FACTORIZATION LADDERS AND EIGENFUNCTIONS

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THE eigenfunctions of a boundary value problem are characterized by two quite distinct properties. They are solutions of ordinary differential equations, and they satisfy prescribed boundary conditions. It is a definite advantage to combine these two requirements into a single problem expressed by a unified formula. The use of integral equations is an example in point. The subject of this paper, namely the Schrödinger-Infeld Factorization Method, which is applicable to certain restricted Sturm-Liouville problems, is based upon another combination of the two properties. The Factorization Method prescribes a manufacturing process. From one eigenfunction we can derive another by means of operators which are constructed directly from the differential equation. Given a starting point, the operators can be applied repeatedly to build up "eigenfunction ladders" containing all the solutions of the problem. The two requirements are thus automatically fulfilled by the process of construction.

Further progress would be made by mechanizing the recursive manufacturing process in the same way that the operators mechanized the fulfilment of the two characteristic properties. We shall show that this is actually possible for a large class of problems, many of which are fundamental in Quantum Mechanics.

The operator formulae we shall treat are first order linear homogeneous differential difference equations. A systematic exposition of the properties of such equations has recently been made available through the work of Truesdell [1]. The F-equation

$$\frac{\partial}{\partial z} F(z, \alpha) = F(z, \alpha + 1)$$

has been made the salient feature of his treatment, and a thorough discussion of solutions $F(z, \alpha)$ has been given. This theory provides a powerful tool for the formal manipulation of special functions. Relations of the form

$$\frac{\partial}{\partial y} f(y, \alpha) = A(y, \alpha)\, f(y, \alpha) + B(y, \alpha)\, f(y, \alpha + 1)$$

are discussed, and a "reducibility condition" on A and B is found, which allows this equation to be transformed into the F-equation. This reduction process is the key to the present treatment of the eigenfunction problem. We shall therefore use the Factorization Method to set up a framework directly from the

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differential equation of Sturm-Liouville type. Then we shall show how the F-equation technique leads directly to explicit formulae for the eigenfunctions of every physical problem tractable by factorization. The complete solutions are found without reference to any theory of the special functions involved, or to the classical methods for the solution of differential equations.

1. Factorization ladders and the reducibility condition. The Factorization Method replaces the second order equation

$$u'' + r(x, m)u + \lambda u = 0, \qquad a \leq x \leq b,\ m = 0, 1, 2, \ldots \tag{1.1}$$

and its associated boundary conditions, which we shall always take as

$$u(a) = 0, \quad u(b) = 0, \quad \int_a^b u^2 dx = 1, \tag{1.2}$$

by a system of two first order equations

$$\left\{k(x, m+1) - \frac{d}{dx}\right\} u_l^m(x) = A_l^{m+1} u_l^{m+1}(x) \tag{1.3a}$$

$$\left\{k(x, m+1) + \frac{d}{dx}\right\} u_l^{m+1}(x) = A_l^{m+1} u_l^m(x) \tag{1.3b}$$

where u_l^m, u_l^{m+1} are contiguous eigenfunctions belonging to the eigenvalue $\lambda_l = L(l+1)$, $l = 0, 1, 2 \ldots$;

$$A_l^{m+1} = [L(l+1) - L(m+1)]^{\frac{1}{2}} \tag{1.4}$$

is a constant which vanishes when $m = l$, and so provides a convenient starting point. We will show that, given a normalized starting point, we can use the recurrence formulae (1, 3) to generate other eigenfunctions, a certain number of which will also be normalized.

THEOREM 1. Suppose that corresponding to $m = l$ or $l + 1$ there exists a normalized eigenfunction

$$Ce^{\pm \int_{x_0}^{x} k(\nu, l+1) d\nu}. \tag{1.5}$$

Then for neighbouring integral values of m there exist normalized eigenfunctions given by (1.3), provided that the corresponding constants A_l^m are real and different from zero.

The proof falls naturally into three stages. First we note that the starting solution vanishes at the endpoints. Suppose that its zero of lesser order is of order $p \geqslant 1$. Now the manufacturing process involves operations of the type $k(x, m) \pm \dfrac{d}{dx}$ which can reduce the order of the zero at an endpoint by one at most, provided that $k(x, m)$ has no singularities more severe than the first order. We shall later verify that this is true for all the functions $k(x, m)$ which are rational in m. Then, if $|m - l| \leqslant p$, u_l^m also vanishes at the endpoints. If p is infinite, the ladder of eigenfunctions vanishing at the endpoints may be indefinitely long.

We see next that on the ladder so formed the normalization is preserved. Using (1.3) we have

$$A_l^{m+1}\int_a^b (u_l^{m+1})^2 dx = \int_a^b u_l^{m+1}\left[k(x, m+1) - \frac{d}{dx}\right] u_l^m dx$$

$$(1.6) \qquad = \int_a^b u_l^m \left[k(x, m+1) + \frac{d}{dx}\right] u_l^{m+1} dx = A_l^{m+1}\int_a^b (u_l^m)^2 dx$$

provided that the product $u_l^m u_l^{m+1}$ vanishes at a and b. The result follows when the non-zero factor A_l^m is cancelled.

With this result it is easy to show that the ladder cannot exist if $L(l+1) - L(m) \leqslant 0$ for any value of m on the ladder. The A_l^m are then imaginary, so that if u_l^m is real, u_l^{m+1} must be pure imaginary. Unless both are identically zero, the normalization condition above would equate a positive and a negative number.

The two Corollaries which follow cover most of the particular cases we shall meet.

