{\infty} (\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n) - 2a\right) \qquad (\text{closed}). \end{aligned} \tag{2.72}$$

* This assumes that the constant a on each side are equal. At this stage, we have no other choice. We have isomorphic copies of the same string modes on the left and the right, for which the values of a are by definition the same. When we have more than one consistent conformal field theory to choose from, then we have the freedom to consider having non-isomorphic sectors on the left and right. This is how the heterotic string is made, for example, as we shall see later.

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Notice that we can denote the (weighted) number of oscillators excited as $N = \sum \alpha_{-n} \cdot \alpha_n$ $(= \sum nN_n)$ on the left and $\bar{N} = \sum \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n$ $(= \sum n\bar{N}_n)$ on the right. N_n and $\bar{N}_n$ are the true count, on the left and right, of the number of copies of the oscillator labelled by n is present.

There is an extra condition in the closed string case. While $L_0 + \bar{L}_0$ generates time translations on the world sheet (being the Hamiltonian), the combination $L_0 - \bar{L}_0$ generates translations in σ. As there is no physical significance to where on the string we are, the physics should be invariant under translations in σ, and we should impose this as an operator condition on our physical states:

$$(L_0 - \bar{L}_0)|\phi\rangle = 0, \tag{2.73}$$

which results in the 'level-matching' condition $N = \bar{N}$, equating the number of oscillators excited on the left and the right. This is indeed the difference between the two equations in (2.70).

In summary then, we have two copies of the open string on the left and the right, in order to construct the closed string. The only extra subtlety is that we should use the correct zero mode relation (2.48) and match the number of oscillators on each side according to the level matching condition (2.73).

2.3.2 The intercept and critical dimensions

Let us consider the spectrum of states level by level, and uncover some of the features, focusing on the open string sector. Our first and simplest state is at level 0, i.e. no oscillators excited at all. There is just some centre of mass momentum that it can have, which we shall denote as k^μ. Let us write this state as $|0;k\rangle$. The first of our constraints (2.70) leads to an expression for the mass:

$$(L_0 - a)|0;k\rangle = 0 \qquad \Rightarrow \alpha' k^2 = a, \qquad \text{so} \quad M^2 = -\frac{a}{\alpha'}. \tag{2.74}$$

This state is a tachyonic state, having negative mass-squared (assuming $a > 0$.

The next simplest state is that with momentum k^μ, and one oscillator excited. We are also free to specify a polarisation vector ζ^μ. We denote this state as $|\zeta, k\rangle \equiv (\zeta \cdot \alpha_{-1})|0;k\rangle$; it starts out the discussion with D independent states. The first thing to observe is the norm of this state:

$$\begin{aligned}\langle\zeta;k||\zeta;k'\rangle &= \langle 0;k|\zeta^* \cdot \alpha_1 \zeta \cdot \alpha_{-1}|0;k'\rangle \\ &= \zeta^*_\mu \zeta_\nu \langle 0;k|\alpha^\mu_1 \alpha^\nu_{-1}|0;k'\rangle \\ &= \zeta \cdot \zeta \langle 0;k|0;k'\rangle = \zeta \cdot \zeta (2\pi)^D \delta^D(k-k'),\end{aligned} \tag{2.75}$$

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where we have used the commutator (2.68) for the oscillators. From this we see that the timelike ζs will produce a state with *negative norm*. Such states cannot be made sense of in a unitary theory, and are often called[†] 'ghosts'.

Let us study the first constraint:

$$(L_0 - a)|\zeta; k\rangle = 0 \qquad \Rightarrow \quad \alpha' k^2 + 1 = a, \qquad M^2 = \frac{1-a}{\alpha'}. \qquad (2.76)$$

The next constraint gives:

$$(L_1)|\zeta; k\rangle = \sqrt{\frac{\alpha'}{2}} k \cdot \alpha_1 \zeta \cdot \alpha_{-1}|0; k\rangle = 0 \qquad \Rightarrow, \qquad k \cdot \zeta = 0. \qquad (2.77)$$

Actually, at level one, we can also make a special state of interest: $|\psi\rangle \equiv L_{-1}|0; k\rangle$. This state has the special property that it is orthogonal to any physical state, since $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^* = \langle 0; k|L_1|\phi\rangle = 0$. It also has $L_1|\psi\rangle = 2L_0|0; k\rangle = \alpha' k^2|0; k\rangle$. This state is called a 'spurious' state.

So we note that there are three interesting cases for the level one physical state we have been considering.

1. $a < 1 \Rightarrow M^2 > 0$:
   - momentum k is timelike,
   - we can choose a frame where it is $(k, 0, 0, \ldots)$,
   - spurious state is not physical, since $k^2 \neq 0$,
   - $k \cdot \zeta = 0$ removes the timelike polarisation; $D - 1$ states left.
2. $a > 1 \Rightarrow M^2 < 0$:
   - momentum k is spacelike,
   - we can choose a frame where it is $(0, k_1, k_2, \ldots)$,
   - spurious state is not physical, since $k^2 \neq 0$,
   - $k \cdot \zeta = 0$ removes a spacelike polarisation; $D - 1$ tachyonic states left, one which is including ghosts.
3. $a = 1 \Rightarrow M^2 = 0$:
   - momentum k is null,
   - we can choose a frame where it is $(k, k, 0, \ldots)$,
   - spurious state is physical *and* null, since $k^2 = 0$,

[†] These are not to be confused with the ghosts of the friendly variety – Faddeev–Popov ghosts. These negative norm states are problematic and need to be removed.

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- $k \cdot \zeta = 0$ and $k^2 = 0$ remove two polarisations; $D - 2$ states left.

So if we choose case (3), we end up with the special situation that we have a massless vector in the D dimensional target spacetime. It even has an associated gauge invariance: since the spurious state is physical and null, and therefore we can add it to our physical state with no physical consequences, defining an equivalence relation:

$$|\phi\rangle \sim |\phi\rangle + \lambda|\psi\rangle \qquad \Rightarrow \qquad \zeta^\mu \sim \zeta^\mu + \lambda k^\mu. \tag{2.78}$$

Case (1), while interesting, corresponds to a massive vector, where the extra state plays the role of a longitudinal component. Case (2) seems bad. We shall choose case (3), where $a = 1$.

It is interesting to proceed to level two to construct physical and spurious states, although we shall not do it here. The physical states are massive string states. If we insert our level one choice $a = 1$ and see what the condition is for the spurious states to be both physical and null, we find that there is a condition on the spacetime dimension‡: $D = 26$.

In summary, we see that $a = 1$, $D = 26$ for the open bosonic string gives a family of extra null states, giving something analogous to a point of 'enhanced gauge symmetry' in the space of possible string theories. This is called a 'critical' string theory, for many reasons. We have the 24 states of a massless vector we shall loosely called the photon, A_μ, since it has a $U(1)$ gauge invariance (2.78). There is a tachyon of $M^2 = -1/\alpha'$ in the spectrum, which will not trouble us unduly. We will actually remove it in going to the superstring case. Tachyons will reappear from time to time, representing situations where we have an unstable configuration (as happens in field theory frequently). Generally, it seems that we should think of tachyons in the spectrum as pointing us towards an instability, and in many cases, the source of the instability is manifest.

Our analysis here extends to the closed string, since we can take two copies of our result, use the appropriate zero mode relation (2.48), and level matching. At level zero we get the closed string tachyon which has $M^2 = -4/\alpha'$. At level zero we get a tachyon with mass given by $M^2 = -4/\alpha'$, and at level 1 we get 24^2 massless states from $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0;k\rangle$. The traceless symmetric part is the graviton, $G_{\mu\nu}$ and the antisymmetric part, $B_{\mu\nu}$, is sometimes called the Kalb–Ramond field, and the trace is the dilaton, Φ.

‡ We get a condition on the spacetime dimension here because level two is the first time it can enter our formulae for the norms of states, via the central term in the Virasoro algebra (2.71).

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2.3.3 A glance at more sophisticated techniques

Later we shall do a more careful treatment of our gauge fixing procedure (2.41) by introducing Faddeev–Popov ghosts (b,c) to ensure that we stay on our chosen gauge slice in the full theory. Our resulting two dimensional conformal field theory will have an extra sector coming from the (b,c) ghosts.

