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LATTICES WHOSE IDEAL LATTICE IS STONE

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1. Introduction

An elementary fact about ideal lattices of bounded distributive lattices is that they belong to the equational class $\mathscr{B}_\omega$ of all distributive p-algebras (distributive lattices with pseudocomplementation). The lattice of equational subclasses of $\mathscr{B}_\omega$ is known to be a chain

$$\mathscr{B}_0 \subset \mathscr{B}_1 \subset \cdots \subset \mathscr{B}_n \subset \cdots \subset \mathscr{B}_\omega$$

of type $\omega+1$, where $\mathscr{B}_0$ is the class of Boolean algebras and $\mathscr{B}_1$ is the class of Stone algebras. G. Grätzer in his book [7] asks after a characterisation of those bounded distributive lattices whose ideal lattice belongs to $\mathscr{B}$ $(n \geqq 1)$. The answer to the problem for the case $n=0$ is well known: the ideal lattice of a bounded lattice L is Boolean if and only if L is a finite Boolean algebra. D. Thomas [**10**] recently solved the problem for the case $n=1$ utilising the order-topological duality theory for bounded distributive lattices and in [**5**] W. Bowen obtained another proof of Thomas's result via a construction of the dual space of the ideal lattice of a bounded distributive lattice from its dual space. In this paper we give a short, purely algebraic proof of Thomas's result and deduce from it necessary and sufficient conditions for the ideal lattice of a bounded distributive lattice to be a relative Stone algebra. Grätzer's problem for the case $n=1$ can be paraphrased as: Characterise those bounded distributive lattices whose congruence kernels form a Stone algebra. We ask and answer the same question for distributive p-algebras and distributive double p-algebras drawing from the main result a characterisation of those double Heyting algebras whose congruence lattice is Stone.

2. Preliminaries

Let $\langle L, \vee, \wedge, 0, 1\rangle$, henceforth simply L, be a bounded distributive lattice. Throughout, we shall write $\mathrm{Cen}(L)$ for the centre of L, $\mathscr{I}(L)$ for its ideal lattice and L/I for $L/\theta(I)$, where $\theta(I)$ is the principal congruence of L generated by $I \in \mathscr{I}(L)$. If L is equipped with a unary operation $*$ characterised by the property:

$$a \wedge x = 0 \quad \text{if and only if} \quad x \leqq a^*$$

then L is called a *distributive p-algebra* or *distributive lattice with pseudocomplementation.*

If, in any such algebra, we write $B(L)=\{x\in L; x=x^{**}\}$ then $\langle B(L), \cup, \wedge, {}^*, 0, 1\rangle$ is a Boolean algebra, called the *skeleton* of L, when the join operation $\cup$ is defined on $B(L)$ by $a\cup b=(a\vee b)^{**}$ and $D(L)=\{x\in L; x^*=0\}$ is a filter in L, called the *dense filter*. A *Stone algebra* is a distributive p-algebra satisfying the identity $x^*\vee x^{**}=1$ and a *relative Stone algebra* is a bounded lattice in which every interval is a Stone algebra. Relative Stone algebras are intimately related to *Heyting algebras*; this is, bounded (distributive) lattices equipped with a binary operation $*$ characterised by the property:

$$a\wedge x\leqq b \quad \text{if and only if} \quad x\leqq a*b.$$

Indeed, the classes of relative Stone algebras and Heyting algebras satisfying the identity $(x*y)\vee(y*x)=1$ are coincident. A distributive p-algebra endowed with a unary operation $^+$ characterised by the property dual to that for * is called a *distributive double p-algebra* and a Heyting algebra endowed with a binary operation $+$ characterised by the property dual to that for $*$ is called a *double Heyting algebra*. If L is a distributive double p-algebra and $a\in L$ then elements $a^{n(*+)}(n<\omega)$ can be defined recursively by

$$a^{0(*+)}=a,\ a^{(k+1)(*+)}=a^{k(*+)*+}.$$

Elements $a^{n(+*)}(n<\omega)$ can also be defined in a similar fashion and the following are known to hold (see [**2**]):

$$x\leqq x^{*+}, (x\vee y)^{*+}=x^{*+}\vee y^{*+}, \mathrm{Cen}(L)=\{x\in L; x=x^{*+}\}.$$

By a *congruence relation* on a distributive p-algebra, distributive double p-algebra, double Heyting algebra we mean a lattice congruence preserving *, * and $^+$, $*$ and $+$, respectively, and by a *congruence kernel* we mean any congruence class containing 0.

