Forum of Mathematics, Sigma (2023), **Vol. 11**:e60 1–30
doi:10.1017/fms.2023.53

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Weights in a Benson-Solomon block

Justin Lynd [1] and Jason Semeraro [2]

[1] Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504; E-mail: lynd@louisiana.edu.
[2] Department of Mathematical Sciences, Loughborough University, LE11 3TU, United Kingdom;
E-mail: j.p.semeraro@lboro.ac.uk.

Received: 26 March 2023; **Revised:** 10 May 2023; **Accepted:** 28 May 2023

2020 Mathematics Subject Classification: *Primary* – 20D06, 20D20; *Secondary* – 20C20, 20C33

Abstract

To each pair consisting of a saturated fusion system over a p-group together with a compatible family of Külshammer-Puig cohomology classes, one can count weights in a hypothetical block algebra arising from these data. When the pair arises from a genuine block of a finite group algebra in characteristic p, the number of conjugacy classes of weights is supposed to be the number of simple modules in the block. We show that there is unique such pair associated with each Benson-Solomon exotic fusion system, and that the number of weights in a hypothetical Benson-Solomon block is 12, independently of the field of definition. This is carried out in part by listing explicitly up to conjugacy all centric radical subgroups and their outer automorphism groups in these systems.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. Associated to each block b of kG, there is a saturated fusion system $\mathcal{F} = \mathcal{F}_S(b)$ over the defect group S of the block in which the morphisms between subgroups are given by conjugation by elements of G preserving the corresponding Brauer pairs [AKO11, Cra11]. Several questions in the modular representation theory of finite groups concern the connection between representation theoretic properties of kGb and the category $\mathcal{F}$. However, it is known that for many purposes $\mathcal{F}$ does not, in general, retain enough information about kGb-mod. For example, it does not determine the number of simple modules in b, in part because it retains too little of the p'-structure of p-local subgroups. On the other hand, the block b also determines a family of degree 2 cohomology classes $\alpha_Q \in H^2(\mathrm{Aut}_{\mathcal{F}}(Q), k^\times)$, for $Q \in \mathcal{F}^c$ an $\mathcal{F}$-centric subgroup, by work of Külshammer and Puig (see [AKO11, IV.5.5]). This family is expected to supply the missing information away from the prime p. The Külshammer-Puig classes are compatible in the sense that, by [Lin19, Theorem 8.14.5], they determine an element

$$\alpha \in \lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}},$$

where $[S(\mathcal{F}^c)]$ is the partially ordered set of $\mathcal{F}$-isomorphism classes of chains $\sigma = (X_0 < X_1 < \cdots < X_n)$ of $\mathcal{F}$-centric subgroups, and $\mathcal{A}^2_{\mathcal{F}}$ is the covariant functor which sends a chain σ to $H^2(\mathrm{Aut}_{\mathcal{F}}(\sigma), k^\times)$. Here, $\mathrm{Aut}_{\mathcal{F}}(\sigma) \leqslant \mathrm{Aut}_{\mathcal{F}}(X_n)$ is the group of automorphisms in $\mathcal{F}$ of X_n preserving all members X_i of the chain. For example, if b is the principal block of kG, then α is always the trivial class [AKO11, IV.5.32].

Thus, by a *Külshammer-Puig pair*, we mean a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a saturated fusion system on a p-group S and α is an element of $\lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$. Given such a pair $(\mathcal{F}, \alpha)$ arising from a block b, the quantity

$$\mathbf{w}(\mathcal{F},\alpha) := \sum_{Q\in\mathcal{F}^{cr}/\mathcal{F}} z(k_{\alpha_Q}\operatorname{Out}_{\mathcal{F}}(Q)),$$

counts the number of kGb-weights. Here, $k_{\alpha_Q}\operatorname{Out}_{\mathcal{F}}(Q)$ is the algebra obtained from the group algebra $k\operatorname{Out}_{\mathcal{F}}(Q)$ by twisting with α_Q [AKO11, IV.5.36], $z(-)$ denotes the number of projective simple modules, and the sum is taken over a set of representatives for the conjugacy classes of $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups. Thus, Alperin's Weight conjecture says that $\mathbf{w}(\mathcal{F},\alpha)$ is the number of simple kGb-modules [AKO11, IV.5.46].

There is always a natural map $H^2(\mathcal{F}^c,k^\times)\to\lim_{[S(\mathcal{F}^c)]}\mathcal{A}^2_{\mathcal{F}}$, and the *gluing problem* asks whether this map is surjective (see [Lin09] and [Lib11] for further details). Linckelmann has shown that Alperin's conjecture has a structural reformulation in terms of algebras constructed from p-local finite groups, provided the gluing problem always has a solution [Lin04]. However, while the weight conjecture has relevance for actual blocks only, the gluing problem is a question about the Külshammer-Puig pair itself and can be considered: (1) when $\mathcal{F}$ is the fusion system of a block, but of no block with the specified compatible family α, and (2) when $\mathcal{F}$ is the fusion system of no block at all. Thus, we are interested in investigating such pairs disembodied from an actual block as a way of gauging the degree to which certain questions, and potential answers to those questions, are p-locally determined. A direct study of Külshammer-Puig pairs might reveal, for example, that there is an exotic pair as in (1) or (2) that does not satisfy the gluing problem. At this stage, such a possibility seems unlikely. On the other hand, and conversely, we would be very interested in a structural explanation why the gluing problem should hold in general, and it seems reasonable to expect that such an explanation would apply to all such pairs, exotic or not.