The central term in the Virasoro algebra (2.71) represents an anomaly in the transformation properties of the stress tensor, spoiling its properties as a tensor under general coordinate transformations. Generally:

$$\left(\frac{\partial\sigma'^{+}}{\partial\sigma^{+}}\right)^2 T'_{++}(\sigma'^{+}) = T_{++}(\sigma^{+}) - \frac{c}{12}\left\{\frac{2\partial_\sigma^3\sigma'\partial_\sigma\sigma' - 3\partial_\sigma^2\sigma'\partial_\sigma^2\sigma'}{2\partial_\sigma\sigma'\partial_\sigma\sigma'}\right\}, \quad (2.79)$$

where here c is a number, the *central charge* which depends upon the content of the theory. In our case, we have D bosons, which each contribute 1 to c, for a total anomaly of D.

The ghosts do two crucial things: They contribute to the anomaly the amount -26, and therefore we can retain all our favourite symmetries for the dimension $D = 26$. They also cancel the contributions to the vacuum energy coming from the oscillators in the $\mu = 0, 1$ sector, leaving $D - 2$ transverse oscillators' contribution.

The regulated value of $-a$ is the vacuum or 'zero point' energy (z.p.e.) of the transverse modes of the theory. This zero point energy is simply the Casimir energy arising from the fact that the two dimensional field theory is in a box. The box is the infinite strip, for the case of an open string, or the infinite cylinder, for the case of the closed string (see figure 2.3).

A periodic (integer moded) boson such as the types we have here, X^μ, each contribute $-1/24$ to the vacuum energy (see insert 2.3 (p. 46) on a quick way to compute this). So we see that in 26 dimensions, with only

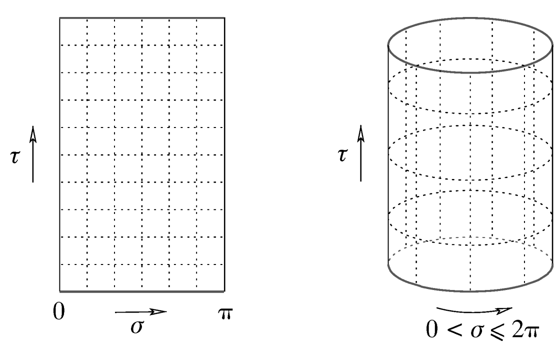

Fig. 2.3. String world-sheets as boxes upon which lives two dimensional conformal field theory.

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Insert 2.3. Zero point energy from the exponential map

After doing the transformation to the z-plane, it is interesting to note that the Fourier expansions we have been working with to define the modes of the stress tensor become Laurent expansions on the complex plane, e.g.

$$T_{zz}(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}.$$

One of the most straightforward exercises is to compute the zero point energy of the cylinder or strip (for a field of central charge c) by starting with the fact that the plane has no Casimir energy. One simply plugs the exponential change of coordinates $z = e^w$ into the anomalous transformation for the energy momentum tensor and compute the contribution to T_{ww} starting with T_{zz}:

$$T_{ww} = -z^2 T_{zz} - \frac{c}{24},$$

which results in the Fourier expansion on the cylinder, in terms of the modes:

$$T_{ww}(w) = -\sum_{m=-\infty}^{\infty} \left(L_m - \frac{c}{24}\delta_{m,0} \right) e^{i\sigma-\tau}.$$

24 contributions to count (see previous paragraph), we get that $-a = 24 \times (-1/24) = -1$. (Notice that from equation (2.69), this implies that $\sum_{n=1}^{\infty} n = -1/12$, which is in fact true (!) in ζ-function regularisation.)

Later, we shall have world-sheet fermions ψ^μ as well, in the supersymmetric theory. They each contribute 1/2 to the anomaly. World sheet superghosts will cancel the contributions from ψ^0, ψ^1. Each anti-periodic fermion will give a z.p.e. contribution of $-1/48$.

Generally, taking into account the possibility of both periodicities for either bosons or fermions:

$$\text{z.p.e.} = \frac{1}{2}\omega \quad \text{for boson;} \qquad -\frac{1}{2}\omega \quad \text{for fermion} \tag{2.80}$$
$$\omega = \frac{1}{24} - \frac{1}{8}(2\theta-1)^2 \qquad \begin{cases} \theta = 0 & \text{(integer modes)} \\ \theta = \frac{1}{2} & \text{(half–integer modes).} \end{cases}$$

This is a formula that we shall use many times in what is to come.

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2.4 The sphere, the plane and the vertex operator

The ability to choose the conformal gauge, as first discussed in section 2.2.6, gives us a remarkable amount of freedom, which we can put to good use. The diagrams in figure 2.3 represent free strings coming in from $\tau = -\infty$ and going out to $\tau = +\infty$. Let us first focus on the closed string, the cylinder diagram. Working with Euclidean signature by taking $\tau \to -i\tau$, the metric on it is

$$ds^2 = d\tau^2 + d\sigma^2, \qquad -\infty < \tau < +\infty \qquad 0 < \sigma \leq 2\pi.$$

We can do the change of variables

$$z = e^{\tau - i\sigma}, \tag{2.81}$$

with the result that the metric changes to

$$ds^2 = d\tau^2 + d\sigma^2 \longrightarrow |z|^{-2} dz d\bar{z}.$$

This is conformal to the metric of the complex plane: $d\hat{s}^2 = dzd\bar{z}$, and so we can use this as our metric on the world-sheet, since a conformal factor $e^{\phi} = |z|^{-2}$ drops out of the action, as we already noticed.

The string from the infinite past $\tau = -\infty$ is mapped to the origin while the string in the infinite future $\tau = +\infty$ is mapped to the 'point' at infinity. Intermediate strings are circles of constant radius $|z|$. See figure 2.4. The more forward-thinking reader who prefers to have the $\tau = +\infty$ string at the origin can use the complex coordinate $\tilde{z} = 1/z$ instead.

One can even ask that *both* strings be placed at finite distance in z. Then we need a conformal factor which goes like $|z|^{-2}$ at $z = 0$ as before, but like $|z|^2$ at $z = \infty$. There is an infinite set of functions which do that, but one particularly nice choice leaves the metric:

$$ds^2 = \frac{4R^2 dz d\bar{z}}{(R^2 + |z|^2)^2}, \tag{2.82}$$

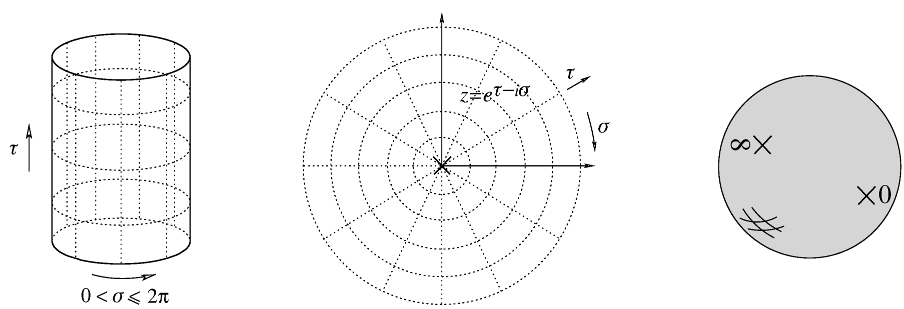

Fig. 2.4. The cylinder diagram is conformal to the complex plane and the sphere.

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which is the familiar expression for the metric on a round S^2 with radius R, resulting from adding the point at infinity to the plane. See figure 2.4. The reader should check that the precise analogue of this process will relate the strip of the open string to the upper half plane, or to the disc. The open strings are mapped to points on the real axis, which is equivalent to the boundary of the disc. See figure 2.5.

We can go even further and consider the interaction with three or more strings. Again, a clever choice of function in the conformal factor can be made to map any tubes or strips corresponding to incoming strings to a point on the interior of the plane, or on the surface of a sphere (for the closed string) or the real axis of the upper half-plane of the boundary of the disc (for the open string). See figure 2.6.

2.4.1 States and operators

There is one thing which we might worry about. Have we lost any information about the state that the string was in by performing this reduction of an entire string to a point? Should we not have some sort of marker with which we label each point with the properties of the string it came from? The answer is in the affirmative, and the object which should be inserted at these points is called a 'vertex operator'. Let us see where it comes from.

