On the central limit theorem and iterated logarithm law for stationary processes .

C.C. Heyde

It has recently emerged that a convenient way to establish
central limit and iterated logarithm results for processes with
stationary increments is to use approximating martingales with
stationary increments. Functional forms of the limit results can
be obtained via a representation for the increments of the
stationary process in terms of stationary martingale differences
plus other terms whose sum telescopes and disappears under
suitable norming. Results based on the most general form of such
a representation are here obtained.

1. Introduction and principal results

In this paper our setting is that of a probability space (Ω, B, P)
with an ergodic one-to-one bimeasurable measure preserving transformation
T . Let $L_2(P)$ be the Hilbert space of random variables with finite
second moment. Define U on $L_2(P)$ by $UX(\omega) = X(T\omega)$ for $X \in L_2(P)$,
$\omega \in \Omega$, and write $X_k = U^k X_0$ for some particular $X_0 \in L_2(P)$ with

$EX_0 = 0$. Set also, $S_0 = 0$, $S_n = \sum_{k=1}^{n} X_k$ for $n \geq 1$ and $\sigma_n^2 = ES_n^2$.

We shall be concerned with giving invariance principles for both the
central limit and iterated logarithm law for appropriate random functions

---

Received 3 September 1974.

in C or D defined from the partial sums S_k . Here $C = C[0, 1]$ is the space of continuous functions on $[0, 1]$ and $D = D[0, 1]$ is the space of continuous functions on $[0, 1]$ which are right continuous with left hand limits. In both cases we employ the supremum metric which we denote by ρ .

Let $\{\theta_n(\cdot)\}$ be a sequence of random functions on $[0, 1]$ defined by

$$\theta_n(t) = S_j/\sigma_n , \quad j/n \leq t < (j+1)/n , \quad j = 0, 1, \ldots, n-1 ,$$

and

$$\theta_n(1) = S_n/\sigma_n .$$

Also, let $\{\eta_n(\cdot)\}$ be a sequence of random functions on $[0, 1]$ defined by

$$\eta_n(t) = \left(2\sigma_n^2 \log \log\sigma_n^2\right)^{-\frac{1}{2}} \left(S_i + (nt-k)X_k\right) ,$$

$$k \leq nt \leq k+1 , \quad k = 0, 1, \ldots, n-1 .$$

Let K be the set of absolutely continuous $x \in C$ such that

$$x(0) = 0$$

and

$$\int_0^1 [\dot{x}(t)]^2 dt \leq 1 ,$$

where $\dot{x}$ denotes the derivative of x determined almost everywhere with respect to Lebesgue measure. Also define

$$g = \sup\left\{n : \sigma_n^2 \leq e\right\} .$$

Let F_0 be a σ-field such that $F_0 \subset B$ and $F_0 \subset T^{-1}(F_0)$ and write $F_k = T^{-k}(F_0)$, $F_{-\infty} = \bigcap_{k=-\infty}^{\infty} F_k$ and $F_\infty = \sigma$-field generated by $\bigcup_{k=-\infty}^{\infty} F_k$.

Our object is the following theorem.

THEOREM. *If*

(1) $$\sum_{n=1}^{\infty} \left\{ \overline{\lim_{m \to \infty}} \, E \left(\sum_{r=n}^{m} x_r \right)^2 + \overline{\lim_{m \to \infty}} \, E \left(\sum_{r=n}^{m} x_{-r} \right)^2 \right\} < \infty \ ,$$

where

$$X_r = E\left(X_{-r} | F_0\right) - E\left(X_{-r} | F_{-1}\right) \ ,$$

and

(2) $E\left(X_0 | F_\infty\right) = X_0$ *almost surely,* $E\left(X_0 | F_{-\infty}\right) = 0$ *almost surely,*

then $\lim\limits_{n \to \infty} \sigma_n / \sqrt{n} = \sigma$ *exists for* $0 \leq \sigma < \infty$. *If* $\sigma > 0$ *then* $\theta_n \xrightarrow{d} W$ *in*

the sense (D, ρ) *where* W *is a standard Wiener process. Also,* $g < \infty$,

$\{\eta_n; \ n > g\}$ *is relatively compact and the set of its limit points*

coincides with K .