In this paper, we consider Külshammer-Puig pairs associated with the exotic family $\operatorname{Sol}(q)$ of Benson-Solomon 2-fusion systems [LO02, AC10]. Although these are defined for any odd prime power q, the fusion systems $\operatorname{Sol}(q)$ and $\operatorname{Sol}(q')$ are isomorphic if q^2-1 and q'^2-1 have the same 2-part. A Benson-Solomon system is known not to be the fusion system of any genuine block. This is a result of Kessar for the smallest such system [Kes06], while Craven extended Kessar's proof to the general case [Cra11, Theorem 9.34]. Thus, there exists no genuine "Benson-Solomon block" of the title. For the purposes of this paper, we simply regard a Külshammer-Puig pair of the form $(\mathcal{F},\alpha)$ with $\mathcal{F}$ a Benson-Solomon system as an avatar of the nonexistent block, one which allows us still to compute some local invariants, such as the number of weights, that such a block would have if it existed.[1]

Our first theorem determines the possible Külshammer-Puig classes that these fusion systems support.

Theorem 1.1. *Let $\mathcal{F}=\operatorname{Sol}(q)$. Then*

$$\lim_{[S(\mathcal{F}^c)]}\mathcal{A}^2_{\mathcal{F}} \cong \lim_{[S(\mathcal{F}^{cr})]}\mathcal{A}^2_{\mathcal{F}} = 0.$$

That is, each Benson-Solomon system supports a unique Külshammer-Puig pair.

Theorem 1.1 is shown by explicitly computing the $\mathcal{F}$-conjugacy classes of centric radical subgroups along with their outer automorphism groups in $\mathcal{F}$. The results of [AC10, Section 10] go a long way towards accomplishing such a task, but more details are required for the present applications. In Section 2 we refine the results of [AC10] to prove the following.

Theorem 1.2. *Let $\mathcal{F}=\operatorname{Sol}(q)$. Representatives for the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric radical subgroups, together with their $\mathcal{F}$-outer automorphism groups, are listed in Tables 1 and 4.*

Theorem 1.3. *The number of weights in the unique pair of Theorem 1.1 is*

$$\mathbf{w}(\operatorname{Sol}(q),0) = 12,$$

independently of q.

[1]See later work of Kessar, Malle, and the second author [KMS20, Section 6] for evidence that a Benson-Solomon block should be "the principal block of a $\mathbb{Z}_2$-spets for the 2-adic reflection group G_{24}".

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

The Benson-Solomon systems are finite versions of the simply connected 2-compact group $DI(4)$ of Dwyer and Wilkerson [Ben94]. As such, they have an associated 2-adic reflection group $W = W(\mathrm{Sol}(q)) \cong C_2 \times \mathrm{GL}_3(2)$, which in $\mathrm{Sol}(q)$, appears as the automorphism group of a finite 2-torus of rank 3, and $\mathbf{w}(\mathrm{Sol}(q), 0) = 12 = |\operatorname{Irr}(W)|$. This was later proved for all finite versions of simply connected ℓ-compact groups as long as ℓ is "very good" for W [KMS20, Theorem 1]. But note that the prime 2 is bad for $W(\mathrm{Sol}(q))$.

We prove Theorem 1.3 in Section 4 by explicitly computing $z(k\operatorname{Out}_{\mathcal{F}}(Q))$ for each of the groups Q appearing in Tables 1 and 4 of Theorem 1.2.

Beyond the weight conjecture, and assuming its validity, we have in mind other counting questions that can be considered for Külshammer-Puig pairs without reference to a group or a block. For example, Malle and Robinson recently conjectured that if b is a p-block associated to a finite group G, then the number of simple kG-modules in b should be bounded by $p^{s(S)}$, where S is a defect group of b and $s(S)$ denotes the sectional rank of S, namely, the largest rank of an elementary abelian section [MR17]. Moreover, they verified their conjecture in a large number of cases where the weight conjecture holds. In Lemma 2.19, we observe that the sectional rank of S is 6, and so the following conjecture, which was suggested to us by Kessar and Linckelmann, also holds easily for $\mathrm{Sol}(q)$.

Conjecture 1.4. *Let $(\mathcal{F}, \alpha)$ be a Külshammer-Puig pair, where $\mathcal{F}$ is a saturated fusion system on S. Then $\mathbf{w}(\mathcal{F}, \alpha) \leq p^{s(S)}$.*

This conjecture is just one small example in a host of other conjectures which are certain purely local analogues of the various local-to-global conjectures in the modular representation theory of finite groups. The local conjectures by their nature do not discriminate between realizable and exotic Külshammer-Puig pairs. They are discussed more fully in a sequel to this paper [KLLS19].

Outline and notation for the tables

After recalling certain initial results about fusion systems and the 2-local structure of $\mathrm{SL}_2(q)$, we set up in Section 2 notation for working in the Benson-Solomon systems and identify the important subgroups of the Aschbacher-Chermak free amalgamated product which realizes the systems. Section 2.7 provides an initial classification of some centric radical subgroups, namely, the centric radical subgroups lying above the 2-torsion in a maximal torus.