COROLLARY 1. If $L(m)$ is an increasing function of m for $m \geqslant 0$, then the permitted values of l and m are

$$l = 0, 1, 2, 3 \ldots$$
$$m = 0, 1 \ldots l.$$

We obtain a finite ladder of eigenfunctions $u_l^0, u_l^1 \ldots u_l^l$, which will be called an Infeld ladder [2]. At the top of each ladder we have the starting point

$$(1.7) \qquad u_l^l = Ce^{\int_{x_0}^x k(\nu, l+1) d\nu}.$$

COROLLARY 2. If $L(m)$ is a decreasing function of m for $m \geqslant 0$ then the permitted values for l and m are

$$l = 0, 1, 2 \ldots$$
$$m = l, l+1, l+2 \ldots.$$

An infinite ladder is obtained for each value of l. As Schrödinger used infinite ladders in the solution of the Kepler problem we shall refer to these as Schrödinger ladders. In this case

$$(1.8) \qquad u_l^l = Ce^{-\int_{x_0}^x k(\nu, l) d\nu}.$$

The solutions of (1.1) satisfying (1.2) are not necessarily the only solutions of physical interest. If the discrete spectrum function $L(m)$ has a maximum value, there may be a continuous spectrum for values of λ exceeding this value. A proof of the uniqueness of the discrete solutions here discussed can be given for the particular cases, but it involves an appeal to the indicial equation of Frobenius.

We now derive the condition that the recurrence formulae be reducible to the F-equation. Both (1.3a) and (1.3b) are of the more general form

$$(1.9) \qquad \frac{d}{dx}f(x, m) = K(x, m)f(x, m) + A(m)f(x, m+1)$$

where $A(m)$ does not depend upon x. This equation is easily transformed into

$$\frac{d}{dx}\left[e^{-\int_{x_0}^{x} K(\nu,m)d\nu} f(x, m)\right] = A(m)e^{\int_{x_0}^{x}\Delta K(\nu,m)d\nu}\left[e^{-\int_{x_0}^{x} K(\nu,m+1)d\nu} f(x, m+1)\right],$$

where $\Delta K(\nu, m) = K(\nu, m+1) - K(\nu, m)$.

Suppose now that the coefficient $A(m)e^{\int_{x_0}^{x}\Delta K(\nu, m)d\nu}$ is the product of a function of x and a function of m; its two factors can be absorbed by redefining the dependent and independent variables of (1.9). Then we have

$$\log A(m) + \int_{x_0}^{x} \Delta K(\nu, m)d\nu = M(m) + \int K(x)dx.$$

By differentiation, $\Delta K(\nu, m) = K(x)$ must be independent of m. Summing this difference equation, we have

$$K(x, m) = mK(x) + K_0(x), \tag{1.10}$$

where $K(x)$ and $K_0(x)$ are arbitrary functions. We shall now show that this condition is necessary and sufficient that the recurrence formula (1.9) be reducible to the F-equation. For, when (1.10) is satisfied, and only then, may we define by the formula

$$z = \int_{x_1}^{x} e^{\int_{x_0}^{x}\Delta K(\nu, m)d\nu} dx \tag{1.11}$$

a quantity z independent of m. If also

$$F(z, m) = \left(\prod_{r=m_0}^{m-1} A(r)\right)^{-1} e^{-\int_{x_0}^{x} K(\nu, m)d\nu} f(x, m), \tag{1.12}$$

the equation (1.9) becomes

$$\frac{d}{dz} F(z, m) = F(z, m+1), \tag{1.13}$$

which is the F-equation. We have proved

THEOREM 2. The reducibility condition for the Factorization Method is that $k(x, m)$ be linear in m.

The formula we shall actually use is, of course, not (1.13), but its n-fold iterate.

THEOREM 3. Any solution $F(z, m)$ of (1.13) satisfies

$$\frac{d^n}{dz^n} F(z, m) = F(z, m+n) \qquad (n = 0, 1, 2 \ldots). \tag{1.14}$$

This quite trivial formula enables us to exhibit finite expressions for a large class of eigenfunctions.

We turn now to the analysis by which the factorizable second order equations can be enumerated. Let us begin with the two adjoint linear differential recursion formulae (1.3), which are sufficient to characterize the eigenvalue problem completely. Eliminating either u_l^m or u_l^{m+1}, we are led to the second order differential equation in one of the two forms:

$$(1.15a)\quad \left\{k(x, m+1)+\frac{d}{dx}\right\}\left\{k(x, m+1)-\frac{d}{dx}\right\} u_l^m = \left\{L(l+1)-L(m+1)\right\} u_l^m$$

$$(1.15b)\quad \left\{k(x, m) - \frac{d}{dx}\right\}\left\{k(x, m) + \frac{d}{dx}\right\} u_l^m = \left\{L(l+1)- L(m)\right\} u_l^m .$$

These two equations are the same, and we find in consequence the differential difference equation

$$(1.16)\quad k^2(x, m+1) - k^2(x, m) + k'(x, m+1) + k'(x, m) = L(m) - L(m+1).$$

We shall take the view that the entire Factorization Theory is implicitly contained in this fundamental equation. To a given function $k(x, m)$ corresponds a factorizable Sturm-Liouville equation if the expression formed with $k(x, m)$ according to (1.16) is independent of x. A discrete spectrum function $L(m)$ is thus determined by the same equation. The constants A_l^m appearing in the recurrence formulae are then given by (1.4).

The equation (1.16) was derived by Stevenson and Infeld [3], who have tabulated a number of solutions $k(x, m)$ with their corresponding $L(m)$. They have shown that there are no polynomial solutions in powers of m or m^{-1} of degree higher than the first. A solution such as $(-1)^m f(x)$ leads to $L(m) = 0$, and is of no further interest here. If $k(x, m)$ and $L(m)$ constitute a solution of (1.16), it is easily verified that

$$(1.17)\quad \begin{array}{ll} -k(-x, m),\ L(m) & k(-x, -m),\ L(-m) \\ -k(x, -m),\ L(-m) & -ik(ix, m),\ -L(m) \end{array}$$

also constitute solutions. It is now sufficient to consider only positive values of m.

To find the solutions of (1.16), let us first assume with Stevenson and Infeld [3] that

$$(1.18)\quad k(x, m) = k_0(x) + mk_1(x).$$

Substitution of this form into (1.16), combined with the requirement that all coefficients of m in the resulting expression be constant, leads to ordinary differential equations for $k(x)$ and $K(x)$. These are

$$(1.19a)\quad k_1^2 + k_1' = -\alpha^2$$

$$(1.19b)\quad k_0 k_1 + k_0' = -\gamma\alpha^2$$

where α^2 and $\gamma\alpha^2$ are independent constants. Direct solution of these equations leads to the functions[1] (a, b, c, d). Employing the fourth transformation in (1.17) above, we have also (e) and (f) as real solutions of (1.16). These six functions will lead to reducible recurrence formulae.

