Can. J. Math., Vol. XXV, No. 6, 1973, pp. 1174–1182

POLYNOMIAL IDEALS IN GROUP RINGS

M. M. PARMENTER, I. B. S. PASSI, AND S. K. SEHGAL

1. Introduction. Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in n non-commuting variables $x_1, x_2, \ldots, x_n$ and their inverses with coefficients in the ring Z of integers, i.e. an element of the integral group ring of the free group on $x_1, x_2, \ldots, x_n$. Let R be a commutative ring with unity, G a multiplicative group and $R(G)$ the group ring of G with coefficients in R. If $g_1, g_2, \ldots, g_n \in G$, then the expression $f(g_1, g_2, \ldots, g_n)$ can be regarded as an element of $R(G)$. We denote the 2-sided ideal of $R(G)$ generated by $f(g_1, g_2, \ldots, g_n)$, $g_1, \ldots, g_n \in G$, by $\mathfrak{A}_{f,R}$ and call the 2-sided ideals of $R(G)$ that are so defined, *polynomial ideals*. We wish to study the elements of $Z(G)$ which are mapped under the homomorphism i_R: $Z(G) \rightarrow R(G)$ induced by $n \rightarrow n1_R$, 1_R = identity of R, into $\mathfrak{A}_{f,R}$. We prove (Theorem 4.1) that $i_R^{-1}(\mathfrak{A}_{f,R})$ depends only on $\mathfrak{A}_{f,Z}$, $i_{Z/p^nZ}{}^{-1}(\mathfrak{A}_{f,Z/p^nZ})$ and the behaviour of the elements $p1_R$, p a prime. It is obvious that the powers $\Delta_R{}^n(G)$ of the augmentation ideal $\Delta_R(G)$ of $R(G)$ are polynomial ideals. We show that the Lie ideals $\Delta_R{}^{(n)}(G)$ defined inductively by

$$\Delta_R{}^{(1)}(G) = \Delta_R(G), \quad \Delta_R{}^{(n)}(G) = [\Delta_R(G), \Delta_R{}^{(n-1)}(G)]R(G)$$

where $[M, N]$ denotes the R-submodule of $R(G)$ generated by $mn - nm$, $m \in M$, $n \in N$, are also polynomial ideals.

An application of our result to the polynomial ideals $\Delta_R{}^n(G)$ and $\Delta_R{}^{(n)}(G)$ yields the dimension subgroups $D_{n,R}(G) = G \cap (1 + \Delta_R{}^n(G))$ and the Lie dimension subgroups $D_{(n),R}(G) = G \cap (1 + \Delta_R{}^{(n)}(G))$ in terms of $D_{n,Z}(G)$, $D_{n,Z/p^rZ}(G)$ and $D_{(n),Z}(G)$, $D_{(n),Z/p^rZ}(G)$ respectively. Our approach unifies and completes the work of Parmenter [**5**] and Sandling [**7**] on dimension subgroups and Lie dimension subgroups over arbitrary rings of coefficients.

We next study the series

$$\Delta_R{}^{(1)}(G) \supseteq \Delta_R{}^{(2)}(G) \supseteq \ldots \supseteq \Delta_R{}^{(i)}(G) \supseteq \ldots .$$

The group rings $R(G)$ with $\Delta_R{}^{(i)}(G) = 0$ for some i are easily characterized. For non-abelian groups G, this happens if and only if G is nilpotent, G' is a finite p-group and p is nilpotent in R. We also investigate the property "$\cap \Delta_R{}^{(i)}(G) = 0$". If R is of characteristic a power of p, p prime, then R(G) has this property if and only if G is residually "nilpotent with derived group a p-group of bounded exponent". We give a partial answer to this question

Received June 7, 1972 and in revised form, December 7, 1972. This research was supported by NRC Grant A-5300.

when R has characteristic zero. For applications of these results and the connection with the underlying Lie algebra of a group algebra see [**8**].

2. Polynomial maps and polynomial ideals. Let G be a group, and R a commutative ring with unity.

2.1. *Definition.* If $f(x_1, x_2, \ldots, x_n)$ is a polynomial in n non-commuting variables and their inverses with integer coefficients, then a map $\theta: G \to M$, M an R-module, is called an *f_R-polynomial map* if the linear extension θ^* of θ to $R(G)$ vanishes on $\mathfrak{A}_{f,R}$, the polynomial ideal determined by f.