Section 3 contains the proof of Theorem 1.2, where the smallest Benson-Solomon system is handled separately (Section 3.1) from the larger ones (Section 3.2). The results are summarized in Tables 1 and 4. Those tables give a list of subgroups whose notation was fixed previously in Notation 2.11, Notation 2.12, Section 2.6, (3.1), or Notation 3.3.

Theorem 1.3 is proved in Section 4. Finally, in Section 5, we compute the Schur multipliers of the outer automorphism groups to give a proof of Theorem 1.1.

2. The Benson-Solomon fusion systems

2.1. Fusion system preliminaries

Throughout this paper, our group-theoretic nomenclature is standard and follows [Wil09], and we are usually consistent with the fusion-theoretic terminology and notation of [AKO11]. One exception to this is that we use exponential notation for images of subgroups and elements under a morphism in a fusion system, as described below. A *fusion system* on a finite p-group S is a category with object set the set of subgroups of S and with morphisms that are injective group homomorphisms, subject to two weak axioms. The standard example of a fusion system is that of a finite group G with Sylow p-subgroup S, where the morphisms are the conjugation homomorphisms between subgroups of S induced by elements of the group G, and which is denoted $\mathcal{F}_S(G)$. Due to the validity of Sylow's theorem in G and its p-local subgroups, the standard example satisfies two additional *saturation axioms*,

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

the Sylow and Extension axioms [BLO03, Definition 1.2]. All fusion systems in this paper are assumed to be (or known already to be) saturated unless otherwise stated, and we will sometimes drop that adjective and speak simply of a fusion system when there is no cause for confusion. For this subsection, we fix a saturated fusion system $\mathcal{F}$ over the p-group S. By analogy with the standard example, two subgroups of S are said to be *$\mathcal{F}$-conjugate* if they are isomorphic in the category $\mathcal{F}$. For a morphism $\varphi\colon P \to Q$ in $\mathcal{F}$, we write P^φ for the image of φ. Similarly, x^φ denotes the image of an element x under a morphism whose domain contains x.

Definition 2.1. Fix a subgroup $P \leqslant S$. We say that P is

(a) *fully $\mathcal{F}$-normalized* if $|N_S(P)| \geqslant |N_S(Q)|$ whenever Q is $\mathcal{F}$-conjugate to P,
(b) *$\mathcal{F}$-centric* if $C_S(Q) = Z(Q)$ for each $\mathcal{F}$-conjugate Q of P,
(c) *$\mathcal{F}$-radical* if $O_p(\mathrm{Out}_\mathcal{F}(P)) = 1$,
(d) *$\mathcal{F}$-centric radical* if it is both $\mathcal{F}$-centric and $\mathcal{F}$-radical, and
(e) *weakly $\mathcal{F}$-closed* if P is the only $\mathcal{F}$-conjugate of P,
(f) *strongly $\mathcal{F}$-closed* if each $\mathcal{F}$-conjugate of a subgroup of P is contained in P.

Denote by $\mathcal{F}^c$, $\mathcal{F}^r$, and $\mathcal{F}^{cr}$ the collection of $\mathcal{F}$-centric, $\mathcal{F}$-radical, and $\mathcal{F}$-centric radical subgroups of S, respectively.

The collections $\mathcal{F}^c$, $\mathcal{F}^r$, and $\mathcal{F}^{cr}$ are all closed under $\mathcal{F}$-conjugacy. Also, the $\mathcal{F}$-centric subgroups are closed under passing to overgroups.

Remark 2.2. Let G be a finite group with Sylow p-subgroup S. A p-subgroup P of G is said to be *p-radical* in G if $O_p(N_G(P)/P) = 1$. By contrast, a subgroup P is $\mathcal{F}_S(G)$-radical if and only if $O_p(N_G(P)/PC_G(P)) = 1$. The collection of p-radical subgroups of G contained in S does *not* coincide, in general, with the collection of $\mathcal{F}_S(G)$-radical subgroups.

For example, let $p = 3$ and $G = G_1 \times G_2$ with $G_i \cong D_6$. The subgroup $P = S \cap G_1$ has order 3 with $N_G(P)/P \cong C_2 \times D_6$, so P is not 3-radical in G. However, $\mathrm{Out}_{\mathcal{F}_S(G)}(P) = N_G(P)/PC_G(P) \cong C_2$, so P is $\mathcal{F}_S(G)$-radical. Conversely, take $p = 2$, but instead $G = D_{24}$, and P of order 4 in the cyclic maximal subgroup. Then $\mathrm{Out}_{\mathcal{F}_S(G)}(P) \cong C_2$ so P is not $\mathcal{F}_S(G)$-radical, but $N_G(P)/P \cong D_6$, so P is 2-radical in G. This distinction is important in Lemma 2.7 below, where both concepts appear simultaneously. It is also relevant in Chevalley groups $G = G(q)$ with q odd, which have an element in the Weyl group inverting a split maximal torus. When such a torus has a nontrivial odd order normal subgroup (often the case), a Sylow 2-subgroup T of such a torus is 2-radical in G but not radical in $\mathcal{F}_S(G)$, where S is a Sylow 2-subgroup of $G(q)$ containing T. This situation occurs, for example, when $G(q) = \mathrm{Spin}_7(q)$, q odd, $q \neq 3, 5$.