This result extends those of Theorem 3 of Scott [5] (central limit
case) and Theorem 2 of Heyde and Scott [3] (iterated logarithm case) which
were both given with the condition

(3) $$\sum_{m=1}^{\infty} \left\{ \left[E\left[E\left(X_0 | F_{-m}\right)\right]^2 \right]^{\frac{1}{2}} + \left[E\left[X_0 - E\left(X_0 | F_m\right)\right]^2 \right]^{\frac{1}{2}} \right\} < \infty$$

replacing those above. The idea in each case involves a representation for
the X's of the form

(4) $$X_0 = Y_0 + UZ_0 - Z_0$$

where the Y_0, $Z_0 \in L_2(P)$ and $\left\{U^k Y_0\right\}$ forms a stationary ergodic sequence
of martingale differences. The martingale approximation idea is due to
Gordin [1] and the conditions of the theorem appear to be the most general
under which (4) will hold as above. Of course the limit behaviour of S_n

is then easy to study via the corresponding behaviour of $\sum\limits_{k=1}^{n} U^k Y_0$, since

$$\dot{S}_n = \sum_{k=1}^{n} U^k X_0 = \overline{\sum_{k=1}^{n} U^k Y_0} + U^{n+1} Z_0 - UZ_0 \ ,$$

and the effect of $U^{n+1} Z_0 - UZ_0$ disappears under suitable norming. It

should be remarked that $\overline{\lim\limits_{n\to\infty}}\, E\left(\sum\limits_{k=1}^{n} U^k \left(X_0 - Y_0\right)\right)^2 < \infty$ for $Y_0 \in L_2(P)$ with

$E\left(Y_0 | F_{-1}\right) = 0$ almost surely, if and only if $X_0 - Y_0$ is representable in

the form (4) (via Theorem 18.2.2 of Ibragimov and Linnik [4]).

That the representation (4) holds under the asserted conditions of the
theorem can easily be extracted from the proof of Theorem 3 of [5].
Condition (1) gives

$$\lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, E\left(\sum_{r=n}^{m} x_r\right)^2 = \lim_{n\to\infty} \overline{\lim_{m\to\infty}}\, E\left(\sum_{r=n}^{m} x_{-r}\right)^2 = 0 \ ,$$

so that $\sum\limits_{r=0}^{\infty} x_r$ and $\sum\limits_{r=0}^{\infty} x_{-r}$ converge in $L_2(P)$. Then

$$\overline{\lim_{m\to\infty}}\, E\left(\sum_{r=n}^{m} x_r\right)^2 + \overline{\lim_{m\to\infty}}\, E\left(\sum_{r=n}^{m} x_{-r}\right)^2 = E\left(\sum_{r=n}^{\infty} x_r\right)^2 + E\left(\sum_{r=n}^{\infty} x_{-r}\right)^2 \ ,$$

and our condition (1) is just the condition (46) of [5]. Once the
representation (4) is obtained the remainder of the proof is exactly that
of Theorem 3 of [5] in the central limit case and Theorem 2 of [3] in the
case of the iterated logarithm law.

We remark that (1) arises via the restriction $Z_0 \in L_2(P)$. In fact,
from [5], p. 133,

$$EZ_0^2 = \sum_{n=1}^{\infty} E\left\{\left(\sum_{r=n}^{\infty} x_r\right)^2 + \left(\sum_{r=n}^{\infty} x_{-r}\right)^2\right\} \ .$$

Note also that under the conditions of the theorem we have that (4) holds
and $\lim\limits_{n\to\infty} \sigma_n/\sqrt{n} = \sigma$ exists for $0 \leq \sigma < \infty$. Now $Z_0 \in L_2(P)$, so

$\sigma^2 = EY_0^2$, and we can only obtain $\sigma = 0$ in the case where

$X_n = U^{n+1} Z_0 - U^n Z_0$, some $Z_0 \in L_2(P)$.