Another trial solution for $k(x, m)$ is

$$(1.20)\quad k(x, m) = k_0(x) + mk_1(x) + m^{-1}k_{-1}(x).$$

[1]See Table I.

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TABLE I

No.	$k(x, m)$	$L(m)$	Range (x)	Ladder Infeld—Class I Schröd.—Class II	Reducibility
a	$a(m+\gamma)\cot ax + \dfrac{C}{\sin ax}$	$a^2(m+\gamma)^2$	$\left(0, \dfrac{\pi}{a}\right)$	I	Reducible
b	$\dfrac{m+\gamma}{x} + \dfrac{bx}{2}$	$-\ 2b(m+\gamma)$	$(0, \infty)$	I $b<0$ II $b>0$	Reducible
c	ax	$-2am$	$(-\infty, \infty)$	I $a<0$ II $a>0$	Reducible
d	$Ce^{\beta x} - \beta(m+\gamma)$	$-\ \beta^2(m+\gamma)^2$	$(-\infty, \infty)$	II	Reducible
e	$\beta(m+\gamma)\coth \beta x + \dfrac{C}{\sinh \beta x}$	$-\ \beta^2(m+\gamma)^2$	$(-\infty, 0)$	II	Reducible
f	$\beta(m+\gamma)\tanh \beta x + \dfrac{C}{\cosh \beta x}$	$-\ \beta^2(m+\gamma)^2$	$(-\infty, \infty)$	II	Reducible
g	$\dfrac{m+\gamma}{x} - \dfrac{\nu}{m+\gamma}$	$-\ \dfrac{\nu^2}{(m+\gamma)^2}$	$(0, \infty)$	I	d — Irreducible but can be transformed to
h	$a(m+\gamma)\cot ax - \dfrac{\nu}{m+\gamma}$	$a^2(m+\gamma)^2 - \dfrac{\nu^2}{(m+\gamma)^2}$	$\left(0, \dfrac{\pi}{a}\right)$	I	f — Irreducible but can be transformed to
i	$a(m+\gamma)\coth ax - \dfrac{\nu}{m+\gamma}$	$-a^2(m+\gamma)^2 - \dfrac{\nu^2}{(m+\gamma)^2}$	$(0, \infty)$	I	e — Irreducible but can be transformed to
j	$a(m+\gamma)\tanh dx - \dfrac{\nu}{m+\gamma}$	$-a^2(m+\gamma)^2 - \dfrac{\nu^2}{(m+\gamma)^2}$	$(-\infty, \infty)$	II	a — Irreducible but can be transformed to

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By the same procedure it is found that $k_0(x)=0$, and that $k_{-1}(x)$ is a constant; $k_1(x)$ must again satisfy the equation (1.19a). In this way (*g*) and (*h*) are determined, and (*i*), (*j*) follow from another application of (1.17). These are examples of irreducible recursion formulae. Clearly the differential equation which determines $k_1(x)$ (or $k_0(x)$) permits only first order singularities; a fact which we used in the proof of Theorem 1. We tabulate these functions, their associated spectrum functions $L(m)$, and the fundamental ranges appropriate to each.

2. Eigenfunctions on an Infeld ladder. We shall now consider the case $L(m)$ an increasing function of m, which by Corollary 1 leads to finite ladders. If $k(x, m)$ is linear in m, we have two reducible recursion formulae, and should expect to find two closed formulae for u_l^m. The two necessary starting points are easily found, but, as we shall see, one of them is sufficient to fix only the form of the eigenfunctions as transforms of certain polynomials. We shall treat first (1.3a), then (1.3b), defining functions $F_l(z, m)$ and $G_l(z, m)$ as the corresponding solutions of the F-equation. The details of the two reductions are different, so that it is advisable to set both down explicitly.

The recurrence relation (1.3a)

$$A_l^{m+1}u_l^{m+1} = \left(k(x, m+1) - \frac{d}{dx}\right) u_l^m$$

is equivalent to

$$\frac{d}{dx}\left[e^{-\int_{x_0}^{x} k(\nu, m+1)d\nu} u_l^m(x)\right] = -A_l^{m+1} e^{\int_{x_0}^{x}\Delta k d\nu}\left[e^{-\int_{x_0}^{x} k(\nu, m+2)d\nu} u_l^{m+1}(x)\right].$$

According to (1.11) we define

$$z = \int_{x_1}^{x} e^{\int_{x_0}^{x}\Delta k d\nu}\, dx \tag{2.1}$$

and

$$F_l(z, m) = (-1)^m \left(\prod_{r=m+1}^{l} A_l^r\right)^{-1} e^{-\int_{x_0}^{x} k(\nu, m+1)d\nu} u_l^m(x). \tag{2.2}$$

It is then found that

$$\frac{dF_l(z, m)}{dz} = F_l(z, m+1) \tag{2.3}$$

which is the F-equation in the variables z and m.

Because A_l^{l+1} is zero, $F_l(z, l)$ has a zero derivative, and is a constant. $F_l(z, l-1)$ is a linear function of z, and in general $F_l(z, m)$ is a polynomial of degree $l - m$ in z. The bottom of the ladder ($m = 0$), is reached with $F_l(z, 0)$ a polynomial of degree l containing $l + 1$ unspecified constants of integration. The complete knowledge of $F_l(z, 0)$ would enable us to determine the $F_l(z, m)$ by differentiation.

The ladder $u_l^l \ldots u_l^0$ is actually constructed with the formula (1.3b)

$$A_l^{m+1} u_l^m = \left(k(x, m+1) + \frac{d}{dx}\right) u_l^{m+1},$$

which can be written

$$\frac{d}{dx}\left[e^{\int_{x_0}^x k(\nu, m)d\nu} u_l^m(x)\right] = A_l^m e^{\int_{x_0}^x \Delta k d\nu}\left[e^{\int_{x_0}^x k(\nu, m-1)d\nu} u_l^{m-1}(x)\right].$$

As in (2.1), define
$$z = \int_{x_1}^x e^{\int_{x_0}^x \Delta k d\nu} dx,$$

and

(2.4) $$G_l(z, m) = \left(\prod_{r=0}^m A_l^r\right)^{-1} e^{\int_{x_0}^x k(\nu, m)d\nu} u_l^m(x).$$

The recurrence formula is then

(2.5) $$\frac{d}{dz} G_l(z, m) = G_l(z, m-1);$$

the F-equation in the variables z and $-m$.