Definition 2.3. Fix a subgroup $P \leqslant S$.

(a) The *normalizer* $N_\mathcal{F}(P)$ of P is the fusion system on $N_S(P)$ consisting of those morphisms $\varphi\colon Q \to R$ in $\mathcal{F}$ for which there exists an extension $\tilde{\varphi}\colon PQ \to PR$ of φ in $\mathcal{F}$, such that $P^{\tilde{\varphi}} = P$.
(b) The *centralizer* $C_\mathcal{F}(P)$ of P is the fusion system on $C_S(P)$ consisting of those morphisms $\varphi\colon Q \to R$ in $\mathcal{F}$ for which there exists an extension $\tilde{\varphi}\colon PQ \to PR$ of φ in $\mathcal{F}$, such that the restriction $\tilde{\varphi}|_P$ is the identity on P.
(c) The subgroup $P \leqslant S$ is *normal* in $\mathcal{F}$ if $\mathcal{F} = N_\mathcal{F}(P)$.
(d) $\mathcal{F}$ is *constrained* if $\mathcal{F}$ has a centric normal subgroup.

These centralizer and normalizer fusion systems are not always saturated, but they are both saturated provided P is fully $\mathcal{F}$-normalized.

Lemma 2.4. *If P is $\mathcal{F}$-centric, then $C_\mathcal{F}(P) = \mathcal{F}_{Z(P)}(Z(P))$.*

Proof. Assume that P is $\mathcal{F}$-centric. The centralizer system $C_\mathcal{F}(P)$ is a fusion system over the abelian group $C_S(P) = Z(P)$, and $Z(P)$ is normal in $C_\mathcal{F}(P)$ from the definitions. As each morphism between subgroups of $Z(P)$ in $C_\mathcal{F}(P)$ extends to act as the identity on P, each such morphism is an identity map. ◻

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Lemma 2.5. *Suppose that $P \leqslant S$ is normal in $\mathcal{F}$. Then P is contained in every $\mathcal{F}$-centric radical subgroup.*

Proof. Let $Q \in \mathcal{F}^{cr}$. Then $\mathrm{Aut}_{PQ}(Q)$ is normal in $\mathrm{Aut}_{\mathcal{F}}(Q)$, and so $\mathrm{Aut}_{PQ}(Q) \leqslant \mathrm{Inn}(Q)$ since Q is radical. Then $P \leqslant PQ \leqslant QC_S(Q) = Q$ with the equality because Q is centric. □

The next two lemmas give applications of the Extension axiom. The second is useful for locating the $\mathcal{F}$-centric radicals that contain a given weakly $\mathcal{F}$-closed subgroup.

Lemma 2.6. *Let $P' \leqslant S$ be fully $\mathcal{F}$-normalized, and let P be a subgroup of S which is $\mathcal{F}$-conjugate to P'. Then there exists a morphism $\alpha \in \mathrm{Hom}_{\mathcal{F}}(N_S(P), N_S(P'))$, such that $P^\alpha = P'$.*

Proof. See [AKO11, I.2.6(c)]. □

Lemma 2.7. *Let W be an $\mathcal{F}$-centric and weakly $\mathcal{F}$-closed subgroup of S. For any subgroup P of S containing W, restriction induces an isomorphism*

$$\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_W(P) \longrightarrow N_{\mathrm{Out}_{\mathcal{F}}(W)}(\mathrm{Out}_P(W))$$

and therefore an isomorphism

$$\mathrm{Out}_{\mathcal{F}}(P) \longrightarrow N_{\mathrm{Out}_{\mathcal{F}}(W)}(\mathrm{Out}_P(W))/\mathrm{Out}_P(W).$$

Hence, the map $P \mapsto \mathrm{Out}_P(W)$ is a bijection between the collection of $\mathcal{F}$-centric radical subgroups containing W and the collection of subgroups of $\mathrm{Out}_S(W)$ which are p-radical in the group $\mathrm{Out}_{\mathcal{F}}(W)$.

Proof. Consider the restriction map $\rho\colon \mathrm{Aut}_{\mathcal{F}}(P) \to N_{\mathrm{Aut}_{\mathcal{F}}(W)}(\mathrm{Aut}_P(W))$, under which $\mathrm{Aut}_W(P)$ maps onto $\mathrm{Inn}(W)$ and under which $\mathrm{Inn}(P)$ maps onto $\mathrm{Aut}_P(W)$. Since W is weakly closed, it is fully $\mathcal{F}$-normalized by Lemma 2.6. A direct application of the Extension axiom [BLO03, Definition 1.2(II)] then gives that ρ is surjective. Since W is $\mathcal{F}$-centric, the centralizer in $\mathcal{F}$ of the centric subgroup W is the fusion system of $Z(W)$ by Lemma 2.4, so the kernel of ρ is $\mathrm{Aut}_{Z(W)}(P)$, which is contained in $\mathrm{Aut}_W(P) \subseteq \mathrm{Inn}(P)$. The induced map