As we learned in the previous subsection, we can work on the complex plane with coordinate z. In these coordinates, our mode expansions (2.46) and (2.47) become:

$$X^\mu(z,\bar{z}) = x^\mu - i\left(\frac{\alpha'}{2}\right)^{1/2}\alpha_0^\mu \ln z\bar{z} + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\left(z^{-n}+\bar{z}^{-n}\right), \tag{2.83}$$

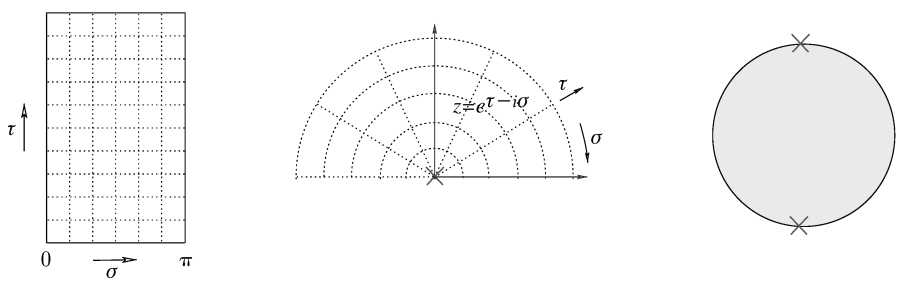

Fig. 2.5. The strip diagram is conformal to the upper half of the complex plane and the disc.

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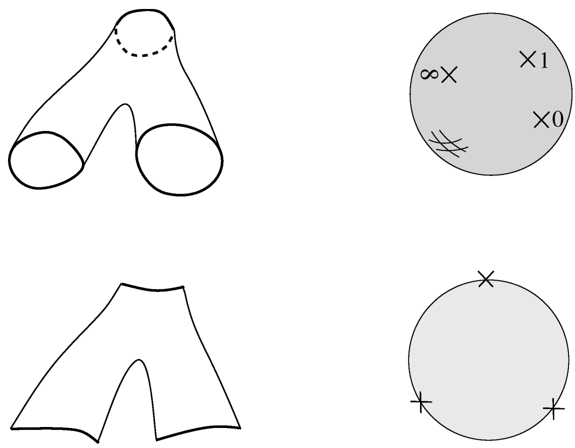

Fig. 2.6. Mapping any number of external string states to the sphere or disc using conformal transformations.

for the open string, and for the closed:

$$\begin{aligned} X^\mu(z,\bar z) &= X^\mu_{\rm L}(z) + X^\mu_{\rm R}(\bar z) \\ X^\mu_{\rm L}(z) &= \frac{1}{2}x^\mu - i\left(\frac{\alpha'}{2}\right)^{1/2}\alpha^\mu_0 \ln z + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq0}\frac{1}{n}\alpha^\mu_n z^{-n} \\ X^\mu_{\rm R}(\bar z) &= \frac{1}{2}x^\mu - i\left(\frac{\alpha'}{2}\right)^{1/2}\tilde\alpha^\mu_0 \ln \bar z + i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_{n\neq0}\frac{1}{n}\tilde\alpha^\mu_n \bar z^{-n}, \end{aligned} \tag{2.84}$$

where we have used the zero mode relations (2.48). In fact, notice that:

$$\begin{aligned} \partial_z X^\mu(z) &= -i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_n \alpha^\mu_n z^{-n-1} \\ \partial_{\bar z} X^\mu(\bar z) &= -i\left(\frac{\alpha'}{2}\right)^{1/2}\sum_n \tilde\alpha^\mu_n \bar z^{-n-1}, \end{aligned} \tag{2.85}$$

and that we can invert these to get (for the closed string)

$$\alpha^\mu_{-n} = \left(\frac{2}{\alpha'}\right)^{1/2}\oint\frac{dz}{2\pi}z^{-n}\partial_z X^\mu(z) \qquad \tilde\alpha^\mu_{-n} = \left(\frac{2}{\alpha'}\right)^{1/2}\oint\frac{dz}{2\pi}\bar z^{-n}\partial_{\bar z} X^\mu(z), \tag{2.86}$$

which are non-zero for $n \geq 0$. This is suggestive: equations (2.85) define left–moving (holomorphic) and right-moving (anti-holomorphic) fields. We previously employed the objects on the left in (2.86) in making states by acting, e.g. $\alpha^\mu_{-1}|0;k\rangle$. The form of the right hand side suggests that this is equivalent to performing a contour integral around an insertion of a pointlike operator at the point z in the complex plane (see figure 2.7). For example, α^{μ}_{-1} is related to the residue $\partial_z X^{\mu}(0)$, while the α^{μ}_{-m} correspond to higher derivatives $\partial_z^m X^{\mu}(0)$. This is course makes sense, as higher levels correspond to more oscillators excited on the string, and hence higher frequency components, as measured by the higher derivatives. The state with no oscillators excited (the tachyon), but with some momentum k, simply corresponds in this dictionary to the insertion of

$$|0;k\rangle \qquad \Longleftrightarrow \qquad \int d^2z :e^{ik\cdot X}: \tag{2.87}$$

We have integrated over the insertions' position on the sphere since the result should not depend upon our parameterisation. This is reasonable, as it is the simplest form that allows the right behaviour under translations: A translation by a constant vector, $X^{\mu} \to X^{\mu} + A^{\mu}$, results in a multiplication of the operator (and hence the state) by a phase $e^{i\mathbf{k}\cdot\mathbf{A}}$. The normal ordering signs :: are there to remind us that the expression means to expand and keep all creation operators to the left, when expanding in terms of the $\alpha_{\pm m}$s.

The closed string level one vertex operator corresponds to the emission or absorption of $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ:

$$\zeta_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle \qquad \Longleftrightarrow \qquad \int d^2z \ :\zeta_{\mu\nu}\partial_z X^{\mu}\partial_{\bar{z}}X^{\nu}e^{ik\cdot X}: \tag{2.88}$$

where the symmetric part of $\zeta_{\mu\nu}$ is the graviton and the antisymmetric part is the antisymmetric tensor.

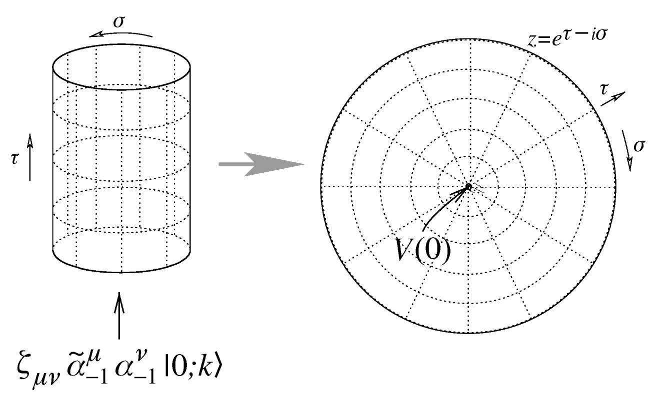

Fig. 2.7. The correspondence between states and operator insertions. A closed string (graviton) state $\zeta_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0;k\rangle$ is set up on the closed string at $\tau = -\infty$ and it propagates in. This is equivalent to inserting a graviton vertex operator $V^{\mu\nu}(z) =: \zeta_{\mu\nu}\partial_z X^{\mu}\partial_{\bar{z}}X^{\nu}e^{ik\cdot X}:$ at $z = 0$.

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For the open string, the story is similar, but we get two copies of the relations (2.86) for the single set of modes α^{μ}_{-n} (recall that there are no $\tilde{\alpha}$s). This results in, for example the relation for the photon:

$$\zeta_\mu \alpha^{\mu}_{-1}|0;k\rangle \qquad \Longleftrightarrow \qquad \int dl \ :\zeta_\mu \partial_t X^\mu e^{ik\cdot X}:, \tag{2.89}$$

where the integration is over the position of the insertion along the real axis. Also, ∂_t means the derivative tangential to the boundary. The tachyon is simply the boundary insertion of the momentum $:e^{ik\cdot X}:$ alone.

2.5 Chan–Paton factors

Let us endow the string endpoints with a slightly more interesting property. We can add non-dynamical degrees of freedom to the ends of the string without spoiling spacetime Poincaré invariance or world-sheet conformal invariance. These are called 'Chan–Paton' degrees of freedom[22] and by declaring that their Hamiltonian is zero, we guarantee that they stay in the state that we put them into. In addition to the usual Fock space labels we have been using for the state of the string, we ask that each end be in a state i or j for i, j from 1 to N (see figure 2.8). We use a family of $N\times N$ matrices, λ^a_{ij}, as a basis into which to decompose a string wavefunction

$$|k;a\rangle = \sum_{i,j=1}^{N} |k,ij\rangle\lambda^a_{ij}. \tag{2.90}$$

These wavefunctions are called 'Chan–Paton factors'. Similarly, all open string vertex operators carry such factors. For example, consider the tree-level (disc) diagram for the interaction of four oriented open strings in figure 2.9. As the Chan–Paton degrees of freedom are non-dynamical, the right end of string number 1 must be in the same state as the left end of string number 2, etc., as we go around the edge of the disc. After summing over all the possible states involved in tying up the ends, we are left with a trace of the product of Chan–Paton factors,

$$\lambda^1_{ij}\lambda^2_{jk}\lambda^3_{kl}\lambda^4_{li} = \mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4). \tag{2.91}$$

Fig. 2.8. An open string with Chan–Paton degrees of freedom.