Note that the new independent variable z is the same in both cases. In practice, $k(x, m)$ has no singularities except at the ends of the fundamental interval, so that Δk is continuous in the open interval; z is then a monotone strictly increasing function of x in the same interval. The arbitrary constants x_0 and x_1 are to be chosen so that z will appear free of unnecessary constants.

From (1.7) it is known that

$$u_l^l = Ce^{\int_{x_0}^x k(\nu, l+1)d\nu}$$

which implies

(2.6) $$G_l(z, l) = C_l\left(\prod_{r=0}^l A_l^r\right)^{-1} e^{\int_{x_0}^x k(\nu, l) + k(\nu, l+1)d\nu}$$
$$= C_l\left(\prod_{r=0}^l A_l^r\right)^{-1} g_l(z)$$

where

(2.7) $$g_l(z) = \exp\left[\int_{x_0}^x k(\nu, l) + k(\nu, l+1)d\nu\right]$$

is chosen to take the value 1 where $x = x_0$. With this definition we have the normalization

(2.8) $$C_l = \left[\int_{z_0}^{z_1} g_l(z)dz\right]^{-\frac{1}{2}}$$

where (z_0, z_1) is the fundamental interval for z.

The functions $G_l(z, m)$ are the successive derivatives of $G_l(z, l)$, and can be calculated once $g_l(z)$ is known as a function of z. In practice it is easy to find the expression for $g_l(z)$, which we shall call the generating function of the lth ladder.

It is now possible to compare the expressions (2.2) and (2.4) for $u_l^m(x)$, and to find the explicit form for $F_l(z, 0)$. Thus

$$
\begin{aligned}
F_l(z,0) &= \left(\prod_{r=1}^{l} A_l^r\right)^{-1} e^{-\int_{x_0}^{x} k(\nu,1)d\nu} u_l^0(x) \\
&= \left(\prod_{r=l}^{l} A_l^r\right)^{-1} e^{-\int_{x_0}^{x} k(\nu,1)+k(\nu,0)d\nu} G_l(z,0) \qquad (2.9)\\
&= C_l A_l^0 \left(\prod_{r=0}^{l} A_l^r\right)^{-2} \frac{1}{g_0(z)} \frac{d^l}{dz^l} g_l(z).
\end{aligned}
$$

Then we have

$$(2.10) \qquad F_l(z,m) = C_l A_l^0 \left(\prod_{r=0}^{l} A_l^r\right)^{-2} \frac{d^m}{dz^m} \frac{1}{g_0(z)} \frac{d^l}{dz^l} g_l(z).$$

The closed formulae for $u_l^m(x)$ which we seek are:
from (2.2)

$$(2.11) \qquad u_l^m(x) = (-1)^m \left(\prod_{r=m+1}^{l} A_l^r\right) e^{\int_{x_0}^{x} k(\nu,m+1)d\nu} F_l(z,m)$$

and from (2.4)

$$(2.12) \qquad u_l^m(x) = \left(\prod_{r=0}^{m} A_l^r\right) e^{-\int_{x_0}^{x} k(\nu,m)d\nu} G_l(z,m).$$

The latter can be expressed entirely in terms of the generating function $g_l(z)$ if we make use of (2.6) and (2.7). The final result is, then:

$$(2.13) \qquad u_l^m(x) = C_l \left(\prod_{r=m+1}^{l} A_l^r\right)^{-1} \frac{1}{\sqrt{g_{m-\frac{1}{2}}(z)}} \frac{d^{l-m}}{dz^{l-m}} g_l(z)$$

where

$$g_l(z) = \exp\left[\int_{x_0}^{x} k(\nu,l+1) + k(\nu,l)d\nu\right];$$

$$A_l^m = \left[L(l+1) - L(m)\right]^{\frac{1}{2}}; \text{ and } C_l = \left[\int_{z_0}^{z_1} g_l(z)\, dz\right]^{-\frac{1}{2}}.$$

The normalization of (2.13) over x is equivalent to the normalization over z of the orthogonal polynomials

$$(2.14) \qquad \frac{1}{g_m(z)} \frac{d^{l-m}}{dz^{l-m}} g_l(z) \qquad (l = m, m+1 \ldots)$$

with weight function $g_m(z)$.

According to our programme, we have mechanized the construction of the eigenfunctions in two different ways. However, one of these is illusory, because there is no independent starting point for it. The equivalence of the two formulae for u_l^m implies that

$$(2.15) \qquad F_l(z,m) g_m(z) / G_l(z,m)$$

is a constant, independent of z. This is easily verified by induction on m.

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We can now apply these results to the k-functions of reducible Infeld type. Taking first

$$k(x, m) = a(m+\gamma) \cot ax + \frac{C}{\sin ax} \qquad 0 \leq x \leq \frac{\pi}{a} \tag{2.16a}$$

we find from (1.16), (2.1), (2.6) and (2.7)

$$\begin{aligned} L(m) &= a^2(m+\gamma)^2 \\ r(x, m) &= -\frac{a^2(m+\gamma+1)(m+\gamma) - C^2 - 2Ca(m+\gamma+\frac{1}{2}) \cos ax}{\sin^2 ax} \\ az &= -\cos ax \qquad\qquad -1 \leq az \leq 1 \\ g_l(z) &= (1-az)^{l+\gamma+\frac{1}{2}-\frac{C}{a}}(1+az)^{l+\gamma+\frac{1}{2}+\frac{C}{a}} \\ F_l(z, 0) &= \text{const.}\, P_l(az)^{\left(l+\gamma+\frac{1}{2}-\frac{C}{a},\, l+\gamma+\frac{1}{2}+\frac{C}{a}\right)} \end{aligned} \tag{2.17}$$

where $P_l^{\beta,\gamma}(az)$ denotes the Jacobian polynomial of degree l [4]. When the necessary condition $\left|\frac{C}{a}\right| < l + \gamma + \frac{3}{2}$ is satisfied, the normalized eigenfunctions are given by

$$\begin{aligned} u_l^m(x) = \frac{a^{l-m+\frac{1}{2}}}{2^{l+\gamma+\frac{1}{2}}} &\left[\frac{(l+\gamma+1)\,\Gamma(l+m+2\gamma+2)}{\Gamma\left(l+\gamma+\frac{C}{a}+\frac{3}{2}\right)\Gamma\left(l+\gamma-\frac{C}{a}+\frac{3}{2}\right)\Gamma(l-m+1)}\right]^{\frac{1}{2}} \\ &+ (\sin ax)^{m+\gamma+1}\left(\tan\frac{ax}{2}\right)^{\frac{C}{a}} P_{l-m}^{\left(m+\gamma+\frac{1}{2}+\frac{C}{a},\, m+\gamma+\frac{1}{2}-\frac{C}{a}\right)}(\cos ax). \end{aligned} \tag{2.18}$$