We shall show that a useful sufficient condition for (1), in the
presence of (2) is

(5) $\sum\limits_{n=1}^{\infty} \sum\limits_{k=0}^{\infty} \left| E\left(X_k E\left(X_0 | F_{-n}\right)\right) + E\left(X_{-k} X_0\right) - E\left(X_{-k} E\left(X_0 | F_{n-1}\right)\right) \right| < \infty \ .$

To obtain this we first note that

$$(6) \quad E\left\{\left(\sum_{r=n}^{m} x_r\right)^2 + \left(\sum_{r=n}^{m} x_{-r}\right)^2\right\} =$$

$$= \sum_{r=n}^{m} E\left(x_r^2 + x_{-r}^2\right) + 2 \sum_{k=1}^{m-n} \sum_{r=n}^{m-k} E\left(x_r x_{r+k} + x_{-r} x_{-r-k}\right) \ .$$

Now for any k, r ,

$$\begin{aligned}
E(x_r x_{r+k}) &= E\big(E(X_{-r}|F_0)E(X_{-r-k}|F_0)\big) - E\big(E(X_{-r}|F_{-1})E(X_{-r-k}|F_0)\big) \\
&= E\big(X_{-r-k}E(X_{-r}|F_0)\big) - E\big(X_{-r-k}E(X_{-r}|F_{-1})\big) \\
&= E\big(X_{-k}E(X_0|F_r)\big) - E\big(X_{-k}E(X_0|F_{r-1})\big) \ ,
\end{aligned}$$

so that for $k \geq 0$,

$$\sum_{r=n}^{m-k} E(x_r x_{r+k}) = E\big(X_{-k}E(X_0|F_{m-k})\big) - E\big(X_{-k}E(X_0|F_{n-1})\big) \ ,$$

$$\sum_{r=n}^{m-k} E(x_{-r} x_{-r-k}) = E\big(X_k E(X_0|F_{-n})\big) - E\big(X_k E(X_0|F_{-m+k-1})\big) \ .$$

Standard martingale results give $E(X_0|F_{-n}) \xrightarrow{L_2} E(X_0|F_{-\infty}) = 0$ almost

surely, and $E(X_0|F_n) \xrightarrow{L_2} E(X_0|F_\infty) = X_0$ almost surely, so that

$E\big(E(X_0|F_{-n})\big)^2 \to 0$ and $E\big(X_0 - E(X_0|F_n)\big)^2 \to 0$ as $n \to \infty$, and hence

$$(7) \quad \lim_{m\to\infty} \sum_{r=n}^{m} E\left(x_r^2 + x_{-r}^2\right) = EX_0^2 - E\big(X_0 E(X_0|F_{n-1})\big) + E\big(X_0 E(X_0|F_{-n})\big) \ .$$

Also,

$$\sum_{k=1}^{m-n} \sum_{r=n}^{m-k} E(x_r x_{r+k} + x_{-r} x_{-r-k})$$

$$= \sum_{k=1}^{m-n} \left\{ E\big(X_k E(X_0|F_{-n})\big) + E(X_{-k}X_0) - E\big(X_{-k}E(X_0|F_{n-1})\big) \right\}$$

$$- \sum_{k=1}^{m-n} \left\{ E\big(X_k E(X_0|F_{-m+k-1})\big) + E(X_{-k}X_0) - E\big(X_{-k}E(X_0|F_{m-k})\big) \right\}$$

and under the condition (5),

(8) $\lim\limits_{m\to\infty} \sum\limits_{k=1}^{m-n} \sum\limits_{r=n}^{m-k} E\left(x_r x_{r+k} + x_{-r} x_{-r-k}\right)$

$$= \sum_{k=1}^{\infty} \left\{ E\left(X_k E\left(X_0 | F_{-n}\right)\right) + E\left(X_{-k} X_0\right) - E\left(X_{-k} E\left(X_0 | F_{n-1}\right)\right) \right\} \ .$$

It is then clear from (6), (7), and (8), that (1) holds under the condition (5).