We note that a polynomial map $\theta: G \to M$ of degree $\leqq n$ in the sense of Passi [**6**] is an f_R-polynomial map for $f = (x_1 - 1)(x_2 - 1) \ldots (x_{n+1} - 1)$.

We assume throughout that a polynomial $f(x_1, x_2, \ldots, x_n)$ has content zero, i.e. the sum of its coefficients is zero.

2.2 Proposition. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be polynomial ideals of $R(G)$. Then $\mathfrak{A} + \mathfrak{B}$ and $\mathfrak{A}\mathfrak{B}$ are also polynomial ideals.*

Proof. Let $\mathfrak{A} = \mathfrak{A}_{f_1(x_1,x_2,\ldots,x_n),R}$ and $\mathfrak{B} = \mathfrak{A}_{f_2(x_1,x_2,\ldots,x_m),R}$. Then

(i) $\mathfrak{A} + \mathfrak{B} = \mathfrak{A}_{f(x_1,x_2,\ldots,x_{m+n}),R}$ where

$$f(x_1, x_2, \ldots, x_{m+n}) = f_1(x_1, x_2, \ldots, x_n) + f_2(x_{n+1}, x_{n+2}, \ldots, x_{m+n})$$

and

(ii) $\mathfrak{A}\mathfrak{B} = \mathfrak{A}_{g(x_1,x_2,\ldots,x_{m+n})}$ where

$$g(x_1, x_2, \ldots, x_{m+n}) = f_1(x_1, x_2, \ldots, x_n)f_2(x_{n+1}, x_{n+2}, \ldots, x_{m+n}).$$

That the right hand side in (ii) is contained in the left hand side is obvious. For the converse, notice that $f(g_1, g_2, \ldots, g_n)\cdot g = gf(g_1^g, g_2^g, \ldots, g_n^g)$, where the g_i's and $g \in G$ and $g_i^g = g^{-1}g_ig$.

Let $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_n \supseteq \ldots$ be the lower central series of G. If N is a normal subgroup of G, we denote by $\Delta_R(G, N)$ the kernel of the natural ring homomorphism $R(G) \to R(G/N)$. It may be noted that

$$\Delta_R(G, N) = \Delta_R(N)\cdot R(G).$$

2.3. Proposition. *The ideals $\Delta_R(G, G_n)$ are polynomial ideals.*

Proof. The ideal $\Delta_R(G, G_n)$ is generated by $(g_1, g_2, \ldots, g_n) - 1$, the g_i's in G, where

$$(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2 \quad \text{and} \quad (g_1, g_2, \ldots, g_n) = ((g_1, g_2, \ldots, g_{n-1}), g_n).$$

Thus, if $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n) - 1$, then $\Delta_R(G, G_n) = \mathfrak{A}_{f,R}$.

We recall

2.4. THEOREM (Sandling [7]).

$$\Delta_R^{(n)}(G) = \Delta_R(G_n)R(G) + \sum \Pi \Delta_R(G_{n_j})R(G),$$

where the sum is over all n_j, $n \geqq n_j > 1$, *for which* $\sum(n_j - 1) = n - 1$.

It is clear from Theorem 2.4 that

$$\Delta_R^{(m)}(G) \cdot \Delta_R^{(n)}(G) \subseteq \Delta_R^{(n+m-1)}(G) \quad \text{for all } m, n \geqq 1.$$

2.5. PROPOSITION. $\Delta_R^{(n)}(G)$ *is a polynomial ideal for all* $n \geqq 1$.

Proof. This follows from Theorem 2.4 and the Propositions 2.2. and 2.3.

2.6. PROPOSITION. *Let* M *be an abelian group,* N *an* R*-module,* $\theta: G \to M$ *an* f_Z*-polynomial map,* $\phi: M \to N$ *a homomorphism. Then the map*

$$\phi \circ \theta: G \to N$$

is an f_R*-polynomial map.*

Proof. $\mathfrak{A}_{f,R}$ is generated as an R-module by the elements $gf_1f_2 \ldots f_k$ where $g \in G$ and the f_i's are the values of $f = f(x_1, x_2, \ldots, x_n)$ (regarded as elements of $R(G)$) when the x_i's take values from G. Since

$$(\phi \circ \theta)^*(gf_1f_2 \ldots f_k) = \phi(\theta^*(gf_1f_2 \ldots f_k)) = 0,$$

the result follows (* denotes the linear extension of the map to the group ring).