Several particular cases of this formula can be noted. To obtain the spherical harmonics, set $C = 0$, $a = 1$, $\gamma = 0$. The cases $C = \pm\frac{1}{2}$, $a = 1$, $\gamma = -\frac{1}{2}$ are closely related to Weyl's spherical harmonics with spin, and are in fact the radial and angular eigenfunctions of the Dirac electron equations in a spherical universe. Similarly the cases $C = \pm\frac{1}{2}$, $a = 1$, $\gamma = 0$ lead to the angular wave functions of an electron in a magnetic dipole field.

The second type has a semi-infinite range for x.

$$k(x, m) = \frac{m+\gamma}{x} + \frac{bx}{2} \qquad (0 \leq x < \infty) \quad b < 0. \tag{2.19}$$

The formulae

$$\begin{aligned} r(x, m) &= -\frac{(m+\gamma)(m+\gamma+1)}{x^2} - \frac{b^2x^2}{4} + b(m+\gamma+\tfrac{1}{2}), \\ L(m) &= -2b(m+\gamma), \\ 2z &= x^2 \qquad\qquad 0 \leq z < \infty, \\ g_l(z) &= z^{l+\gamma+\frac{1}{2}} e^{bz}, \\ F_l(z, 0) &= \text{const.}\, L_l^{(\gamma+\frac{1}{2})}(-bz) \qquad \text{(cf. [4])}, \end{aligned} \tag{2.20}$$

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follow from the definitions. For the normalized eigenfunctions we have

$$u_l^m(x) = \frac{1}{\Gamma(l-m+1)^{\frac{1}{2}}\,\Gamma(l+\gamma+\frac{3}{2})^{\frac{1}{2}}}\left(\frac{x^2}{2}\right)^{\frac{m+\gamma+1}{2}} e^{\frac{bx^2}{4}}\,L_{l-m}{}^{(m+\gamma+\frac{1}{2})}\left(-\frac{bx^2}{2}\right). \tag{2.21}$$

Bessel's Equation is a limiting case of each of these types, for in this case

$$k(x, m) = \frac{m-\frac{1}{2}}{x} \tag{2.22}$$

$$L(m) = 0.$$

In (2.16) set $c = 0$, $\gamma = -\frac{1}{2}$ and let a tend to zero. Similarly, set $b = 0$, $\gamma = -\frac{1}{2}$ in (2.19). The ladders break down because $L(m)$ does not vary with m. However, the Bessel equation can still be factored into a pair of adjoint recurrence formulae. In fact,

$$\left\{\frac{m+\frac{1}{2}}{x} - \frac{d}{dx}\right\} Z_m = Z_{m+1}, \tag{2.23}$$

and

$$\left\{\frac{m-\frac{1}{2}}{x} + \frac{d}{dx}\right\} Z_m = Z_{m-1}$$

differ only by the sign of m.

The third type is comparatively simple, in so far as $k(x, m)$ is independent of m. Choosing the origin so as to obtain the simplest formulas, we have

$$k(x, m) = -x \qquad -\infty \leq x \leq \infty. \tag{2.24}$$

Then

$$\begin{aligned} r(x, m) &= -2m - x^2 + 1, \\ L(m) &= 2m, \\ z &= x, \\ g_l(z) &= e^{-z^2}, \\ F_l(z, 0) &= \text{const. } H_l(z), \qquad \text{(cf.[4]).} \end{aligned} \tag{2.25}$$

The normalized eigenfunctions are

$$u_l^m(x) = \pi^{-\frac{1}{4}}\, 2^{-\frac{l-m}{2}}\, \Gamma(l-m+1)^{-\frac{1}{2}}\, e^{-\frac{x^2}{2}} H_{l-m}(x). \tag{2.26}$$

A factorization of this type leads to the solution of the problem of the harmonic oscillator.

In this section we have considered finite ladder problems on finite, semi-finite and infinite ranges. From the three examples it is seen how the various classical orthogonal polynomials emerge in a very natural way from the analysis when $k(x, m)$ is linear in m.

3. Eigenfunctions on a Schrödinger Ladder. We turn now to the solutions $k(x, m)$ of (1.16) corresponding to $L(m)$ a decreasing function of m. The theory of the last section needs some modification, for the roles of the two recurrence formulae are now interchanged. The formulae (2.1) to (2.5) with $(l+1)$ re-

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placed by l carry over except for a constant factor in (2.2) and (2.4). For a starting point we have now, according to (1.8),

$$u_l^l = \bar{C}e^{-\int_{x_0}^{x} k(\nu, l)d\nu}$$

which implies

$$F_l(z, l) = (-1)^l C_l e^{-\int_{x_0}^{x} k(\nu, l+1) + k(\nu, l)d\nu}$$

$$= (-1)^l C_l g_l^{-1}(z). \tag{3.1}$$

Thus the generating function of the infinite ladder for the eigenvalue $L(l)$ has the form of the reciprocal of the generating function of the finite ladder with eigenvalue $L(l+1)$. The normalization integral is

$$\bar{C}_l = \left[\int_{z_0}^{z_1} \frac{dz}{g_l(z)}\right]^{-\frac{1}{2}}. \tag{3.2}$$

The analogue of the final formula (2.13) is now to be found with (1.3a) and it turns out to be

$$u_l^m(x) = \bar{C}_l \left(\prod_{r=l+1}^{m} A_l^r\right)^{-1} (-1)^m \sqrt{g_{m+\frac{1}{2}}(z)}\, \frac{d^{m-l}}{dz^{m-l}}\left(\frac{1}{g_l(z)}\right). \tag{3.3}$$

The normalization of the u's again corresponds to the z normalization of the orthogonal functions

$$g_m(z)\, \frac{d^{m-l}}{dz^{m-l}} \frac{1}{g_l(z)} \qquad (l = 0, 1 \ldots m), \tag{3.4}$$

and weight function $\dfrac{1}{g_m(z)}$.