All undefined terms as well as general lattice theoretic results and facts about distributive p-algebras may be found in [**1**] or [**7**].

3. Grätzer's problem for $n=1$

The key to the solution of the problem is the following simple observation:

Lemma 1. *An ideal I in a bounded distributive lattice L is complemented if and only if it is of the form $(z]$, for some $z\in\mathrm{Cen}(L)$.*

Proof. Clearly, if $z\in\mathrm{Cen}(L)$ then $(z]$ has complement $(z']$ in $\mathscr{I}(L)$. Conversely, if $I\in\mathrm{Cen}(\mathscr{I}(L))$ then $I\vee I^*=L$ so that $1=z\vee w$, for some $z\in I$, $w\in I^*$ which since $I^*=\{x\in L;\ x\wedge i=0 \text{ for all } i\in I\}$, shows that $z\in\mathrm{Cen}(L)$ and $z'=w$. For any $x\in I^*$, we have $x\wedge z=0$ so that $x\leqq z^*=z'=w$ and, therefore, $I^*\subseteq(w]$. Thus, $I^*=(w]$, since $w\in I^*$, and it follows that $I=(z]$.

Theorem 2. *The ideal lattice of a bounded distributive lattice L is a Stone algebra if and only if L is a Stone algebra whose centre is complete.*

Proof. If $\mathscr{I}(L)$ is Stone than $(a]^* \in \mathrm{Cen}(\mathscr{I}(L))$, for any $a \in L$, and so $(a]^* = (z]$, for some $z \in \mathrm{Cen}(L)$, by Lemma 1. However, $(a]^* = \{x \in L; x \wedge a = 0\}$ and so a^* exists and belongs to $\mathrm{Cen}(L)$, for any $a \in L$. In other words, L is a Stone algebra. In order to show that $\mathrm{Cen}(L)$ is a complete lattice, it is enough by Lemma 1 to show that if $Z \subseteq \mathrm{Cen}(L)$ then $\bigcap\{(z]; z \in Z\}$ is of the form I^*, for some $I \in \mathscr{I}(L)$. We claim that $I = \bigvee\{(z']; z \in Z\}$ is an ideal satisfying our needs. Indeed, $x \in I^*$ if and only if $x \wedge a = 0$, for all $a \in \bigvee\{(z']; z \in Z\}$, and by distributivity this is equivalent to $x \wedge z' = 0$, for all $z \in Z$, since $\bigvee\{(z']; z \in Z\}$ consists of all finite joins of elements in the set union $\bigcup\{(z']; z \in Z\}$. This, in turn, holds if and only if $x \in \bigcap\{(z]; z \in Z\}$. Thus, $I^* = \bigcap\{(z]; z \in Z\}$.

Conversely, suppose that L is a Stone algebra and that $\mathrm{Cen}(L)$ is complete. In order to show that $\mathscr{I}(L)$ is Stone it is enough to show that $I^* \in \mathrm{Cen}\,\mathscr{I}(L)$, for any $I \in \mathscr{I}(L)$. First, observe that $i^* \in \mathrm{Cen}(L)$ for any $i \in I$, since L is Stone, and so the existence of $z = \bigwedge\{i^*; i \in I\}$, taken in $\mathrm{Cen}(L)$, is guaranteed. We claim that $I^* = (z]$. Indeed, $x \in I^*$ if and only if x is a lower bound in L of $\{i^*; i \in I\}$ or, equivalently, x^{**} is a lower bound in L of $\{i^*; i \in I\}$. This, in turn, is equivalent to $x \leqq z$, since $x \leqq x^{**} \in \mathrm{Cen}(L)$ and $z^{**} = z \in \mathrm{Cen}(L)$. Thus, $I^* = (z] \in \mathrm{Cen}(\mathscr{I}(L))$.