3. The second dimension subgroups of rationals mod 1. We denote by T the group of rationals mod 1. Since the dimension conjecture holds for abelian groups, Sandling's theorem [7, Theorem 6.1 of Chapter I] applies to give the dimension subgroups of T with arbitrary coefficient rings. In view of the important role that $D_{2,R}(T)$ plays in this work, we give an independent proof for this case.

3.1. THEOREM. $D_{2,R}(T) = \sum_{p \in \sigma(R)} Z(p^\infty)$, *where*

$$\sigma(R) = \{p | p^nR = p^{n+1}R \text{ for some } n, p \text{ prime}\}.$$

(If $\sigma(R)$ is empty, the right hand side is to be interpreted as the identity subgroup.)

Proof. Let $p \in \sigma(R)$, $t \in Z(p^\infty)$. Then

$$\begin{aligned} t - 1 &= x^{p^n} - 1 \text{ for some } x \in Z(p^\infty), \text{ since } Z(p^\infty) \text{ is divisible} \\ &\equiv p^n(x-1) \bmod \Delta_R^2(Z(p^\infty)) \\ &\equiv rp^{n+m}(x-1) \text{ where } x^{p^m} = 1 \text{ and } r \in R \\ &\equiv 0, \text{ since } p^m(x-1) \in \Delta_R^2(T). \end{aligned}$$

Hence $\sum_{p \in \sigma(R)} Z(p^\infty) \subseteq D_{2,R}(T)$.

Next let $t \in D_{2,R}(T)$. Then for any prime p, t_p, the p-primary component of t, is in $D_{2,R}(Z(p^\infty))$. This follows from the projection of T on its direct summand $Z(p^\infty)$. Let H be the subgroup generated by the elements of $Z(p^\infty)$ which appear in an expression of $t_p - 1$ as an element of $\Delta_R{}^2(Z(p^\infty))$. Then H is a cyclic group of order p^r, say, and $t_p \in D_{2,R}(H)$. If for no n, $p^nR = p^{n+1}R$, then the rings R/p^nR, $n \geqq 1$, are of increasing characteristic and as $D_{2,R}(H) \subseteq D_{2,R/p^nR}(H)$, we have $t_p \in D_{2,R/p^nR}(H) = D_{2,Z/p^nZ}(H)$, where Z_{p^n} denotes the ring of integers mod n (see Theorem 4.1, case I, or [**7**, Chapter I, Corollary 6.4]). Now $D_{2,Z/p^rZ}(H) = H^{p^r} = (1)$. Hence, if $p^nR \neq p^{n+1}R$ for all n, then $t_p = 1$ and the proof is complete.

4. Main result.

4.1. THEOREM. *Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in n non-commuting variables and their inverses with coefficients in Z, G a group, and R a commutative ring with unity. Then*

(i) *if the characteristic of $R = n > 0$, $i_R{}^{-1}(\mathfrak{A}_{f,R}) = i_{Z_n}{}^{-1}(\mathfrak{A}_{f,Z_n})$, where Z_n is the ring of integers* mod n;

(ii) *if the characteristic of $R = 0$,*

$$i_R{}^{-1}(\mathfrak{A}_{f,R}) = \sum_{p \in \sigma(R)} \tau_p(Z(G) \bmod \mathfrak{A}_{f,Z}) \cap i_{Z/p^eZ}{}^{-1}(\mathfrak{A}_{f,Z/p^eZ})$$

where $\sigma(R)$ is the set of primes p for which $p^nR = p^{n+1}R$ for some n, p^e is the smallest power of p for which this holds and for a ring S i_S: $Z(G) \to S(G)$ is the ring homomorphism induced by $m \to m1_S$, $m \in Z$, 1_S = identity of S. Here $\tau_p(Z(G)$ mod $\mathfrak{A}_{f,Z})$ stands for the p-torsion subgroup of $Z(G)$ mod $\mathfrak{A}_{f,Z}$ and if $\sigma(R)$ is empty, then the right hand side of the above equation is to be interpreted as $\mathfrak{A}_{f,Z}$.