We shall apply these formulae to the linear k-functions for which $L(m)$ is a decreasing function of m. The first of these is the exponential solution

$$k(x, m) = Ce^{\beta x} - \beta(m + \gamma) \qquad -\infty \leq x < \infty. \tag{3.5}$$

With

$$L(m) = -\beta^2(m + \gamma)^2$$

$$r(x, m) = 2\beta C(m + \gamma + \tfrac{1}{2})\, e^{\beta x} - C^2 e^{2\beta x},$$

$$-\beta z = e^{-\beta x},$$

$$g_l(z) = (-\beta z)^{2(l+\gamma+\frac{1}{2})} e^{-\frac{2C}{\beta^2 z}}, \tag{3.6}$$

the eigenfunctions are given according[2] to (3.3) by

$$u_l^m(x) = \text{const.}\, e^{(l+\gamma)\beta x} e^{-\frac{C}{\beta} e^{\beta x}} L_{m-l}^{2(l+\gamma)}\left(\frac{2l}{\beta}\, e^{\beta x}\right). \tag{3.7}$$

We shall later apply this formula in connection with the Euclidean Kepler Problem.

The first of the hyperbolic solutions is

$$k(x, m) = \beta(m + \gamma) \coth \beta x + \frac{C}{\sinh \beta x} \qquad -\infty < x \leq 0, \tag{3.8}$$

[2]See Appendix.

which leads to

$$L(m) = -\beta^2(m+\gamma)^2$$

$$r(x,m) = -\frac{\beta^2(m+\gamma)(m+\gamma+1)+C^2+2\beta C(m+\gamma+\frac{1}{2})\cosh\beta x}{\sinh^2\beta x}$$

(3.9) $$\beta z = \cosh\beta x \qquad 1 \leq \beta z < \infty$$

$$g_l(z) = (\beta z - 1)^{l+\gamma+\frac{1}{2}-\frac{C}{\beta}}(\beta z+1)^{l+\gamma+\frac{1}{2}+\frac{C}{\beta}}.$$

The eigenfunctions are

(3.10) $$u_l^m(x) = \text{const.}\ (\sinh\beta x)^{-m-\gamma}\left(\tanh\frac{\beta x}{2}\right)^{-\frac{C}{\beta}}\cdot$$

$$P_{m-l}^{\left(-m-\gamma-\frac{1}{2}+\frac{C}{\beta},\,-m-\gamma-\frac{1}{2}-\frac{C}{\beta}\right)}(\cosh\beta x).$$

For the normalization, $|m+\gamma| < \dfrac{C}{\beta}$ is necessary. Thus the normalized solutions end at the same "height" on each ladder.

The second hyperbolic solution is

(3.11) $$k(x,m) = \beta(m+\gamma)\tanh\beta x + \frac{C}{\cosh\beta x} \qquad -\infty < x < \infty,$$

$$L(m) = -\beta^2(m+\gamma)^2.$$

We have

$$r(x,m) = -\frac{\beta^2(m+\gamma)(m+\gamma+1)+C^2+2\beta C(m+\gamma+\frac{1}{2})\sinh\beta x}{\cosh^2\beta x},$$

(3.12) $$\beta z = \sinh\beta x,$$

$$g_l(z) = (1+i\beta z)^{l+\gamma+\frac{1}{2}+\frac{iC}{\beta}}(1+i\beta z)^{l+\gamma+\frac{1}{2}-\frac{iC}{\beta}}.$$

The eigenfunctions are

(3.13) $$u_l^m(x) = \text{const.}\ (\cosh\beta x)^{-m-\gamma}\tanh\left(\frac{\beta x}{2}+\frac{i\pi}{4}\right)^{\frac{iC}{\beta}}\cdot$$

$$P_{m-l}^{\left(-m-\gamma-\frac{1}{2}\pm\frac{iC}{\beta}\right)}(i\sinh\beta x).$$

This last expression has only real values.

Lastly we observe that the two functions

(3.14) $$k(x,m) = \frac{m+\gamma}{x}+\frac{bx}{2}, \quad b>0;$$

(3.15) $$= +x$$

also lead to infinite ladders. The formulae are not essentially different from the corresponding results in the last section; so we shall not include them here.

4. Eigenfunctions for the irreducible types. We turn now to the non-linear functions of m listed in section one. The first three of them occur in the radial equations of the Kepler Problem in the Euclidean, spherical, and hyperbolic spaces, respectively. For these solutions the reducibility condition of Theorem 2 is not satisfied, so that we cannot apply the method of the foregoing sections, which consists essentially of reducing to the F-equation the difference equations resulting from the factorization. However it is possible to transform the second order differential equations and to modify them in an artificial way, so that they can be factorized into reducible recurrence formulae of the types we have already treated.

The problem in the Euclidean space is the simplest of the four, and we shall treat it in some detail to illustrate the method employed. We have to solve the equation for the densities f,

$$f'' + \frac{2\nu}{x} f - \frac{(m+\gamma)(m+\gamma+1)}{x^2} f - \frac{\nu^2}{(l+\gamma)^2} f = 0. \tag{4.1}$$

The factorization of (4.1) leads to Table I, g and the method based on the linearity of $k(x, m)$ fails. Perform instead the transformation

$$f = x^{\frac{1}{2}}u; \; x = e^z$$

and obtain the equation

$$u'' + \left[2\nu e^z - \frac{\nu^2 e^{2z}}{(l+\gamma)^2} - \frac{1}{4} \right] u - (m+\gamma)(m+\gamma+1)\, u = 0. \tag{4.2}$$

This equation is quite similar to the equation of the exponential type in the preceding section. Let us introduce a new, artificial paramater "n" in such a way as to make the two equivalent. This procedure was suggested by Hull [5], who pointed out its analogy with Schrödinger's original factorization of this equation.