$$\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_W(P) \longrightarrow N_{\mathrm{Aut}_{\mathcal{F}}(W)}(\mathrm{Aut}_P(W))/\mathrm{Aut}_W(W) \cong N_{\mathrm{Out}_{\mathcal{F}}(W)}(\mathrm{Out}_P(W))$$

is an isomorphism, and therefore upon factoring by $\mathrm{Aut}_P(P)/\mathrm{Aut}_W(P)$, the induced map

$$\mathrm{Out}_{\mathcal{F}}(P) \longrightarrow N_{\mathrm{Aut}_{\mathcal{F}}(W)}(\mathrm{Aut}_P(W))/\mathrm{Aut}_P(W) \cong N_{\mathrm{Out}_{\mathcal{F}}(W)}(\mathrm{Out}_P(W))/\mathrm{Out}_P(W) \tag{2.1}$$

is an isomorphism.

Observe that W is normal in S because it is weakly $\mathcal{F}$-closed. So $\mathrm{Out}_P(W) \cong P/W$ since $C_S(W) \leqslant W$. The map $P \mapsto \mathrm{Out}_P(W)$ is therefore a bijection between the subgroups containing W and the subgroups of $\mathrm{Out}_S(P)$. By (2.1), $\mathrm{Out}_{\mathcal{F}}(P)$ corresponds to $N_{\mathrm{Out}_{\mathcal{F}}(W)}(\mathrm{Out}_P(W))/\mathrm{Out}_P(W)$ under the bijection, so P is $\mathcal{F}$-radical if and only if $\mathrm{Out}_P(W)$ is p-radical in the group $\mathrm{Out}_{\mathcal{F}}(W)$ (Remark 2.2). The last statement now follows because the collection of $\mathcal{F}$-centric subgroups is closed under passing to overgroups. □

2.2. *Quaternion groups and the 2-local structure of* $\mathrm{SL}_2(q)$

It will be convenient to recall here standard facts about the 2-local structure of $\mathrm{SL}_2(q)$, where q is an odd prime power. For reasons that will become apparent in a moment, we set $l \geqslant 0$ and take $q = q_l = 5^{2^l}$ for simplicity of exposition. Given this notation, $\mathrm{SL}_2(q)$ has generalized quaternion Sylow 2-subgroups of order 2^{l+3}, and this can be seen as follows. First, the size of a Sylow 2-subgroup can be deduced from the order $q(q-1)(q+1)$ of $\mathrm{SL}_2(q)$, together with the fact that the 2-adic valuation $v_2(5^{2^l} - 1)$ is $l+2$.

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

By the choice of q, the multiplicative group $\mathbb{F}_q^\times$ contains a primitive root of unity ω of order 2^{l+2}. Thus,

$$a := \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

generate a Sylow 2-subgroup of $\mathrm{SL}_2(q)$ by order considerations. Since a and b satisfy the relations

$$a^{2^{l+2}} = b^4 = 1, \quad a^{2^{l+1}} = b^2, \quad b^{-1}ab = a^{-1}, \tag{2.2}$$

we see that $R := \langle a, b\rangle$ is a generalized quaternion group of order 2^{l+3}. The following lemma records some basic facts about the subgroup structure of a generalized quaternion group.

Lemma 2.8. *The following hold:*

(a) *each element of R is of the form $a^i b^j$ with $0 \le i \le 2^{l+2} - 1$ and $0 \le j \le 1$;*
(b) *each element in $R\backslash\langle a\rangle$ is of order 4;*
(c) *$a^i b$ is conjugate to $a^j b$ if and only if $i \equiv j \bmod 2$, where $0 \le i, j \le 2^{l+2} - 1$;*
(d) *the set $\mathcal{Q}$ of subgroups of R isomorphic with Q_8 is given by $\{ \langle a^{2^l}, a^i b\rangle \mid 0 \le i \le 2^{l+2} - 1 \}$;*
(e) *when $l > 0$, there are two R-conjugacy classes, each of size 2^{l-1}, of subgroups isomorphic to Q_8, and $Q := \langle a^{2^l}, b\rangle$ and $Q' := \langle a^{2^l}, ab\rangle$ are representatives of these classes; and*
(f) *when $l > 0$, $N_S(Q) = \langle Q, a^{2^{l-1}}\rangle$ and $N_S(Q') = \langle Q', a^{2^{l-1}}\rangle$.*

Proof. Part (a) is clear, and (b) follows since, for each i,

$$(a^i b)^2 = a^i b a^i b = b a^{-i} a^i b = b^2$$

has order 2. A general element $a^j b^m$ as in (a) conjugates $a^i b$ to

$$b^{-m} a^{-j} a^i b a^j b^m = (b^{-m} a^{i-2j} b^m) b = \begin{cases} a^{i-2j} b, & \text{if } m = 0 \\ a^{2j-i} b, & \text{if } m = 1 \end{cases}$$

from which the claim in (c) follows.

Let $\mathcal{Q}$ be the set of subgroups of R isomorphic to Q_8 as in (d), and fix $Q \in \mathcal{Q}$. As $\langle a\rangle$ is cyclic of index 2 in R, we have $Q\langle a\rangle = R$, and so $Q \cap \langle a\rangle = \langle a^{2^l}\rangle$ by order considerations. This shows that Q is of the form $\langle a^{2^l}, a^i b\rangle$ for some i. Conversely, for each i, the elements a^{2^l} and $a^i b$ satisfy the relations (2.2), applied with $l = 0$, in place of a and b, respectively. Hence, $\langle a^{2^l}, a^i b\rangle \cong Q_8$, and so $\langle a^{2^l}, a^i b\rangle \in \mathcal{Q}$. This completes the proof of (d).