Proof. Let $\pi(R)$ denote the set of primes p which are invertible in R.

Case 1. *characteristic of $R = n \neq 0$*: In this case the theorem asserts that $i_R{}^{-1}(\mathfrak{A}_{f,R}) = i_{Z_n}{}^{-1}(\mathfrak{A}_{f,Z_n})$, where n is the characteristic of R and Z_n is the ring of integers mod n. Z_n can be regarded as a subring of R and so the right hand side is contained in the left hand side. Let $z \in Z(G)$ be such that $z' = i_R(z) \in \mathfrak{A}_{f,R}$. If $z'' = i_{Z_n}(z) \notin \mathfrak{A}_{f,Z_n}$, then we can define a homomorphism

$$\theta: Z_n(G)/\mathfrak{A}_{f,Z_n} \to T,$$

where T is the additive group of rationals mod 1, such that $\theta(z'' + \mathfrak{A}_{f,Z_n}) \neq 0$. As the image of θ must be contained in Z_n, we have an f_Z-polynomial map ϕ: $G \to Z_n$, $\phi(x) = \theta(x + \mathfrak{A}_{f,Z})$. Composing with the embedding i: $Z_n \to R$, we have (Proposition 2.6) an f_R-polynomial map $\alpha = i \circ \phi$: $G \to R$ such that

$$\alpha^*(z') = i\phi^*(z'') = i\theta(z'' + \mathfrak{A}_{f,Z_n}) \neq 0.$$

This is a contradiction, since $z' \in \mathfrak{A}_{f,R}$. Hence $z'' \in \mathfrak{A}_{f,Z_n}$ and we have $i_R{}^{-1}(\mathfrak{A}_{f,R}) \subseteq i_{Z_n}{}^{-1}(\mathfrak{A}_{f,Z_n})$.

For the rest of the proof we assume that R is of characteristic zero.

Case 2. *$\pi(R)$ = the set of all primes*: In this case Q, the field of rationals, can be regarded as a subring of R. An argument essentially similar to that given in Case 1, with Q in place of both Z_n and T, shows that

$$i_R^{-1}(\mathfrak{A}_{f,R}) = i_Q^{-1}(\mathfrak{A}_{f,Q}).$$

Now $i_Q^{-1}(\mathfrak{A}_{f,Q}) = \sum_{p\in\pi(R)}\tau_p(Z(G) \bmod \mathfrak{A}_{f,Z})$ and we are done.

We next assume that $\pi(R)$ is not the set of all primes.

Case 3. $\sigma(R) = \pi(R)$: We have the natural homomorphism

$$\alpha\colon T \rightarrow \Delta_R(T)/\Delta_R^2(T),$$

by $t \rightarrow t - 1 + \Delta_R(T)$, where T = the (additive) group of rationals mod 1. By Theorem 3.1 $\operatorname{Ker}\alpha = D_{2,R}(T) = \sum_{p\in\sigma(R)}Z(p^\infty)$. As $\pi(R)$ is not the set of all primes, $\operatorname{Ker}\alpha \neq T$. Let $z \in i_R^{-1}(\mathfrak{A}_{f,R})$. We assert that for some integer m, all of whose prime divisors are in $\sigma(R)$, $mz \in \mathfrak{A}_{f,Z}$. For, otherwise, we can find a homomorphism $\gamma\colon Z(G)/\mathfrak{A}_{f,Z} \rightarrow T$ such that

$$\gamma(z + \mathfrak{A}_{f,Z}) \notin \sum_{p\in\sigma(R)}Z(p^\infty).$$

This leads to an f_Z-polynomial map $\bar{\gamma}\colon G \rightarrow T$ such that $\bar{\gamma}^*(z) \neq 0$. Composing $\bar{\gamma}$ with α we obtain an f_R-polynomial map $\theta = \alpha \circ \bar{\gamma}\colon G \rightarrow \Delta_R(T)/\Delta_R^2T)$ into the R-module $\Delta_R(T)/\Delta_R^2(T)$ such that $\theta^*(z) \neq 0$. This contradicts the fact that $z \in i_R^{-1}(\mathfrak{A}_{f,R})$. Hence for some m, $mz \in \mathfrak{A}_{f,Z}$ and all prime divisors of m are in $\sigma(R)$. As $\sigma(R) = \pi(R)$, the proof of this case is complete.