Corollary 3. *For a bounded distributive lattice L, the following are equivalent:*

(i) *$\mathscr{I}(L)$ is a relative Stone algebra,*
(ii) *For any $I \in \mathscr{I}(L)$, L/I is a Stone algebra whose centre is complete.*
(iii) *L is a relative Stone algebra whose centre is complete and, for any $I \in D(\mathscr{I}(L))$, L/I is a Stone algebra whose centre is complete.*

Proof. It is well known (see [**1**]) that for a bounded distributive lattice L to be relative Stone it is necessary and sufficient that every principal filter of L be a Stone algebra. This fact, applied to $\mathscr{I}(L)$ in conjunction with Theorem 2 and the equally well known fact that $\mathscr{I}(L/I) \cong [I)$, for any $I \in \mathscr{I}(L)$, establishes the equivalence of (i) and (ii). Now, if $\mathscr{I}(L)$ is relative Stone and $b \in L$ is arbitrary then $L/(b]$ is a Stone algebra. In particular, it follows that, given any $a \in L$, there exists $\bar{a} \in L$ such that

$$[a]_b \wedge [x]_b = [0]_b \quad \text{if and only if} \quad [x]_b \leqq [\bar{a}]_b,$$

where $[x]_b$ denotes the congruence class of $\theta((b])$ containing x. As a consequence of this and the well known description of principal congruences on distributive lattices we conclude that

$$a \wedge x \leqq b \quad \text{if and only if} \quad x \leqq \bar{a} \vee b$$

and so L is a Heyting algebra in which $a * b = \bar{a} \vee b$, for any $a, b \in L$. Furthermore, the identity $(a * b) \vee (b * a) = 1$ holds in L by virtue of the fact that it holds in $\mathscr{I}(L)$. Indeed, for any $I, J \in \mathscr{I}(L)$, we have $I * J = \{x \in L;\ x \wedge i \in J$, for all $i \in I\}$ and so, in particular, $(a] * (b] = (a * b]$, for any $a, b \in L$. Therefore, $(a * b] \vee (b * a] = L$ and so $(a * b) \vee (b * a) = 1$. Thus, L is relative Stone and the proof that (ii) implies (iii) is complete. Moreover, condition (iii) in conjunction with Theorem 2 shows that $\mathscr{I}(L/I)$ and therefore $[I)$ is a Stone algebra, for any $I \in D(\mathscr{I}(L))$. Thus, (iii) implies (i), since it is well known that for a

Stone algebra to be relative Stone it is necessary and sufficient that each of its principal filters generated by a dense element be a Stone algebra (see [1]).

Corollary 4. *The congruence lattice of a Boolean lattice L is relative Stone if and only if every homomorphic image of L is complete.*

In connection with Corollaries 3 and 4 we point out that if L is a Stone algebra whose centre is complete and $I \in \mathscr{I}(L)$ the L/I is not necessarily Stone nor is its centre necessarily complete. Indeed, if L is the Stone algebra obtained by adjoining a new zero and unit to the four-element Boolean algebra and I is the principal ideal of L generated by its only atom then L/I is isomorphic to the four-element Boolean algebra with a new unit adjoined and so is not Stone. Furthermore, while the field of all subsets of an uncountable set X is complete, its quotient modulo the ideal of all countable subsets of X is not.

4. The problem for distributive p-algebras and double p-algebras

Earlier we pointed out that a subset of a bounded distributive lattice is an ideal if and only if it is a congruence kernel. W. Cornish [6] showed that an ideal I in a distributive p-algebra is a congruence kernel if and only if $i^{**} \in I$ whenever $i \in I$. In addition, Cornish showed that the lattice of congruence kernels of a distributive p-algebra L is isomorphic to the ideal lattice of the skeleton $B(L)$ of L. T. S. Blyth [4] showed that exactly the same is true for pseudo-complemented semilattices with, of course, the appropriate definition of congruence kernel in this context. Thus, we have

Theorem 5. *The lattice of congruence kernels of a pseudo-complemented semilattice or of a distributive p-algebra L is a Stone algebra if and only if the skeleton of L is complete.*