Note that exactly four elements of the form $a^i b$ lie in a given member of $\mathcal{Q}$. Since there are 2^{l+2} choices for i, $\mathcal{Q}$ has cardinality $2^{l+2}/4 = 2^l$. Part (e) now follows from the conjugacy information in (c), while (f) follows from the observation that $b^a = a^{-2} b$ so that $b^{a^i} = a^{-2i} b$. □

Since $v_2(q-1) = l+2$ and ω is a primitive 2^{l+2} root of unity, $\sqrt{\omega} \notin \mathbb{F}_q^\times$. So $p(t) := t^2 - \omega$ is irreducible in $\mathbb{F}_q$, and $\mathbb{F}_0 := \mathbb{F}_q[t]/p(t)$ is a finite field of order q^2 containing $\mathbb{F}_q$. Set $c := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_0)$.Then straightforward computations show that $c^2 = a$ (so c has order 2^{l+3}) and that $c^{-1}bc = ba$ and $cbc^{-1} = ab$. Hence, by Lemma 2.8 (d),(e), c fuses the two conjugacy classes of subgroups of R isomorphic with Q_8.

Finally, we will need the following lemma, which we will usually use in Section 3 without further comment. For a discussion of (2), see, for example [Cra11, Theorem 4.54].

Lemma 2.9. *For $q = 5^{2^l}$, as above, let $\mathcal{F} := \mathcal{F}_R(\mathrm{SL}_2(q))$ be the 2-fusion system of $\mathrm{SL}_2(q)$.*

1. *If $l = 0$, then $\mathcal{F}$ is constrained with centric normal subgroup R, $N_{\mathrm{SL}_2(q)}(R) \cong \mathrm{SL}_2(3)$, and $\mathrm{Out}_{\mathcal{F}}(R) \cong C_3$.*
2. *If $l > 0$, then $\{R, Q, Q'\}$ is a complete set of $\mathcal{F}$-conjugacy class representatives of $\mathcal{F}$-centric radical subgroups. Moreover, $N_{\mathrm{SL}_2(q)}(Q) \cong N_{\mathrm{SL}_2(q)}(Q') \cong \mathrm{GL}_2(3)$, $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathrm{Out}_{\mathcal{F}}(Q') \cong S_3$, and $\mathrm{Out}_{\mathcal{F}}(R) = 1$.*

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

2.3. $\mathrm{Spin}_7(q)$

Let q be an odd prime power, and let V be an odd dimensional vector space over $\mathbb{F}_q$. Let $\mathfrak{q}$ be a nondegenerate quadratic form on V and $\mathfrak{b}$ the associated symmetric bilinear form, which determine each other via $\mathfrak{q}(v) = \mathfrak{b}(v, v)$ and $\mathfrak{b}(v, w) = \frac{1}{2}(\mathfrak{q}(v+w) - \mathfrak{q}(v) - \mathfrak{q}(w))$. Let $(V, \mathfrak{q})$ be the associated geometric space, and $\mathrm{O}(V) = \mathrm{O}(V, \mathfrak{q})$ the isometry group. There are two such forms $\mathfrak{b}$ up to equivalence, and the corresponding isometry groups are isomorphic. We may therefore take $\mathfrak{b}$ to be of square discriminant when convenient. We have $\mathrm{O}(V) = \{\pm 1\} \times \mathrm{SO}(V)$. The spinor norm $\mathrm{SO}(V) \to \mathbb{F}_q^{\times}/\mathbb{F}_q^{\times 2}$ is defined by writing an element of $\mathrm{SO}(V)$ as a product of reflections, and then taking the product of the discriminants of the -1-eigenspaces of those reflections. The kernel of the spinor norm is the simple subgroup $\Omega(V)$. Let $\mathrm{Spin}(V)$ be the perfect double cover of $\Omega(V)$, and write Z for the center of $\mathrm{Spin}(V)$. Thus, $Z = \langle z \rangle$ is of order 2. We sometimes speak of the action of an element of $\mathrm{Spin}(V)$ on V, we mean the action of the image of the element in $\Omega(V)$.

We generally refer to [LO02, Appendix A] and [AC10, Section 4] for information on the construction and subgroup structure of the Spin groups but record the following basic lemma for use in Section 3.

Lemma 2.10. *An involution in* $\Omega(V)$ *lifts to an involution in* $\mathrm{Spin}(V)$ *if and only if the dimension of its* -1*-eigenspace is a multiple of* 4.