Conditions (1) and (5) do not appear to simplify in any really convenient way in general. However, we remark that we are free to choose a convenient F_0 . For example, if F_0 is the σ-field generated by X_k , $k \leq 0$, then the condition (5) becomes just

(9) $$\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| E\left(X_k E\left(X_0 | F_{-n}\right)\right) \right| < \infty \ .$$

In some cases it may be useful to translate (1) (or (5)) into a condition on the Fourier coefficients of the expansion of X_0 in terms of a suitable complete orthonormal set. Certainly (1) represents a significant improvement over (3) and this justifies its use. An example is given in Section 2 to illustrate this point. It should also be remarked that conditions (1) (or (5)) and (3) provide a convenient vehicle for the study of central limit and iterated logarithm results for stationary processes satisfying mixing conditions. Illustrations concerning uniform mixing processes have been given in the papers [3] and [5].

2. An example

To illustrate the improvement of (1) over (2) we introduce the stationary linear process $\{x(n)\}$ given by

$$x(n) - \mu = \sum_{j=-\infty}^{\infty} \alpha(j)\varepsilon(n-j) \ , \quad \sum_{j=-\infty}^{\infty} \alpha^2(j) < \infty \ ,$$

where the $\varepsilon(n)$ are independent and identically distributed with zero mean and variance σ^2 . One widely applicable model which gives rise to this process is the standard version of the mixed autoregression and moving average process. Another is the stationary gaussian process with absolutely continuous spectral density.

Suppose that $x(1)$, $x(2)$, $\ldots$, $x(N)$ is a sample of N consecutive observations on the process $\{x(n)\}$ and $\bar{x}$ denotes the sample mean. It follows from the ergodic theorem that $\bar{x} \to \mu$ almost surely, as $N \to \infty$, and it is of interest to obtain functional central limit and iterated logarithm results which give information on the rate of this convergence. Here we have $X_k = x(k) - \mu$ and we can take F_k as the σ-field generated by $\varepsilon(m)$, $m \leq k$. Then

$$x_r = E\left(X_{-r}|F_0\right) - E\left(X_{-r}|F_{-1}\right) = \alpha(-r)\varepsilon(0) \ ,$$

and the theorem of this paper applies if

$$(10) \qquad \sum_{n=1}^{\infty} \left\{ \left[\sum_{r=n}^{\infty} \alpha(r) \right]^2 + \left[\sum_{r=n}^{\infty} \alpha(-r) \right]^2 \right\} < \infty \ .$$

On the other hand, the corresponding results based on the use of condition (3) hold if

$$(11) \qquad \sum_{n=1}^{\infty} \left[\sum_{|r| \geq n} \alpha^2(r) \right]^{\frac{1}{2}} < \infty \ ,$$

as in the lemma of Heyde [2]. The condition (10) represents a significant improvement over (11) in the case where the α's vary continually in sign. For example, $\alpha(|r|) = (-1)^r r^{-1}$, $r \geq 1$. If the α's are ultimately all positive, an example where (10) holds but (11) does not is provided by $\alpha(|r|) \sim Cr^{-3/2}(\log r)^{-1}$ as $r \to \infty$ for some $C > 0$.

References

[1] M.I. Gordin, "The central limit theorem for stationary processes", *Soviet Math. Dokl.* 10 (1969), 1174-1176.

[2] C.C. Heyde, "An iterated logarithm result for autocorrelations of a stationary linear process", *Ann. Probability* 2 (1974), 328-332.

[3] C.C. Heyde and D.J. Scott, "Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments", *Ann. Probability* 1 (1973), 428-436.

[4]   I.A. Ibragimov and Yu.V. Linnik, *Independent and stationary sequences of random variables* (Walters-Noordhoff, Groningen, 1971).

[5]   D.J. Scott, "Central limit theorems for martingales and for processes with stationary increments using a Skorokhod representation approach", *Adv. Appl. Probability* 5 (1973), 119-137.

Department of Statistics,
Faculty of Economics,
Australian National University,
Canberra, ACT.