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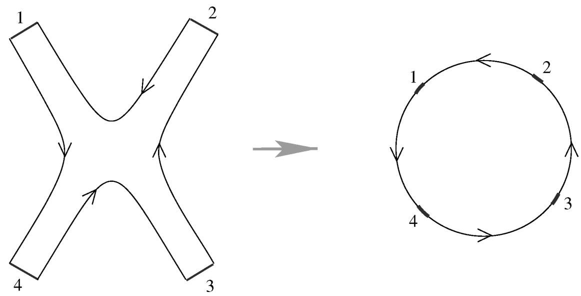

Fig. 2.9. A four-point scattering of open strings, and its conformally related disc amplitude.

All open string amplitudes will have a trace like this and are invariant under a global (on the world-sheet) $U(N)$:

$$\lambda^i \rightarrow U\lambda^i U^{-1}, \tag{2.92}$$

under which the endpoints transform as $\mathbf{N}$ and $\bar{\mathbf{N}}$.

Notice that the massless vector vertex operator $V^{a\mu} = \lambda^a_{ij}\partial_t X^\mu \exp \times (ik\cdot X)$ transforms as the adjoint under the $U(N)$ symmetry. *This means that the global symmetry of the world-sheet theory is promoted to a gauge symmetry in spacetime.* It is a gauge symmetry because we can make a different $U(N)$ rotation at separate points $X^\mu(\sigma,\tau)$ in spacetime.

2.6 Unoriented strings

2.6.1 Unoriented open strings

There is an operation of world-sheet parity Ω which takes $\sigma \rightarrow \pi - \sigma$, on the open string, and acts on $z = e^{\tau - i\sigma}$ as $z \leftrightarrow -\bar{z}$. In terms of the mode expansion (2.83), $X^\mu(z,\bar{z}) \rightarrow X^\mu(-\bar{z},-z)$ yields

$$\begin{aligned} x^\mu &\rightarrow x^\mu \\ p^\mu &\rightarrow p^\mu \\ \alpha^\mu_m &\rightarrow (-1)^m \alpha^\mu_m. \end{aligned} \tag{2.93}$$

This is a global symmetry of the open string theory and so we can, if we wish, also consider the theory that results when it is gauged, by which we mean that only Ω-invariant states are left in the spectrum. We must also consider the case of taking a string around a closed loop. It is allowed to come back to itself only up to an over all action of Ω, which is to swap the ends. This means that we must include unoriented world-sheets in our analysis. For open strings, the case of the Möbius strip is a useful

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example to keep in mind. It is on the same footing as the cylinder when we consider gauging Ω. The string theories which result from gauging Ω are understandably called ‘unoriented string theories’.

Let us see what becomes of the string spectrum when we perform this projection. The open string tachyon is even under Ω and so survives the projection. However, the photon, which has only one oscillator acting, does not:

$$\begin{aligned}\Omega|k\rangle &= +|k\rangle \\ \Omega\alpha^{\mu}_{-1}|k\rangle &= -\alpha^{\mu}_{-1}|k\rangle.\end{aligned} \tag{2.94}$$

We have implicitly made a choice about the sign of Ω as it acts on the vacuum. The choice we have made in writing equation (2.94) corresponds to the symmetry of the vertex operators (2.89): the resulting minus sign comes from the orientation reversal on the tangent derivative $\partial_{\rm t}$ (see figure 2.10).

Fortunately, we have endowed the string's ends with Chan–Paton factors, and so there is some additional structure which can save the photon. While Ω reverses the Chan–Paton factors on the two ends of the string, it can have some additional action:

$$\Omega\lambda_{ij}|k,ij\rangle \ \to\ \lambda'_{ij}|k,ji\rangle, \quad \lambda' = M\lambda^T M^{-1}. \tag{2.95}$$

This form of the action on the Chan–Paton factor follows from the requirement that it be a symmetry of the amplitudes which have factors like those in equation (2.91).

If we act twice with Ω, this should square to the identity on the fields, and leave only the action on the Chan–Paton degrees of freedom. States should therefore be invariant under:

$$\lambda \to MM^{-T}\lambda M^T M^{-1}. \tag{2.96}$$

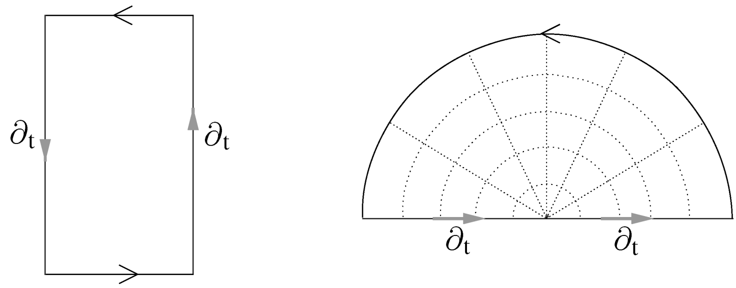

Fig. 2.10. The action of Ω on the photon vertex operator can be deduced from seeing how exchanging the ends of the string changes the sign of the tangent derivative, $\partial_{\rm t}$.

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Now it should be clear that the λ must span a complete set of $N \times N$ matrices: If strings with ends labelled ik and jl are in the spectrum for *any* values of k and l, then so is the state ij. This is because jl implies lj by CPT, and a splitting–joining interaction in the middle gives $ik + lj \to ij + lk$.

Now equation (2.96) and Schur's lemma require MM^{-T} to be proportional to the identity, so M is either symmetric or antisymmetric. This gives two distinct cases, modulo a choice of basis[24]. Denoting the $n \times n$ unit matrix as I_n, we have the symmetric case:

$$M = M^T = I_N. \tag{2.97}$$

In order for the photon $\lambda_{ij}\alpha^\mu_{-1}|k, ij\rangle$ to be even under Ω and thus survive the projection, λ must be antisymmetric to cancel the minus sign from the transformation of the oscillator state. So $\lambda = -\lambda^T$, giving the gauge group $SO(N)$. For the antisymmetric case, we have:

$$M = -M^T = i \begin{bmatrix} 0 & I_{N/2} \\ -I_{N/2} & 0 \end{bmatrix}. \tag{2.98}$$

For the photon to survive, $\lambda = -M\lambda^T M$, which is the definition of the gauge group $USp(N)$. Here, we use the notation that $USp(2) \equiv SU(2)$. Elsewhere in the literature this group is often denoted $Sp(N/2)$.

2.6.2 Unoriented closed strings

Turning to the closed string sector. For closed strings, we see that the mode expansion (2.84) for $X^\mu(z, \bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z})$ is invariant under a world-sheet parity symmetry $\sigma \to -\sigma$, which is $z \to -\bar{z}$. (We should note that this is a little different from the choice of Ω we took for the open strings, but more natural for this case. The two choices are related to each other by a shift of π.) This natural action of Ω simply reverses the left- and right-moving oscillators:

$$\Omega: \qquad \alpha^\mu_n \leftrightarrow \tilde{\alpha}^\mu_n. \tag{2.99}$$

Let us again gauge this symmetry, projecting out the states which are odd under it. Once again, since the tachyon contains no oscillators, it is even and is in the projected spectrum. For the level one excitations:

$$\Omega\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|k\rangle = \tilde{\alpha}^\mu_{-1}\alpha^\nu_{-1}|k\rangle, \tag{2.100}$$

and therefore it is only those states which are symmetric under $\mu \leftrightarrow \nu$ – the

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graviton and dilaton – which survive the projection. The antisymmetric tensor is projected out of the theory.

2.6.3 World-sheet diagrams

As stated before, once we have gauged Ω, we must allow for unoriented world-sheets, and this gives us rather more types of string world-sheet than we have studied so far. Figure 2.11 depicts the two types of one-loop diagram we must consider when computing amplitudes for the open string. The annulus (or cylinder) is on the left, and can be taken to represent an open string going around in a loop. The Möbius strip on the right is an open string going around a loop, but returning with the ends reversed. The two surfaces are constructed by identifying a pair of opposite edges on a rectangle, one with and the other without a twist.

Figure 2.12 shows an example of two types of closed string one-loop diagram we must consider. On the left is a torus, while on the right is a Klein bottle, which is constructed in a similar way to a torus save for a twist introduced when identifying a pair of edges.