The situation for distributive double p-algebras is not so simple but nevertheless tractible. It follows on dualising results in [2] that a subset I of a distributive double p-algebra is a congruence kernel if and only if $i^{*+} \in I$ whenever $i \in I$. Moreover, it is easy to show, utilising the well known description of infinite joins in ideal lattices of distributive lattices and the identity $(x \vee y)^{*+} = x^{*+} \vee y^{*+}$ which holds in any distributive double p-algebra, that the lattice $K(L)$ of congruence kernels of a distributive double p-algebra L is a complete sublattice of the ideal lattice $\mathscr{I}(L)$ of L. Consequently, $K(L) \in \mathscr{B}_\omega$ and, using Lemma 1, it is easy to see that $\mathrm{Cen}(K(L)) = \{(z];\ z \in \mathrm{Cen}(L)\}$. When, then, is $K(L)$ Stone?

Theorem 6. *The lattice $K(L)$ of congruence kernels of a distributive double p-algebra L is a Stone algebra if and only if $\bigwedge_{n<\omega} a^{n(+*)}$ and $\bigwedge S$ exist in L, for any $a \in B(L)$ and $S \subseteq \mathrm{Cen}(L)$.*

Proof. We start with the observation that if $a \in L$ and $I(a) = \{x \in L; x \leqq a^{n(*+)}$, for some $n < \omega\}$ then $I(a) \in K(L)$, since $a^{n(*+)} \leqq a^{m(*+)}$ whenever $n \leqq m$, and claim that $I(a)^* = \bigcap_{n<\omega} (a^{*n(+*)}]$. With the aim of showing that the ideal $\bigcap_{n<\omega} (a^{*n(+*)}] \in K(L)$, let

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$x \in \bigcap_{n<\omega}(a^{*n(+*)}]$ and let $k<\omega$. Then $x \leqq a^{*(k+1)(+*)}$ so that $x^* \geqq a^{(k+1)(*+)**} \geqq a^{(k+1)(*+)}$ and, therefore, $x^{*+} \leqq a^{(k+1)(*+)+} = a^{*k(+*)++} \leqq a^{*k(+*)}$. Thus, $x^{*+} \in \bigcap_{n<\omega}(a^{*n(+*)}]$. Moreover, if $x \in I(a) \cap \bigcap_{n<\omega}(a^{*n(+*)}]$ then $x \leqq a^{n(*+)}$, for some $n<\omega$, and $x \leqq a^{*n(+*)} = a^{n(*+)*}$ so that $x \leqq a^{n(*+)} \wedge a^{n(*+)*} = 0$. Therefore, $I(a) \cap \bigcap_{n<\omega}(a^{*n(+*)}] = \{0\}$. In addition, if $K \in K(L)$ satisfies $I(a) \cap K = \{0\}$ and $k \in K$ then $k \wedge a^{n(*+)} = 0$ for all $n<\omega$; that is, $k \leqq a^{n(*+)*} = a^{*n(+*)}$ for all $n<\omega$. Thus, $K \subseteq \bigcap_{n<\omega}(a^{*n(+*)}]$ and we conclude that $I^* = \bigcap_{n<\omega}(a^{*n(+*)}]$. It follows, now, that if $K(L)$ is a Stone algebra and $a \in L$ then $I(a)^* \in \mathrm{Cen}(K(L))$ and so $I(a)^* = (z]$, for some $z \in \mathrm{Cen}(L)$. Thus, $\bigwedge_{n<\omega} a^{*n(+*)}$ exists, for any $a \in L$. Equivalently, $\bigwedge_{n<\omega} a^{n(+*)}$ exists, for any $a \in B(L)$. Next, it is straightforward to show that if $S \subseteq \mathrm{Cen}(L)$ and $I(S') = \{x \in L; x \leqq s'_1 \vee \cdots \vee s'_n,$ for some $s_i \in S, 1 \leqq i \leqq n\}$ then $I(S') \in K(L)$. Moreover, by distributivity, $I \in K(L)$ satisfies $I \cap I(S') = \{0\}$ if and only if $i \wedge s' = 0$, for all $i \in I$ and $s \in S$: equivalently, if and only if $I \subseteq \bigcap\{(s]; s \in S\}$. Consequently, $I(S')^* = \bigcap\{(s]; s \in S\}$ which, since $I(S')^* = (z]$ for some $z \in \mathrm{Cen}(L)$, shows that $\bigwedge S$ exists, for any $S \subseteq \mathrm{Cen}(L)$.