We find that the equation

$$u_l'' + \left[2\nu \frac{n+\gamma}{l+\gamma} e^z - \frac{\nu^2 e^{2z}}{(l+\gamma)^2} \right] u_l - (m+\gamma+\tfrac{1}{2})^2 u_l = 0 \tag{4.3}$$

is an equation of the type (3.6), leading to recurrence relations

$$\left\{ \frac{\nu e^z}{l+\gamma} - (n+\gamma+\tfrac{1}{2}) \pm \frac{d}{dz} \right\} \begin{matrix} u_{l,\,n+1}{}^m \\ u_{l,\,n}{}^m \end{matrix} = \left[(n+\gamma+\tfrac{1}{2})^2 - (m+\gamma+\tfrac{1}{2})^2 \right]^{\frac{1}{2}} \begin{matrix} u_{l,\,n}{}^m \\ u_{l,\,n+1}{}^m \end{matrix}. \tag{4.4}$$

For a given (l, m) we obtain an infinite ladder with $n = m, m+1, m+2, \ldots$ For $n = l$, (4.4) is (4.3), so $u_{l,\,l}{}^m = u_l{}^m$. To identify (4.4) formally with (3.6) let us make the substitution

(4.5)

Exponential type	x	β	C	m	l	γ
Kepler problem	z	1	$\dfrac{\nu}{l+\gamma}$	n	m	$\gamma - \frac{1}{2}$.

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We can now quote the densities as

$$f_l^m = x^{\frac{1}{2}} u_{l,\,l}^m = C x^{m+\gamma} e^{\frac{-\nu x}{l+\gamma}} L_{l-m}^{2m+2\gamma+1}\left(\frac{2\nu x}{l+\gamma}\right). \tag{4.6}$$

To find the normalization of these functions, we recall that the normalized densities are given by a succession of operations of the form (Table I, g) on a normalized starting function like (1.7). The coefficient of the lowest power of x in a series expansion about $x = 0$ is easily worked out for this function. Comparison with the lowest coefficient in (4.6) leads to the desired normalization. As the formulas are complicated we omit them.

The spherical space [6] with quadratic form

$$ds^2 = R^2(dx^2 + \sin^2 x(d\theta^2 + \sin^2\theta d\phi^2)), \qquad x = \frac{r}{R},$$

requires that we solve the radial equation

$$s'' + (2\nu \cot x)s - \frac{(m+\gamma)(m+\gamma+1)}{\sin^2 x} s + (\lambda + 1)s = 0. \tag{4.7}$$

The factorization of (4.7) directly yields (Table I, h). Again, let us transform by the substitution $s = \sin^{\frac{1}{2}} x\, u$; $\tan \dfrac{x}{2} = e^z$ and introduce an artificial parameter n in such a way that the radial equation is recovered for $n = l$. The result is the pair of recurrence formulae:

$$\left\{ \frac{\nu}{l+\gamma} \frac{1}{\cosh z} - (n + \gamma + \tfrac{1}{2}) \tanh z \pm \frac{d}{dz} \right\} \begin{matrix} u_{l,\,n+1}^m \\ u_{l,\,n}^m \end{matrix} = \left[(n+\gamma+\tfrac{1}{2})^2 - (m+\gamma+\tfrac{1}{2})^2 \right]^{\frac{1}{2}} \begin{matrix} u_{l,\,n}^m \\ u_{l,\,n+1}^m \end{matrix}. \tag{4.8}$$

Returning to the use of x as variable, we may write the second of these as

$$\frac{d}{d(\cot x)}\left[\sin^{n+\gamma+\frac{1}{2}} x \; e^{\frac{-\nu x}{l+\gamma}} u_{l,\,n}^m(x) \right] = \text{const.} \sin^{n+\gamma+\frac{3}{2}} x \; e^{\frac{-\nu x}{l+\gamma}} u_{l,\,n+1}^m(x);$$

and with the starting point

$$u_{l,\,m}^m(x) = C \sin^{m+\gamma+\frac{1}{2}} x \; e^{\frac{-\nu x}{l+\gamma}}$$

we find

$$s_l^m = (\sin x)^{\frac{1}{2}} u_{l,\,l}^m = C \sin^{-l-\gamma} x \; e^{\frac{\nu x}{l+\gamma}} \frac{d^{l-m}}{d(\cot x)^{l-m}} \left[\sin^{2(m+\gamma+1)} x \; e^{-\frac{2\nu x}{l+\gamma}} \right]$$

$$= C \sin^{l+\gamma} x \; e^{-\frac{\nu x}{l+\gamma}} P_{l-m}^{\left(-l-\gamma-\frac{i\nu}{l+\gamma};\; -l-\gamma+\frac{i\nu}{l+\gamma}\right)} (i \cot x). \tag{4.9}$$

The normalization can be effected in the same way. The formulas (4.8) are an example of (h). Allowing ν to become zero we of course recover the spherical harmonics.

The Kepler problem in the hyperbolic space can be similarly treated. We transform the usual radial equation [7] by the substitution

$$s = (\sinh x)^{\frac{1}{2}} u; \; e^z = \tanh \frac{x}{2}$$

and introduce n. The result is the pair of recurrence formulae of the form (i).

$$(4.10) \quad \left\{ \frac{\nu}{l+\gamma} \frac{1}{\sinh z} - (n + \gamma + \tfrac{1}{2}) \coth z \pm \frac{d}{dz} \right\} \begin{matrix} u_{l,\,n+1}{}^m \\ u_{l,\,n}{}^m \end{matrix} = \left[(n + \gamma + \tfrac{1}{2})^2 - (m + \gamma + \tfrac{1}{2})^2 \right]^{\frac{1}{2}} \begin{matrix} u_{l,\,n}{}^m \\ u_{l,\,n+1}{}^m \end{matrix}.$$

Using x as variable, we find the starting point

$$u_{l,\,m}{}^m(x) = C(\sinh x)^{m+\gamma+\frac{1}{2}} e^{\frac{-\nu x}{l+\gamma}}, \qquad 0 \leq x < \infty$$

and the densities

$$(4.11) \quad s_l{}^m = (\sinh x)^{\frac{1}{2}} u_{l,\,l}{}^m = C(\sinh x)^{l+\gamma} e^{-\frac{\nu x}{l+\gamma}} P_{l-m}{}^{\left(-l-\gamma-\frac{\nu}{l+\gamma},\, -l-\gamma+\frac{\nu}{l+\gamma}\right)}(\coth x).$$

The normalization can be achieved only when $(l + \gamma)^2 < \nu$, so that there is a finite number of discrete energy levels in the hyperbolic space. This is just what the behaviour of $L(m)$ given by (Table I, i) predicts. For $(m + \gamma)^2 < \nu$ this function increases, but at $(m + \gamma)^2 = \nu$ it has a maximum and decreases again. The ladders corresponding to larger values of $l + \gamma$ cannot get started because $L(m)$ is a decreasing function for the initial steps down the ladder.