In both the open and closed string cases, the two diagrams can be thought of as descending from the oriented case after the insertion of the normalised projection operator $\frac{1}{2}\text{Tr}(1+\Omega)$ into one-loop amplitudes.

Similarly, the unoriented one-loop open string amplitude comes from the annulus and Möbius strip. We will discuss these amplitudes in more detail later.

The lowest order unoriented amplitude is the projective plane $\mathbb{RP}^2$, which is a disk with opposite points identified (see figure 2.13). Shrinking

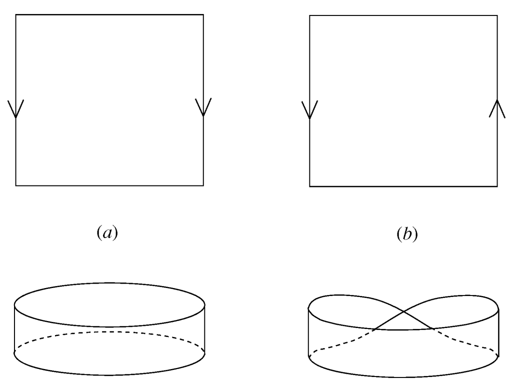

Fig. 2.11. (a) Constructing a cylinder or annulus by identifying a pair of opposite edges of a rectangle. (b) Constructing a Möbius strip by identifying after a twist.

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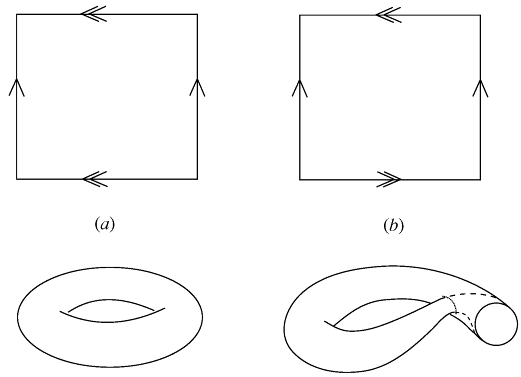

Fig. 2.12. (a) Constructing a torus by identifying opposite edges of a rectangle. (b) Constructing a Klein bottle by identifying after a twist.

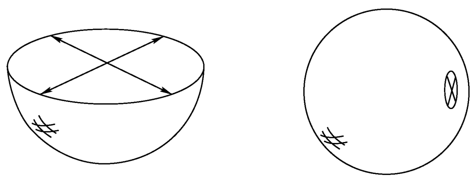

Fig. 2.13. Constructing the projective plane $\mathbb{RP}^2$ by identifying opposite points on the disk. This is equivalent to a sphere with a crosscap insertion.

the identified hole down, we recover the fact that $\mathbb{RP}^2$ may be thought of as a sphere with a crosscap inserted, where the crosscap is the result of shrinking the identified hole. Actually, a Möbius strip can be thought of as a disc with a crosscap inserted, and a Klein bottle is a sphere with two crosscaps. Since a sphere with a hole (one boundary) is the same as a disc, and a sphere with one handle is a torus, we can classify all world-sheet diagrams in terms of the number of handles, boundaries and crosscaps that they have. Insert 2.4 (p.57) summaries all the world-sheet perturbation theory diagrams up to one loop.

2.7 Strings in curved backgrounds

So far, we have studied strings propagating in the (uncompactified) target spacetime with metric $\eta_{\mu\nu}$. While this alone is interesting, it is curved backgrounds of one sort or another which will occupy much of this book, and so we ought to see how they fit into the framework so far.

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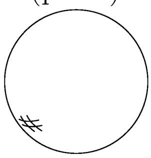
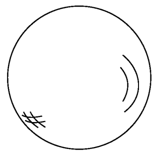
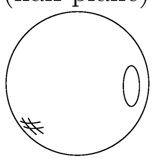
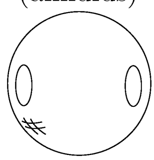
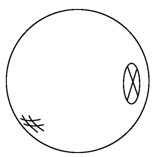
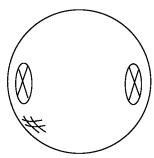
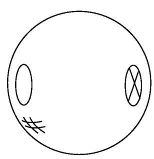
Insert 2.4. World-sheet perturbation theory diagrams

It is worthwhile summarising all of the string theory diagrams up to one-loop in a table. Recall that each diagram is weighted by a factor $g_s^{\chi} = g_s^{2h-2+b+c}$ where h, b, c are the numbers of handles, boundaries and crosscaps, respectively.

	g_s^{-2}	g_s^{-1}	g_s^{0}
closed oriented	sphere S^2 (plane) 	$\cdot$	torus T^2 
open oriented	$\cdot$	disc D_2 (half-plane) 	cylinder C_2 (annulus) 
closed unoriented	$\cdot$	projective plane $\mathbb{RP}^2$ 	Klein bottle KB 
open unoriented	$\cdot$	$\cdot$	Möbius strip MS 

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A natural generalisation of our action is simply to study the 'σ-model' action:

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,(-\gamma)^{1/2}\gamma^{ab}G_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu. \tag{2.101}$$

Comparing this to what we had before (2.21), we see that from the two dimensional point of view this still looks like a model of D bosonic fields X^μ, but with *field dependent* couplings given by the non-trivial spacetime metric $G_{\mu\nu}(X)$. This is an interesting action to study.

A first objection to this is that we seem to have cheated somewhat: strings are supposed to generate the graviton (and ultimately any curved backgrounds) dynamically. Have we cheated by putting in such a background by hand? Or a more careful, less confrontational question might be: is it consistent with the way strings generate the graviton to introduce curved backgrounds in this way?

Well, let us see. Imagine, to start off, that the background metric is only locally a small deviation from flat space: $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$, where h is small.

Then, in conformal gauge, we can write in the Euclidean path integral (2.36):

$$e^{-S_\sigma} = e^{-S}\left(1 + \frac{1}{4\pi\alpha'}\int d^2z h_{\mu\nu}(X)\partial_z X^\mu\partial_{\bar{z}}X^\nu + \cdots\right), \tag{2.102}$$

and we see that if $h_{\mu\nu}(X) \propto g_s\zeta_{\mu\nu}\exp(ik\cdot X)$, where ζ is a symmetric polarisation matrix, we are simply inserting a graviton emission vertex operator. So we are indeed consistent with that which we have already learned about how the graviton arises in string theory. Furthermore, the insertion of the full $G_{\mu\nu}(X)$ is equivalent in this language to inserting an exponential of the graviton vertex operator, which is another way of saying that a curved background is a 'coherent state' of gravitons.

It is clear that we should generalise our success, by including σ-model couplings which correspond to introducing background fields for the antisymmetric tensor and the dilaton:

$$S_\sigma = \frac{1}{4\pi\alpha'}\int d^2\sigma\, g^{1/2}\left\{(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X))\partial_a X^\mu\partial_b X^\nu + \alpha'\Phi R\right\}, \tag{2.103}$$

where $B_{\mu\nu}$ is the background antisymmetric tensor field and Φ is the background value of the dilaton. The coupling for $B_{\mu\nu}$ is a rather straightforward generalisation of the case for the metric. The power of α' is there to counter the scaling of the dimension one fields X^μ, and the antisymmetric tensor accommodates the antisymmetry of B. For the dilaton, a

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coupling to the two dimensional Ricci scalar is the simplest way of writing a reparametrisation invariant coupling when there is no index structure. Correspondingly, there is no power of α' in this coupling, as it is already dimensionless.

N.B. It is worth noting that α' is rather like $\hbar$ for this two dimensional theory, since the action is very large if $\alpha' \to 0$, and so this is a good limit to expand around. In this sense, the dilaton coupling is a one-loop term. Another thing to notice is that the $\alpha' \to 0$ limit is also like a 'large spacetime radius' limit. This can be seen by scaling lengths by $G_{\mu\nu} \to r^2 G_{\mu\nu}$, which results in an expansion in α'/r^2. Large radius is equivalent to small α'.