Case 4. *$\sigma(R) - \pi(R)$ is finite*: We proceed by induction on the order of the set $\sigma(R) - \pi(R)$. When the order is zero, we have the situation of Case 3. Let $p \in \sigma(R)$, $p \notin \pi(R)$ and let p^e be the smallest power of p for which $p^eR = p^{e+1}R$. Then $R \cong R/p^eR \oplus R/J$, where $J = \{r \in R | p^e r = 0\}$, $\sigma(R) = \sigma(R/J)$ and p can be seen to be invertible in R/J [**7**, Chapter I, section 6]. Thus we can assume that the theorem holds for R/J and so if

$$z \in i_R^{-1}(\mathfrak{A}_{f,R}),$$

then

$$z \in i_{R/J}{}^{-1}(\mathfrak{A}_{f,R/J}) = \sum_{q\in\sigma(R/J)=\sigma(R)} \tau_q(Z(G) \bmod \mathfrak{A}_{f,Z}) \cap i_{Z/q^{e(q)}Z}{}^{-1}(\mathfrak{A}_{f,Z/q^{e(q)}Z})$$

where $e(q)$ is the least integer for which

$$q^{e(q)}R/J = q^{e(q)+1}R/J.$$

It is easy to see that for $q \neq p$ $e(q)$ is also the least integer for which

$$q^{e(q)}R = q^{e(q)+1}R.$$

Hence

$$(**) \quad i_R{}^{-1}(\mathfrak{A}_{f,R}) \subseteq \sum_{q\in\sigma(R),q\neq p} \tau_q(Z(G) \bmod \mathfrak{A}_{f,Z}) \cap i_{Z/q^{e(q)}Z}{}^{-1}(\mathfrak{A}_{f,Z/q^{e(q)}Z}) + \tau_p(Z(G) \bmod \mathfrak{A}_{f,Z}).$$

https://doi.org/10.4153/CJM-1973-127-4 Published online by Cambridge University Press

Also

$$z \in i_{R/p^eR}^{-1}(\mathfrak{A}_{f,R/p^eR}) = i_{Z/p^eZ}^{-1}(\mathfrak{A}_{f,Z/p^eZ}).$$

Since for $q \neq p$

$$i_{Z/p^eZ}(\tau_q(Z(G) \bmod \mathfrak{A}_{f,Z}) \subseteq \mathfrak{A}_{f,Z/p^eZ}$$

we get from (**) that

$$z \in \sum_{q \in \sigma(R)} \tau_q(Z(G) \bmod \mathfrak{A}_{f,Z}) \cap i_{Z/q^{e(q)}Z}^{-1}(\mathfrak{A}_{f,Z/q^{e(q)}Z}).$$

Conversely, if

$$z \in \sum_{q \in \sigma(R)} \tau_q(Z(G) \bmod \mathfrak{A}_{f,Z}) \cap i_{Z/q^{e(q)}Z}^{-1}(\mathfrak{A}_{f,Z/q^{e(q)}Z}),$$

then, by induction

$$z \in i_{R/J}^{-1}(\mathfrak{A}_{f,R/J})$$

and also

$$z \in i_{R/p^eR}^{-1}(\mathfrak{A}_{f,R/p^eR}) = i_{Z/p^eZ}^{-1}(\mathfrak{A}_{f,Z/p^eZ}).$$

Hence

$$i_R(z) \in \mathfrak{A}_{f,R}.$$

Case 5. *$\sigma(R)$ is arbitrary*: As in [**7**, p. 62], the general case reduces to Case 3 since one can assume that R is finitely generated and therefore $\sigma(R) - \pi(R)$ is finite. For details of the reduction argument we refer the reader to [**7**].

5. Dimension subgroups over arbitrary rings of coefficients. If N is a normal subgroup of G and p a prime, we denote by $\tau_p(G \bmod N)$ the subgroup of G which is generated by the elements having some pth power in N.