Our last example is a similar equation with no physical analogue, which leads to normalized eigenfunctions on Schrödinger ladders when $(l + \gamma)^2 > \nu$. We can treat this type, given by (I, j), in the same artificial way, finding the formula on an infinite ladder by means of an artificial finite ladder. Let us perform the transformation

$$s = (\cosh x)^{\frac{1}{2}} u \, ; \; e^x = \tan \frac{z}{2}$$

and introduce n as before. We find two relations of the first type discussed, namely (I, a):

$$(4.12) \quad \left\{ \frac{\nu}{l+\gamma} \frac{1}{\sin z} + (n + \gamma + \tfrac{1}{2}) \cot z \pm \frac{d}{dz} \right\} \begin{matrix} u_{l,\,n+1}{}^m \\ u_{l,n}{}^m \end{matrix} = \left[(m + \gamma + \tfrac{1}{2})^2 - (n + \gamma + \tfrac{1}{2})^2 \right]^{\frac{1}{2}} \begin{matrix} u_{l,\,n}{}^m \\ u_{l,\,n+1}{}^m \end{matrix}.$$

The starting point is

$$u_{l,\,m+1}{}^m(x) = C\,(\cosh x)^{-(m+\gamma+\frac{1}{2})} e^{\frac{-\nu x}{l+\gamma}}$$

corresponding to a "density" $s_l{}^m$ which is square integrable over $(-\infty, \infty)$ if $(l + \gamma)^2 > \nu$. The eigenfunctions can be found as functions of z with the aid of (2.18).

This last example serves to complete the pattern of the artificial correspondence of the finite and infinite ladders with the linear and non-linear k-functions. The changes of variable were all such as simply to interchange the roles of the "quantum numbers" l and m in the differential equations. The allowed values "all m less than l" become "all n greater than m," where n takes the place of l. Thus a finite ladder scheme should transform into an infinite ladder scheme, and vice versa.

We can summarize the work as follows. First we found conditions which ensure the existence of ladders of normalized eigenfunctions, and showed that two distinct classes of ladder are possible. Truesdell's reducibility condition was then enunciated. The factorizations were obtained by a method due to Stevenson and Infeld. The problems of Infeld's class led to finite ladders, while those of the class due to Schrödinger involve infinite ladders in general. We applied Truesdell's condition to the recurrence equations so found. It turned out that there are reducible and irreducible cases belonging to each class: four possibilities altogether. The next step was to work out formulae for the normalized eigenfunctions belonging to the reducible problems, and this was done for the Infeld ladders in section **2**. The reducible Schrödinger problems were treated in section **3**. Finally, in the last section we used a device which transforms each of the irreducible types into one of the known reducible problems belonging to the opposite class. An interesting artificial correspondence between the Infeld and Schrödinger ladders was thus established. In all cases treated we were able to find the normalized eigenfunctions explicitly.

Much of the elegance of the Factorization Method is due to the quite elementary character of the methods employed. We needed in addition one simple F-equation formula, and found results which are applicable to the fundamental pure problems in Quantum Mechanics.

I wish to thank Professor Infeld for his advice and encouragement throughout. Professor Truesdell very kindly read the manuscript, and his helpful criticisms are much appreciated.

Appendix

The formula used for the Laguerre function in (3.7) involves the evaluation of an nth derivative of the form

$$D_n{}^\kappa = \frac{d^n}{dv^n}\left[v^{-\kappa} e^{\frac{1}{v}}\right].$$

To evaluate $D_n{}^\kappa$ consider the product $v\,.\,v^{-\kappa-1} e^{\frac{1}{v}}$ and differentiate it n times by Leibnitz' Rule. The result is

$$D_n{}^\kappa = v\frac{d}{dv}D_{n-1}{}^{\kappa+1} + nD_{n-1}{}^{\kappa+1}.$$

Then

$$\frac{d}{dv}(v^n D_{n-1}{}^{\kappa+1}) = v^{n-1}\left[v\frac{d}{dv}D_{n-1}{}^{\kappa+1} + nD_{n-1}{}^{\kappa+1}\right] = v^{-2}v^{n+1}D_n{}^\kappa$$

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whence

$$-\frac{d}{d\left(\frac{1}{v}\right)}(v^n D_{n-1}{}^{\kappa+1}) = v^{n+1} D_n{}^{\kappa}.$$

Setting $v = -\dfrac{1}{y}$ this is a statement that

$$(-y)^{-\alpha-1} D_\alpha^{\kappa-\alpha}\left(-\frac{1}{y}\right)$$

is a solution of the F-equation in the variables (y, α).
Then

$$\frac{d^n}{dy^n}\left(-\frac{1}{y} D_0{}^{\kappa+n}\left(-\frac{1}{y}\right)\right) = \frac{d^n}{dy^n}\left((-y)^{\kappa+n-1} e^{-y}\right)$$

$$= (-1)^{\kappa+n-1} n! y^{\kappa-1} e^{-y} L_n{}^{(-\kappa-1)}(y)$$

using a well-known nth derivative formula for $L_m{}^{-\kappa-1}(y)$. The result is

$$D_n{}^{\kappa}(v) = (-1)^n n!\, v^{+n-\kappa} e^{\frac{1}{v}} L_n{}^{(-\kappa-1)}\left(-\frac{1}{v}\right).$$

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