The next step is to do a full analysis of this new action and ensure that in the quantum theory, one has Weyl invariance, which amounts to the tracelessness of the two dimensional stress tensor. Calculations (which we will not discuss here) reveal that:

$$T^a{}_a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu} g^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R, \quad (2.104)$$

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha'\left(R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\kappa\sigma}H_\nu{}^{\kappa\sigma}\right) + O(\alpha'^2),\\
\beta^B_{\mu\nu} &= \alpha'\left(-\frac{1}{2}\nabla^\kappa H_{\kappa\mu\nu} + \nabla^\kappa\Phi H_{\kappa\mu\nu}\right) + O(\alpha'^2), \qquad (2.105)\\
\beta^\Phi &= \alpha'\left(\frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_\kappa\Phi\nabla^\kappa\Phi - \frac{1}{24}H_{\kappa\mu\nu}H^{\kappa\mu\nu}\right) + O(\alpha'^2),
\end{aligned}$$

with $H_{\mu\nu\kappa} \equiv \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu}$. For Weyl invariance, we ask that each of these β-functions for the σ-model couplings actually vanish. (See insert 3.1 for further explanation of this.) The remarkable thing is that these resemble *spacetime field equations for the background fields.* These field equations can be derived from the following spacetime action:

$$\begin{aligned}
\mathrm{S} = \frac{1}{2\kappa_0^2}\int d^D X(-G)^{1/2}e^{-2\Phi}\Bigg[&R + 4\nabla_\mu\Phi\nabla^\mu\Phi - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\\
&- \frac{2(D-26)}{3\alpha'} + O(\alpha')\Bigg]. \qquad (2.106)
\end{aligned}$$

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N.B. Now we note something marvellous: Φ is a background field which appears in the closed string theory σ-model multiplied by the Euler density. So comparing to equation (2.35) (and discussion following), we recover the remarkable fact that the string coupling g_s is not fixed, but is in fact given by the value of one of the background fields in the theory: $g_s = e^{\langle\Phi\rangle}$. So the only free parameter in the theory is the string tension.

Turning to the open string sector, we may also write the effective action which summarises the leading order (in α') open string physics at tree level:

$$S = -\frac{C}{4}\int d^DX\, e^{-\Phi}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + O(\alpha'), \tag{2.107}$$

with C a dimensionful constant which we will fix later. It is of course of the form of the Yang–Mills action, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field A_μ is coupled in σ-model fashion to the boundary of the world sheet by the boundary action:

$$\int_{\partial\mathcal{M}} d\tau\, A_\mu \partial_t X^\mu, \tag{2.108}$$

mimicking the form of the vertex operator (2.89).

One should note the powers of e^Φ in the above actions. Recall that the expectation value of e^Φ sets the value of g_s. We see that the appearance of Φ in the actions are consistent with this, as we have $e^{-2\Phi}$ in front of all of the closed string parts, representing the sphere (g_s^{-2}) and $e^{-\Phi}$ for the open string, representing the disc (g_s^{-1}).

Notice that if we make the following redefinition of the background fields:

$$\tilde{G}_{\mu\nu}(X) = e^{2\Omega(X)}G_{\mu\nu} = e^{4(\Phi_0-\Phi)/(D-2)}G_{\mu\nu}, \tag{2.109}$$

and use the fact that the new Ricci scalar can be derived using:

$$\tilde{R} = e^{-2\Omega}\Big[R - 2(D-1)\nabla^2\Omega - (D-2)(D-1)\partial_\mu\Omega\partial^\mu\Omega\Big], \tag{2.110}$$

the action (2.106) becomes:

$$\begin{aligned}\mathrm{S} = \frac{1}{2\kappa^2}\int d^DX(-\tilde{G})^{1/2}\Bigg[&\tilde{R} - \frac{4}{D-2}\nabla_\mu\tilde{\Phi}\nabla^\mu\tilde{\Phi} \\ &- \frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)} + O(\alpha')\Bigg],\end{aligned} \tag{2.111}$$

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with $\tilde{\Phi} = \Phi - \Phi_0$. Looking at the part involving the Ricci scalar, we see that we have the form of the standard Einstein–Hilbert action (i.e. we have removed the factor involving the dilaton Φ), with Newton's constant set by

$$\kappa \equiv \kappa_0 e^{\Phi_0} = (8\pi G_{\mathrm{N}})^{1/2}. \tag{2.112}$$

The standard terminology to note here is that the action (2.106) written in terms of the original fields is called the 'string frame action', while the action (2.111) is referred to as the 'Einstein frame action'. It is in the latter frame that one gives meaning to measuring quantities like gravitational mass-energy. It is important to note the means, equation (2.109), to transform from the fields of one to another, depending upon dimension.

2.8 A quick look at geometry

Now that we are firmly in curved spacetime, it is probably a good idea to gather some concepts, language and tools which will be useful to us in many places later on. We have already reminded ourselves in chapter 1 of aspects of the classical differential geometry that is used to formulate the dynamics of gravity, introducing the metric, affine connection, Riemann tensors, etc. We will have reason to use another very pleasant way of writing of the various geometrical objects which appear in dynamical gravity, so we will quickly review it now, visiting a few other useful objects like differential forms along the way.

2.8.1 *Working with the local tangent frames*

We can introduce 'vielbeins' which locally diagonalise the metric[§]:

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x).$$

The vielbeins form a basis for the tangent space at the point x, and orthonormality gives

$$e^a_\mu(x) e^{\mu b}(x) = \eta^{ab}.$$

These are interesting objects, connecting curved and tangent space, and transforming appropriately under the natural groups of each (see figure 2.14). It is a covariant vector under general coordinate transformations $x \to x'$:

$$e^a_\mu \to e'^a_\mu = \frac{\partial x^\nu}{\partial x'^\mu} e^a_\nu,$$

[§] 'Vielbein' means 'many legs', adapted from the German. In $D = 4$ it is called a 'vierbein'. We shall offend the purists henceforth and not capitalise nouns taken from the German language into physics, such as 'ansatz', 'bremsstrahlung' and 'gedankenexperiment'.

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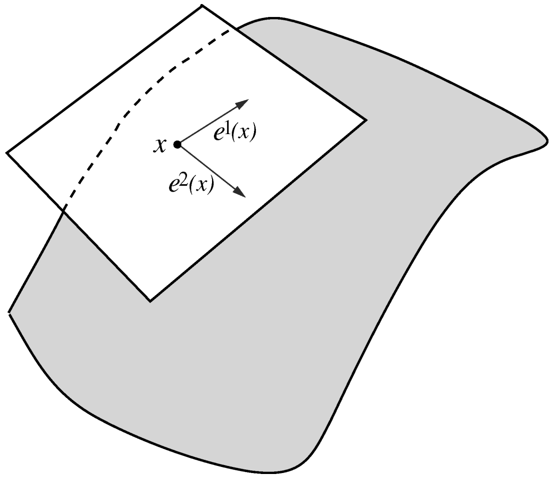

Fig. 2.14. The local tangent frame to curved spacetime is a copy of Minkowski space, upon which the Lorentz group acts naturally.

and a contravariant vector under local Lorentz:

$$e^a_\mu(x) \to e'^a_\mu(x) = \Lambda^a{}_b(x) e^b_\mu(x),$$

where $\Lambda^a{}_b(x)\Lambda^c{}_d(x)\eta_{ac} = \eta_{bd}$ defines Λ as being in the Lorentz group $SO(1, D{-}1)$.

So we have the expected freedom to define our vielbein up to a local Lorentz transformation in the tangent frame. In fact the condition Λ is a Lorentz transformation guarantees that the metric is invariant under local Lorentz:

$$g_{\mu\nu} = \eta_{ab} e'^a{}_\mu e'^b{}_\nu. \tag{2.113}$$

Notice that we can naturally define a family of inverse vielbiens as well, by raising and lowering indices in the obvious way, $e^\mu_a = \eta_{ab} g^{\mu\nu} e^b_\nu$. (We use the same symbol for the vielbien, but the index structure will make it clear what we mean.) Clearly,

$$g^{\mu\nu} = \eta^{ab} e^\mu_a e^\nu_b, \quad e^\mu_b e^a_\mu = \delta^a_b. \tag{2.114}$$

In fact, the vielbien may be thought of as simply the matrix of coefficients of the transformation (discussed in insert 1.2) which finds a locally inertial frame $\xi^a(x)$ from the general coordinates x^μ at the point $x = x_o$:

$$e^a_\mu(x) = \left.\frac{\partial \xi^a(x)}{\partial x^\mu}\right|_{x=x_o},$$

which, by construction, has the transformation properties ascribed to it above.