5.1. THEOREM. (i) *If characteristic of $R = 0$, then*

$$D_{n,R}(G) = \prod_{p \in \sigma(R)} \{\tau_p(G \bmod D_{n,Z}(G)) \cap D_{n,Z/p^eZ}(G)\}$$

where $\sigma(R)$ and p^e are as defined in Theorem 4.1. (If $\sigma(R)$ is empty, then the right hand side is to be interpreted as $D_{n,Z}(G)$.)

(ii) *If characteristic of $R = r > 0$, then $D_{n,R}(G) = D_{n,Z_r}(G)$ for all $n \geqq 1$.*

Proof. Suppose char $R = 0$. Let $g \in D_{n,R}(G)$. Then $g - 1_R \in \Delta_R^n(G)$, where 1_R is the identity of R. Let $f(x_1, x_2, \ldots, x_n) = (x_1 - 1)(x_2 - 1) \ldots (x_n - 1)$. Then $\mathfrak{A}_{f,R} = \Delta_R^n(G)$. Therefore, by Theorem 4.1 we have $g - 1 = \sum_{p \in \sigma(R)} z_p$ where $z_p \in Z(G)$ is such that for some $m = m(p)$, $p^m \cdot z_p \in \Delta_Z^n(G)$ and $i_{Z/p^eZ}(z_p) \in \Delta_{Z/p^eZ}^n(G)$. Let $r = \Pi p^{m(p)}$.T hen r is a σ-number and $r(g-1) \in \Delta_Z^n(G)$. For sufficiently large s, r divides the binomial coefficients $\binom{r^s}{i}$, $i = 1, 2, \ldots, n-1$. Hence

$$g^{r^s} - 1 = \sum_{i=1}^{r^s} \binom{r^s}{i} (g-1)^i \equiv 0 \mod \Delta_Z^n(G).$$

https://doi.org/10.4153/CJM-1973-127-4 Published online by Cambridge University Press

From this it is easy to conclude that

$$g \in \Pi_{p\in\sigma(R)}\{\tau_p(G \bmod D_{n,Z}(G)) \cap D_{n,Z/p^eZ}(G)\}.$$

Conversely, let $g \in \tau_p(G \bmod D_{n,Z}(G)) \cap D_{n,Z/p^eZ}(G)$. Then for some u, $g^{p^u} \in D_{n,Z}(G)$. This means that $g^{p^u} - 1 \in \Delta_Z{}^n(G)$ which shows that for a sufficiently large u, $p^u(g-1) \in \Delta_Z{}^n(G)$.

By Theorem 4.1, we have $g - 1 \in \Delta_R{}^n(G)$, i.e. $g \in D_{n,R}(G)$. This completes the proof of case (i). Case (ii) follows immediately from Theorem 4.1(i).

6. Lie dimension subgroups over arbitrary rings of coefficients.

6.1. THEOREM. (i) *If characteristic of* $R = 0$, *then*

$$D_{(n),R}(G) = \prod_{p\in\sigma(R)} G_2 \cap \{\tau_p(G \bmod D_{(n),Z}(G)) \cap D_{(n),Z/p^eZ}(G)\}$$

for $n \geqq 2$, *where* $\sigma(R)$ *and* p^e *are as defined in Theorem* 4.1. (If $\sigma(R)$ is empty, the right hand side is to be interpreted as $D_{(n),Z}(G)$.)

(ii) *If characteristic of* $R = r > 0$, *then* $D_{(n),R}(G) = D_{(n),Z_r}(G)$ *for all* $n \geqq 1$.

Proof. Suppose char $R = 0$. Since $\Delta_R{}^{(2)}(G) = \Delta_R(G, G_2)$, it is clear that $D_{(n),R}(G) \subseteq G_2$ for $n \geqq 2$. Let $g \in D_{n,R}(G)$, $n \geqq 2$. As $\Delta_R{}^{(n)}(G)$ is a polynomial ideal, Theorem 4.1 says that for some σ-number r, $r(g-1) \in \Delta_Z{}^{(n)}(G)$. Theorem 2.4 shows that $(g-1)^m \in \Delta_R{}^{(m+1)}(G)$ for all m. Hence, choosing s sufficiently large, we can conclude that $g^{r^s} - 1 \in \Delta_R{}^{(n)}(G)$ which yields that g is a σ-element mod $D_{(n),Z}(G)$. Hence $g = g_1g_2 \ldots g_k$ where each g_i is a power of g and is a p-element mod $D_{(n),Z}(G)$ for some $p \in \sigma(R)$. Thus

$$g \in \prod_{p\in\sigma(R)} G_2 \cap \{\tau_p(G \bmod D_{(n),Z}(G)) \cap D_{(n),Z/p^eZ}(G)\}.$$