Proof. See [LO02, Lemma A.4(b)]. □

From now, take V to be of dimension 7. To help motivate some of the definitions in the next subsection, we describe very roughly the structure of the normalizer of a four subgroup containing Z in $\mathrm{Spin}_7(q)$. For more information and proofs, we refer the reader to Proposition 2.5(b) of [LO02] (which views $\mathrm{Spin}(V)$ classically) or Lemma 4.3(b) of [HLL23] (for a Lie theoretic approach). Lemma 2.10 implies that $\mathrm{Spin}_7(q) := \mathrm{Spin}(V)$ has two classes of involutions, namely, those with representatives given by the central involution $z \in Z(\mathrm{Spin}_7(q))$ and by the preimage of an involution with -1-eigenspace of dimension 4. Let V_1 be a nondegenerate subspace of dimension 4 (and Witt index 2), and let V_2 be its orthogonal complement. Let $z_1 \in \mathrm{Spin}(V)$ be an element whose image in $\Omega(V)$ is an involution with -1-eigenspace V_1 (thus, z_1 is an involution by Lemma 2.10). Setting $U = \langle z, z_1 \rangle$, the normalizer $B := N_{\mathrm{Spin}(V)}(U)$ contains the normal subgroup $C_B(V_2)C_B(V_1)$ with index 4, isomorphic to the commuting product

$$\mathrm{Spin}(V_1) * \mathrm{Spin}(V_2) \cong (\mathrm{SL}_2(q) \times \mathrm{SL}_2(q) \times \mathrm{SL}_2(q))/\langle(-1, -1, -1)\rangle.$$

There is a four group complementing $C_B(V_2)C_B(V_1)$ in B, which contains an involution interchanging the first two $\mathrm{SL}_2(q)$'s and centralizing the third (and whose image in $\Omega_7(q)$ acts as -1 on V_2), and which contains an involution acting simultaneously as a diagonal automorphism on each $\mathrm{SL}_2(q)$ factor.

All additional information about $\mathrm{Spin}_7(q)$ that we require directly will be collected later in Lemmas 2.14 through 2.18, in Proposition 3.2, and in the proof of Lemma 3.8.

2.4. Construction of $\mathrm{Sol}(q)$

Following work of Solomon [Sol74], the Benson-Solomon systems were predicted to exist by Benson [Ben98c], and then later constructed by Levi and Oliver [LO02, LO05]. They are exotic in the sense that they are not of the form $\mathcal{F}_S(G)$ for any finite group G with Sylow 2-subgroup S. They are also not the fusion system of any 2-block of a finite group [Kes06], [Cra11, Section 9.4], an a priori stronger statement. After Levi and Oliver, Aschbacher and Chermak gave a different construction of the Benson-Solomon systems as the fusion system of a certain free amalgamated product of two finite groups having Sylow 2-subgroup isomorphic to $\mathrm{Spin}_7(q)$ [AC10]. We primarily view $\mathrm{Sol}(q)$ through the lens of [AC10], so we consider it as the 2-fusion system of an amalgamated product $G = H *_B K$, where $H := \mathrm{Spin}_7(q)$.

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

The isomorphism type of the Benson-Solomon system $\mathrm{Sol}(q)$ depends not on q, but only (uniquely) on the 2-adic valuation of q^2-1 by [COS08, Theorem 3.4]. For reasons of exposition, it will be helpful therefore to fix the following choice of q: unless otherwise specified, for the remainder of this section and the next, we

let l be a fixed but arbitrary nonnegative integer, and set $q = 5^{2^l}$.

We have described how B arises as a subgroup of H in Subsection 2.3 (but the explicit embedding $B \hookrightarrow H$ in the amalgam is not the "obvious" one). We now take a more abstract approach to obtain a working description of K in Aschbacher-Chermak free amalgamated product, as follows. Consider the natural inclusion $\mathrm{SL}_2(q) \le \mathrm{SL}_2(q^2)$ induced by an inclusion of fields, and define $N := N_{\mathrm{SL}_2(q^2)}(\mathrm{SL}_2(q))$ so that $|N : \mathrm{SL}_2(q)| = 2$ and N and $\mathrm{SL}_2(q)$ both have generalized quaternion Sylow 2-subgroups, as explained more fully in Subsection 2.2. Form the wreath product $W := N \wr S_3$, and let $N_0 := N_1 \times N_2 \times N_3$ and $X := S_3$ be the base and acting group respectively. Note that $O^2(N_0) \trianglelefteq W$ is a direct product $\widehat{L}_1 \times \widehat{L}_2 \times \widehat{L}_3$ of three copies of $\mathrm{SL}_2(q)$ permuted transitively by X.

Define $\widehat{K} := O^2(N_0)C_{N_0}(X)X$ regarded as the group generated by the wreath product $O^2(N_0) \rtimes X$, and an element of $N_0 \backslash O^2(N_0)$ acting in the same way simultaneously on each factor $\widehat{L}_i$ of $O^2(N_0)$. Thus, $Z(O^2(N_0)) = Z(O^2(N_0)C_{N_0}(X)) = \langle(\pm 1, \pm 1, \pm 1)\rangle$ and $Z(\widehat{K}) = \langle(-1,-1,-1)\rangle$. Here, we write 1 for the identity matrix. Finally, set

$$K := \widehat{K}/Z(\widehat{K}).$$

We will write $[a_1, a_2, a_3]$, for example, for the image in K of an element (a_1, a_2, a_3) of $O^2(N_0)C_{N_0}(X)$.

Notation 2.11. We fix the following notation for certain subgroups of K.