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As a not-unrelated aside, note that the prototype contravariant vector in curved spacetime is in fact the object whose components are the infinitessimal coordinate displacements, dx^μ, since by the elementary chain rule, under $x \to x'$:

$$dx^\mu \to dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu. \tag{2.115}$$

They are often thought of as the coordinate basis elements, $\{dx^\mu\}$, for the 'cotangent' space at the point x, and are a natural dual coordinate basis to that of the tangent space, the objects $\{\partial/\partial x^\mu\}$, via the perhaps obvious relation:

$$\frac{\partial}{\partial x^\mu} \cdot dx^\nu = \delta^\nu_\mu. \tag{2.116}$$

Of course, the $\{\partial/\partial x^\mu\}$ are the prototype covariant vectors:

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu}. \tag{2.117}$$

The things we usually think of as vectors in curved spacetime have a natural expansion in terms of these bases:

$$V = V^\mu \frac{\partial}{\partial x^\mu}, \qquad \text{or} \quad V = V_\mu dx^\mu,$$

where the latter is sometimes called a 'covector', and is also in fact a one-form.

2.8.2 *Differential forms*

Since we've seen some one-forms appearing, let's pause to introduce them properly, if briefly. As might be apparent, it is the dx^μ which are useful for constructing p-forms, objects whose components are rank p tensors which are totally antisymmetric¶.

As already stated, the dx^μ are themselves the basis for one-forms. Any one-form A has components A_μ and is expanded $A = A_\mu dx^\mu$. To make higher rank forms, we need the idea of the *wedge* product $\wedge$. The basis for two-forms for example, is made by the antisymmetric tensor product

$$dx^\mu \wedge dx^\nu \equiv dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu = -dx^\nu \wedge dx^\mu,$$

and we may then define a two-form F to have totally antisymmetric components $F_{\mu\nu}$, so that $F = (F_{\mu\nu}/2)dx^\mu \wedge dx^\nu$. After noting paranthetically

¶ We will not give an exhaustive account of these objects here, but enough detail to get an intuitive feel for what we need. We shall uncover more features as we need them.

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and for completeness that ordinary functions are zero-forms, the generalisation to higher rank forms is obvious: we make a basis for a p-form by making a totally antisymmetric combination of tensor multiplications of the one-forms, by adding together the results of taking products in all possible permutations, including a result with a minus sign if the permutation is odd, and a plus sign if it is even, giving us for example:

$$\begin{aligned}
&dx^{\mu_1}\wedge dx^{\mu_2}\wedge dx^{\mu_3}\\
&\quad\equiv dx^{\mu_1}\otimes dx^{\mu_2}\otimes dx^{\mu_3}+dx^{\mu_2}\otimes dx^{\mu_3}\otimes dx^{\mu_1}+dx^{\mu_3}\otimes dx^{\mu_1}\otimes dx^{\mu_2}\\
&\quad -dx^{\mu_1}\otimes dx^{\mu_3}\otimes dx^{\mu_2}-dx^{\mu_3}\otimes dx^{\mu_2}\otimes dx^{\mu_1}-dx^{\mu_2}\otimes dx^{\mu_1}\otimes dx^{\mu_3}.
\end{aligned}$$

So in general we have, for rank p:

$$dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p},$$

with which we can define a p-form $G_{(p)}$ with totally antisymmetric components $G_{\mu_1\mu_2\cdots\mu_p}$. We have:

$$G_{(p)}=\frac{1}{p!}G_{\mu_1\mu_2\cdots\mu_p}dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}.$$

It is natural to define the 'exterior derivative' which makes a $(p+1)$-form from a p-form:

$$dG_{(p)}=\frac{1}{p!}\frac{\partial}{\partial x^{\nu}}\left(G_{\mu_1\mu_2\cdots\mu_p}\right)dx^{\nu}\wedge dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_p}.$$

Notice that d^2 always gives zero, since (as the reader should check) this would give a symmetric combination of partial derivatives, which is being summed with the antisymmetric basis, which can't help but give zero.

A form G which can be written everywhere as the result of having acted with d on a form of lower rank is said to be 'exact'. A form H for which $dH=0$ is 'closed'. Exact forms are trivially closed, since $d^2=0$, and so the interesting exercise is to find the closed forms on a space which are not exact. This is a problem of *cohomology*, and we shall have some more to say about this matter in chapter 9.

Forms are extremely natural objects to integrate over some manifold, M. In fact, a manifold of dimension p has a natural form defined on it, of rank p, which is simply the volume form, $\omega=dx^1\wedge\cdots\wedge dx^p$. All p-forms on M are made by taking this object and multiplying it by some function. So the meaning of integrating a p-form on a manifold of dimension p is simply the standard multiple integration of the function:

$$\begin{aligned}
\int_M F_{(p)}&\equiv\int_M\frac{1}{p!}F_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\\
&=\int_M F_{1\cdots p}dx^1\wedge\cdots\wedge dx^p=\int_M F_{1\cdots p}d^px,
\end{aligned}$$

where the reader should notice that this required no metric on the manifold to be defined at all. Putting this observation together with the statements about cohomology, it should be apparent that forms give tools for computing topological properties of manifolds, since they can be integrated on various submanifolds to give numbers, and we never have to specify a metric.

The wedge or exterior product between a p-form and a q-form, which gives a $(p+q)$ form, is straightforward to define. On components, the result is:

$$(A_{(p)} \wedge B_{(q)})_{\mu_1 \cdot \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}.$$

It is worth noting that

$$A_{(p)} \wedge B_{(q)} = (-1)^{pq} B_{(q)} \wedge A_{(p)}.$$

More subtle is the observation that the space of independent p-forms on a D-dimensional spacetime is in fact of the same dimension as that of the $D-p$-forms. There is a map which takes one into the other, called 'Hodge duality', which takes any p-form and gives back a $(D-p)$-form. On the basis it is:

$$\begin{aligned} &{}^*(dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}) = \\ &\qquad \frac{(-g)^{1/2}}{(D-p)!} \epsilon^{\mu_1 \mu_2 \cdots \mu_p}{}_{\mu_{p+1}\mu_{p+2}\cdots\mu_D} dx^{\mu_{p+1}} \wedge dx^{\mu_{p+2}} \wedge \cdots \wedge dx^{\mu_D}, \end{aligned}$$

from which its action on components of any form gives:

$${}^*G_{\mu_1 \cdots \mu_{D-p}} = \frac{(-g)^{1/2}}{p!} \epsilon_{\mu_1 \cdots \mu_{D-p}}{}^{\nu_1 \cdots \nu_p} G_{\nu_1 \cdots \nu_p}.$$

Notice that it is the totally antisymmetric tensor (normalised to unity for its non-zero components) which appears in this definition, and indices are raised and lowered with the metric.

A most useful object is the *'inner product'* between two p-forms, $A_{(p)}$ and $B_{(p)}$, which yields a number. It is defined as:

$$(A_{(p)}, B_{(p)}) \equiv \int_{\mathcal{M}} A_{(p)} \wedge {}^*B_{(p)} = p! \int_{\mathcal{M}} (-g)^{1/2} A_{\mu_1 \mu_2 \cdots} B^{\mu_1 \mu_2 \cdots} dx^1 \wedge \cdots dx^D.$$

2.8.3 Coordinate vs. orthonormal bases

Yet another way of thinking of the vielbiens is as a means of converting that coordinate basis into a basis for the tangent space which is orthonormal, via $\{e^a = e^a_\mu(x) dx^\mu\}$. We see that we have defined a natural family of

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Insert 2.5. Yang–Mills theory with forms

Just in case differential forms which we are briefly introducing have not been encountered before, let us familiarise ourselves with how they work using Yang–Mills theory as an example. The gauge potential, which is valued in the Lie algebra of some gauge group G can be written as a matrix-valued one-form: $A = t^a A^a_\mu dx^\mu$, where the t^a are generators of the Lie algebra. (The index a here is a label of generators in the adjoint representation of the Yang–Mills gauge group G.) Recall also that the generators of the Lie algebra satisfy

$$[t^a, t^b] = i f^{ab}{}_c t^c,$$

where the $f^{ab}{}_c$ are the '*structure constants*'. We shall discuss some Lie algebra and group theory more carefully in section 4.6.1.
We write the Yang–Mills field strength as a matrix-valued 2-form:

$$\begin{aligned} F &= dA + A \wedge A = F^a t^a = \frac{1}{2} t^a F^a_{\mu\nu} dx^\mu \wedge dx^\nu, \\ \text{where} \quad F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i f^a{}_{bc} A^b_\mu A^c_\nu. \end{aligned}$$

Note that we'll sometimes suppress the $\wedge$ and write $F = dA + A^2$ for short.
A gauge transformation is

$$A \to \Sigma A \Sigma^{-1} - d\Sigma \Sigma^{-1}, \quad \Sigma \in G,$$

or infinitessimally, writing $\Sigma = e^{-\Lambda}$, it is:

$$\delta A = d\Lambda + [A, \Lambda].$$

The field strength transforms under this as

$$F \to \Sigma F \Sigma^{-1}; \quad \text{or} \quad \delta F = [F, \Lambda].$$

The action for the theory is

$$S_{\rm YM} = \int d^D x \sqrt{-g} \left(-\frac{1}{4 g^2_{\rm YM}} {\rm Tr}(F^2) \right),$$

where by ${\rm Tr}(F^2)$ we mean $F^a_{\mu\nu} F^{b\mu\nu} {\rm Tr}(t^a t^b)$ and the trace is on the gauge indices. Here $g^2_{\rm YM}$ is the Yang–Mills coupling.