Conversely, let $g \in G_2 \cap \{\tau_p(G \bmod D_{(n),Z}(G) \cap D_{(n),Z/p^eZ}(G)\}$. Then $g^{p^r} - 1 \in \Delta_Z{}^{(n)}(G)$ for some r. As $(g-1)^m \in \Delta_R{}^{(m+1)}(G)$ $(g \in G_2)$, we can find an s such that $p^s(g-1) \in \Delta_Z{}^{(n)}(G)$. Hence, by Theorem 4.1, $g - 1 \in \Delta_R{}^{(n)}(G)$ and so $g \in D_{(n),R}(G)$. This completes the proof of case (i). Case (ii) follows from Theorem 4.1(i).

7. Lie powers of the augmentation ideal. Let G be a group, R a commutative ring with unity. In this section we study the Lie ideals $\Delta_R{}^{(n)}(G)$. (See section 1 for definition.) We recall $(*)$ that

$$\Delta_R{}^{(n)}(G) \cdot \Delta_R{}^{(m)}(G) \subseteq \Delta_R{}^{(n+m-1)}(G) \quad \text{for all } n, m \geqq 1.$$

Evidently $\Delta_R{}^{(2)}(G) = 0$ if and only if G is abelian.

7.1. THEOREM. $\Delta_R{}^{(n)}(G) = 0$ *for some* $n > 2$ *and* $\Delta_R{}^{(2)}(G) \neq 0$ *if and only if* G *is nilpotent*, G_2 *is a finite* p-*group* $\neq (1)$ *and* p *is nilpotent in* R.

Proof. Suppose $\Delta_R{}^{(n)}(G) = 0$. Then $D_{(n),R}(G) = (1)$ and so G is nilpotent.

Also $(\Delta_R^{(2)}(G))^{n-1} = 0$ and therefore $(\Delta_R(G_2))^{n-1} = 0$ which implies ([**1**; **3**] or [**7**, Chapter 2, Lemma 1.1]) that G_2 is a finite p-group and p is nilpotent in R. Conversely, suppose G is a nilpotent group with $|G_2| = p^r$, $r \geqq 1$. Then $G_n = (1)$ and $(\Delta(G_2))^n = 0$ for sufficiently large n and consequently $\Delta_R^{(n^2)}(G) = 0$ (Theorem 2.4).

7.2. *Remark.* The converse in the above proof can also be seen directly by inducting on the order of G_2. The theorem for finite groups is due to R. Sandling [**7**].

7.3. *Notation.* Let p be a prime. We denote by K_p the class of those nilpotent groups whose derived groups are p-groups of finite exponent and by RK_p the class of groups which are residually in K_p.

7.4. THEOREM. *Let R be a commutative ring with unity having characteristic a power of p, p prime. Then $\bigcap_n \Delta_R^{(n)}(G) = 0$ if and only if $G \in RK_p$.*

Proof. Suppose $\bigcap_n \Delta_R^{(n)}(G) = 0$. Then the Lie dimension subgroups $D_{(n),R}(G)$ have the property that $\bigcap_n D_{(n),R}(G) = (1)$. Notice that $G/D_{(n),R}(G)$ is a nilpotent group, since $G_n \subseteq D_{(n),R}(G)$. Let $g \in G_2$. Then $g - 1 \in \Delta_R^{(2)}(G)$ and $(g-1)^r \in \Delta_R^{(n)}(G)$ for $r \geqq n - 1 > 0$. If the characteristic of R is p^m, we choose t so that $p^t > np^m$. Then $p^m | \binom{p^t}{r}$, for $r = 1, 2, \ldots, n-1$ and it follows that

$$g^{p^t} - 1 = \sum_{r=1}^{p^t} \binom{p^t}{r} (g-1)^r \in \Delta_R^{(n)}(G).$$

Hence $G_2^{p^t} \subseteq D_{(n),R}(G)$. This proves that $G/D_{(n),R}(G)$ is a nilpotent group whose derived group is a p-group of bounded exponent, i.e. $G/D_{(n),R}(G) \in K_p$. As $\bigcap_n D_{(n),R}(G) = (1)$, it follows that $G \in RK_p$.