Conversely, suppose that, for any $a \in B(L)$ and $S \subseteq \mathrm{Cen}(L)$, $\bigwedge_{n<\omega} a^{n(+*)}$ and $\bigwedge S$ exist in L. We start by showing that all such meets necessarily belong to $\mathrm{Cen}(L)$. Indeed, if $a \in B(L)$, $k<\omega$ and $m(a) = \bigwedge_{n<\omega} a^{n(+*)}$ then $m(a) \leqq a^{(k+1)(+*)}$ so that $m(a)^* \geqq a^{(k+1)(+*)*} = a^{k(+*)+**} \geqq a^{k(+*)+}$ and, therefore, $m(a)^{*+} \leqq a^{k(+*)++} \leqq a^{k(+*)}$. It follows that $m(a)^{*+} \leqq m(a)$ and so $m(a) \in \mathrm{Cen}(L)$, since $x \leqq x^{*+}$ holds for any $x \in L$. Moreover, if $S \subseteq \mathrm{Cen}(L)$ and $m(S) = \bigwedge S$ then, since $m(S) \leqq s$ implies $m^{*+}(S) \leqq s$, for any $s \in S$, we have $m^{*+}(S) \leqq m(S)$ and so $m(S) \in \mathrm{Cen}(L)$. It follows, now, that if $i \in I \in K(L)$ then $m(i^*)$ and, therefore, $z = \bigwedge\{m(i^*); i \in I\}$ exists and is central. We claim that $I^* = (z]$. Indeed, if $x \in I^*$, $i \in I$ and $n<\omega$ then $x \wedge i^{n(*+)} = 0$ so that $x \leqq i^{n(*+)*} = i^{*n(+*)}$. Therefore, $x \leqq m(i^*)$, for all $i \in I$, and so $x \leqq z$. Hence, $I^* \subseteq (z]$. For the reverse inclusion, first observe that if $x \leqq z$, $i \in I$ and $m<\omega$ then $x^{m(*+)} \leqq z^{m(*+)} = z \leqq i^*$, since $z \in \mathrm{Cen}(L)$, and so $i \leqq i^{**} \leqq x^{m(*+)*}$. It follows, now, that $I(x) \cap I = \{0\}$; because if $i \in I(x) \cap I$ then $i \leqq x^{n(*+)}$, for some $n<\omega$, and so $i \leqq x^{n(*+)} \wedge x^{n(*+)*} = 0$. Therefore, $x \in I(x) \subseteq I^*$. Thus, we have shown that I^* is complemented in $K(L)$, for any $I \in K(L)$. Equivalently, $K(L)$ is a Stone algebra.

In [9] P. Köhler call s an ideal I in a double Heyting algebra L *normal* if $a \in I$ implies $a^{*+} \in I$ and proves that the congruence lattice of L is isomorphic to the lattice of normal ideals of L. Thus, we have

Corollary 7. *The congruence lattice of a double Heyting algebra L is a Stone algebra if and only if* $\bigwedge_{n<\omega} a^{n(+*)}$ *and* $\bigwedge S$ *exist, for any* $a \in B(L)$ *and* $S \subseteq \mathrm{Cen}(L)$.

A double p-algebra is called *regular* if it satisfies the identity $(x \wedge x^+) \vee (y \vee y^*) = y \vee y^*$. In [8], T. Katriňák shows that a regular double p-algebra is, in fact, a double Heyting algebra in which $x * y$ and $x + y$ are double p-algebra polynomials. As a consequence, double Heyting algebra congruences and double p-algebra congruences coincide for regular double p-algebras. Thus, Corollary 7 is a generalisation and, simultaneously, an improvement of the main result of [3].

Added in proof. Recently, I learnt that Theorem 2 was proved by T. Katriňák in his paper "Notes on Stone lattices I", *Mat. Časopsis Sloven. Akad. Vied.* **16** (1966), 128–142 (in Russian). A version of Theorem 2 for join similattices with 1 appears in his paper "Pseudocomplementäre Halbverbände", *Mat. Časopsis Sloven. Akad. Vied.* **18** (1968), 121–143.

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