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one-forms. Similarly, using the inverse vielbiens, we can make an orthonormal basis for the dual tangent space via $e_a = e^\mu_a \partial/\partial x^\mu$.

As an example, for the two-sphere, S^2, of radius R, the metric in standard polar coordinates (θ, ϕ) is $ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$ and so we have:

$$e^1_\theta = R, \;\; e^2_\phi = R\sin\theta, \;\; \text{i.e. } e^1 = Rd\theta, \;\; e^2 = R\sin\theta\, d\phi. \tag{2.118}$$

The things we think of as vectors, familiar from flat space, now have two natural settings. In the local frame, there is the usual vector property, under which the vector has Lorentz contravariant components $V^a(x)$. But we can now relate this component to another object which has an index which is contravariant under general coordinate transformations, V^μ. These objects are related by our handy vielbiens: $V^a(x) = e^a_\mu(x)V^\mu$.

2.8.4 The Lorentz group as a gauge group

The standard covariant derivative which we defined earlier in equation (1.9), e.g. on a contravariant vector V^μ, has a counterpart for $V^a = e^a_\mu V^\mu$:

$$D_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\kappa}V^\kappa \quad \Rightarrow \quad D_\nu V^a = \partial_\nu V^a + \omega^a{}_{b\nu}V^b,$$

where $\omega^a{}_{b\nu}$ is the *spin connection*, which we can write as a 1-form in either basis:

$$\omega^a{}_b = \omega^a{}_{b\mu}dx^\mu = \omega^a{}_{b\mu}e^\mu_c e^c_\nu dx^\nu = \omega^a{}_{bc}e^c.$$

We can think of the two Minkowski indices (a, b) from the space tangent structure as labelling components of ω as an $SO(D{-}1, 1)$ matrix in the fundamental representation. So in the analogy with Yang–Mills theory, (see insert 2.5), ω_μ is rather like a gauge potential and the gauge group is the Lorentz group.

Actually, the most natural appearance of the spin connection is in the *structure equations* of Cartan. One defines the torsion T^a, and the curvature $R^a{}_b$, both two-forms, as follows:

$$\begin{aligned} T^a &\equiv \frac{1}{2}T^a{}_{bc}e^a \wedge e^b = de^a + \omega^a{}_b \wedge e^b \\ R^a{}_b &\equiv \frac{1}{2}R^a{}_{bcd}e^c \wedge e^d = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \end{aligned} \tag{2.119}$$

Now consider a Lorentz transformation $e^a \to e'^a = \Lambda^a{}_b e^b$. It is amusing to work out how the torsion changes. Writing the result as $T'^a = \Lambda^a{}_b T^b$, the reader might like to check that this implies that the spin connection must transform as (treating everything as $SO(1, D-1)$ matrices):

$$\omega \to \Lambda\omega\Lambda^{-1} - d\Lambda \cdot \Lambda^{-1}, \;\; \text{i.e.} \quad \omega_\mu \to \Lambda\omega_\mu\Lambda^{-1} - \partial_\mu\Lambda \cdot \Lambda^{-1}, \tag{2.120}$$

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or infinitessimally we can write $\Lambda = e^{-\Theta}$, and it is:

$$\delta\omega = d\Theta + [\omega, \Theta]. \tag{2.121}$$

A further check shows that the curvature two-form does

$$R \to R' = \Lambda R \Lambda^{-1}, \quad \text{or} \quad \delta R = [R, \Theta], \tag{2.122}$$

which is awfully nice. This shows that the curvature two-form is the analogue of the Yang–Mills field strength two-form in insert 2.5. The following rewriting makes it even more suggestive:

$$R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu} dx^\mu \wedge dx^\nu, \quad R^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + [\omega_\mu, \omega_\nu]^a{}_b.$$

2.8.5 Fermions in curved spacetime

Another great thing about this formalism is that it allows us to discuss fermions in curved spacetime. Recall first of all that we can represent the Lorentz group with the Γ-matrices as follows. The group's algebra is:

$$[J_{ab}, J_{cd}] = -i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{db} J_{ac}), \tag{2.123}$$

with $J_{ab} = -J_{ba}$, and we can define via the Clifford algebra:

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \quad J^{ab} = -\frac{i}{4}\left[\Gamma^a, \Gamma^b\right], \tag{2.124}$$

where the curved space Γ-matrices are related to the familiar flat (tangent) spacetime ones as $\Gamma^a = e^a_\mu(x)\Gamma^\mu(x)$, giving $\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$. With the Lorentz generators defined in this way, it is now natural to couple a fermion ψ to spacetime. We write a covariant derivative as

$$D_\mu \psi(x) = \partial_\mu \psi(x) + \frac{i}{2} J_{ab} \omega^{ab}{}_\mu(x) \psi(x), \tag{2.125}$$

and since the curved space Γ-matrices are now covariantly constant, we can write a sensible Dirac equation using this: $\Gamma^\mu D_\mu \psi = 0$.

2.8.6 Comparison to differential geometry

Let us make the connection to the usual curved spacetime formalism now, and fix what ω is in terms of the vielbiens (and hence the metric). Asking that the torsion vanishes is equivalent to saying that the vielbeins are covariantly constant, so that $D_\mu e^a_\nu = 0$. This gives $D_\mu V^a = e^{a\nu} D_\mu V_\nu$,

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allowing the two definitions of covariant derivatives to be simply related by using the vielbeins to convert the indices.

The fact that the metric is covariantly constant in terms of curved spacetime indices relates the affine connection to the metric connection, and in this language makes ω^{ab} antisymmetric in its indices. Finally, we get that

$$\omega^a{}_{b\mu} = e^a_\nu \nabla_\mu e^\nu_b = e^a_\nu(\partial_\mu e^\nu_b + \Gamma^\nu_{\mu\kappa} e^\kappa_b).$$

We can now write covariant derivatives for objects with mixed indices (appropriately generalising the rule for terms to add depending upon the index structure), for example, on a vielbien:

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\kappa_{\mu\nu} e^a_\kappa + \omega_\mu{}^a{}_b e^b_\nu. \tag{2.126}$$

Revisiting our two-sphere example, with bases given in equation (2.118), we can see that

$$\begin{aligned} 0 &= de^1 + \omega^1{}_2 \wedge e^2 = 0 + \omega^1{}_2 \wedge e^2, \\ 0 &= de^2 + \omega^2{}_1 \wedge e^1 = R\cos\theta d\theta \wedge d\phi + \omega^2{}_1 \wedge e^1, \end{aligned} \tag{2.127}$$

from which we see that $\omega^1{}_2 = -\cos\theta\, d\phi$. The curvature is:

$$R^1{}_2 = d\omega^1{}_2 = \sin\theta d\theta \wedge d\phi = \frac{1}{R^2} e^1 \wedge e^2 = R^1{}_{212} e^1 \wedge e^2. \tag{2.128}$$

Notice that we can recover our friend the usual Riemann tensor if we pulled back the tangent space indices (a, b) on $R^a{}_{b\mu\nu}$ to curved space indices using the vielbiens e^μ_a.

One last thing to note is the usefulness of forms for writing volume elements for integration:

$$dV \equiv e = e^1 \wedge e^2 \wedge \cdots \wedge e^D = (-g)^{1/2} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^D = (-g)^{1/2} d^D x.$$

Commonly, we will take the totally antisymmetric symbol ϵ and make a tensor out of it by multiplying by $(-g)^{1/2}$, defining:

$$\varepsilon_{\mu_1\cdots\mu_D} = (-g)^{1/2} \epsilon_{\mu_1\cdots\mu_D},$$

and the reader should check that this is a tensor, noting that the factor of the tensor density $(-g)^{1/2}$ will produce just the right non-tensorial parts to cancel those of the permutation symbol.

We can write the Einstein–Hilbert Lagrangian as:

$$\mathcal{L} \sim eR, \tag{2.129}$$

where R is the Ricci scalar, with $de^a + \omega e^a = 0$ as an additional condition.