Conversely, as the class K_p is closed under finite direct sums it is enough [**4**] to prove that

$$G \in K_p \Rightarrow \bigcap_n \Delta_R^{(n)}(G) = 0.$$

For a nilpotent group G, Theorem 2.4 gives

$$\bigcap_n \Delta_R^{(n)}(G) \subseteq \bigcap_m \Delta_R^{m}(G_2) \cdot R(G).$$

If now G_2 is a nilpotent p-group of bounded exponent, then, since $R(G)$ is a free $R(G_2)$-module, we can conclude that $\bigcap_m \Delta_R^m(G_2) \cdot R(G) = 0$ [**2**]. Hence $\bigcap_n \Delta_R^{(n)}(G) = 0$.

We now consider the case when the characteristic of R is 0.

7.5. *Definition.* An element $g \in G_2$ is called a *generalized Lie p-element* if for every n, there exists $r(n)$ such that $g^{p^{r(n)}} \in D_{(n),Z}(G)$ or, equivalently, there exists $s(n)$ such that $p^{s(n)}(g-1) \in \Delta_Z^{(n)}(G)$.

7.6. THEOREM. *Let G be a group having a non-identity generalized Lie p-element $g \in G_2$. Let R be a commutative ring with unity such that the charac-*

https://doi.org/10.4153/CJM-1973-127-4 Published online by Cambridge University Press

teristic of R is zero. Then $\bigcap_n \Delta_R^{(n)}(G) = 0$ if and only if $G \in RK_p$ and $\bigcap_n p^n R = 0$.

Proof. Suppose first that $\bigcap_n \Delta_R^{(n)}(G) = 0$. Let

$$D_{(n),m,p,R}(G) = \{x \in G | x - 1 \in \Delta_R^{(n)}(G) + p^m \Delta_R(G)\}.$$

Then we assert that

$$\bigcap_{n,m} D_{(n),m,p,R}(G) = (1).$$

For, let $h \in \bigcap_{n,m} D_{(n),m,p,R}(G)$. Then it is easy to prove that

$$(g - 1)(h - 1) \in \Delta_R^{(n)}(G)$$

for all n. As the characteristic of R is 0, this yields $h = 1$. The groups $G/D_{(n),m,p,R}(G)$ can be seen to be in class K_p. Hence $G \in RK_p$. If $r \in \bigcap_n p^n R$, then $r(g - 1) \in \bigcap_n \Delta_R^{(n)}(G) = 0$. Hence $r = 0$. This proves $\bigcap_n p^n R = 0$.

Conversely, assume that $G \in RK_p$ and $\bigcap_n p^n R = 0$. As the class K_p is closed under finite direct sums [**4**], we can assume without loss of generality that $G \in K_p$. An application of Hartley's result [**2**, Theorem E] yields this case as it does the converse part of Theorem 7.4.

References

1. I. G. Connell, *On the group ring*, Can. J. Math. *15* (1963), 650–685.
2. B. Hartley, *The residual nilpotence of wreath products*, Proc. London Math. Soc. *20* (1970), 365–392.
3. G. Losey, *On group algebras of p-groups*, Michigan Math. J. *7* (1960), 237–240.
4. J. N. Mital, *On residual nilpotence*, J. London Math. Soc. *2* (1970), 337–345.
5. M. M. Parmenter, *On a theorem of Bovdi*, Can. J. Math. *23* (1971), 929–932.
6. I. B. S. Passi, *Polynomial maps on groups*, J. Algebra *9* (1968), 121–151.
7. R. Sandling, *The modular group rings of p-groups*, Ph.D. Thesis, University of Chicago, 1969.
8. S. K. Sehgal, *Lie properties in group algebras*, Proceedings of the conference on "Orders, Group Rings and Related Topics", Ohio State University, 1972, Springer-Verlag (to appear).

University of Alberta,
Edmonton, Alberta;
University of Kurukshetra,
Kurukshetra, India