(a) $L_i \cong \mathrm{SL}_2(q)$ for $i = 1, 2, 3$ are the images in K of the subgroups $\widehat{L}_i$ of $\widehat{K}$;
(b) $L_0 := L_1L_2L_3$;
(c) $X \cong S_3$ is the image in K of the subgroup with the same name;
(d) $\tau \in X$ is the permutation $(1, 2)$ on the indices of the L_i;
(e) S is a Sylow 2-subgroup of K containing τ;
(f) $U = Z(L_0) = \langle[\pm 1, \pm 1, \pm 1]\rangle \cong C_2 \times C_2$; and
(g) $B := L_0S$.

Thus, the subgroup B in Notation 2.11(g) is a subgroup of K of index 3, and $B \cap X = \langle\tau\rangle$. As was shown in [AC10], there is a four subgroup $U \leqslant H$, such that $B \cong N_H(U)$, and a choice of injection $\iota : B \hookrightarrow H$, such that the free amalgamated product $G = H *_B K$ has finite Sylow 2-subgroup S and determines a saturated fusion system $\mathrm{Sol}(q)$ over S that was constructed by Levi and Oliver by different means [LO02, LO05]. An incorrect choice of ι can lead to a fusion system which is not saturated (see [AC10, Section 5] and [LO05] for more details, but generally, this subtlety will be unimportant in our computations).

It will be helpful to introduce some more notation. Some of it follows the notation of [AC10, Section 10] in preparation for the application in Section 3 of some of the results there.

Notation 2.12. We fix the following additional notation for subgroups and elements of K.

(a) $R_i \cong Q_{2^{l+3}}$ is a Sylow 2-subgroup of L_i for $i = 1, 2, 3$, chosen so that $X \cong S_3$ acts on the set $\{R_1, R_2, R_3\}$;
(b) $R_0 := R_1R_2R_3 \in \mathrm{Syl}_2(L_0)$;
(c) $\mathcal{Q}_i$ is the set of subgroups of R_i isomorphic to Q_8 for $i = 1, 2, 3$; thus, $\mathcal{Q}_i = \{R_i\}$ if $l = 0$, while $\mathcal{Q}_i$ is a union of two R_i-conjugacy classes of subgroups if $l > 0$ by Lemma 2.8(e);
(d) when $l > 0$, $Q_i, Q_i' \in \mathcal{Q}_i$ are representatives for the two R_i-conjugacy classes of subgroups chosen so that $X \cong S_3$ acts by permuting the sets $\{Q_1, Q_2, Q_3\}$ and $\{Q_1', Q_2', Q_3'\}$;
(e) $\mathbf{c} := [c, c, c]$, where c is as in Section 2.2, so that $\mathbf{c}$ acts simultaneously on $L_i \cong SL_2(q)$ by conjugation in the way described there;

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

(f) $\mathbf{d} := [b, b, b]\mathbf{c} \in K$, where b is as in Section 2.2, an involution commuting with τ; and
(g) $\tau' = \mathbf{d}\tau$.

Note that $\langle \mathbf{d}, \tau \rangle$ is a four group which intersects R_0 trivially. Thus, refining Notation 2.11(e), we fix the following Sylow 2-subgroup of K throughout the remainder of this section and in Section 3:

$$S = R_0 \langle \mathbf{d}, \tau \rangle.$$

Then S is isomorphic with a Sylow 2-subgroup of H, R_0 is normal in S with complement $\langle \mathbf{d}, \tau \rangle$, R_3 is normal in S, and $\mathbf{d}$ interchanges the two R_i-conjugacy classes of subgroups isomorphic with Q_8 when $l > 0$. Finally, we define

$$\mathcal{K} := \mathcal{F}_S(K), \quad \mathcal{H} := \mathcal{F}_S(H) \quad \text{and} \quad \mathcal{F} := \mathcal{F}_S(G).$$

We note that $\mathcal{F}$ is the fusion system generated by $\mathcal{H}$ and $\mathcal{K}$ by [Sem14, Theorem 3.3], namely, $\mathcal{F}$ is the smallest fusion system on S containing all morphisms in $\mathcal{H}$ and $\mathcal{K}$.

2.5. *The torus of $\mathcal{F}$*

The next lemma calls attention to the 2-power torsion subgroup $T \leqslant S$ in a maximal torus of H. As a subgroup of K, it may be generated by the elements $[a, 1, 1]$, $[1, a, 1]$, $[c, c, c]$ in the notation of Section 2.2, and it is inverted by the involution $\mathbf{d}$. In the lemma and elsewhere, we refer to Borel and parabolic subgroups of $C_2 \times \mathrm{GL}_3(2)$, as an algebraic group over $\mathbb{F}_2$ with unipotent radical C_2. Thus, a Borel subgroup is the stabilizer of a maximal flag in the three-dimensional $\mathbb{F}_2$-representation of $C_2 \times \mathrm{GL}_3(2)$ which is natural for $\mathrm{GL}_3(2)$ and has C_2 in its kernel, and the two maximal parabolic subgroups are similarly stabilizers of proper subspaces and isomorphic to $C_2 \times S_4$.

Lemma 2.13. *There is a unique subgroup T of S isomorphic to $(C_{2^{l+2}})^3$. The centralizer $C_H(T)$ is a split maximal torus of H; in particular, $C_S(T) = T$. The subgroup T is $\mathcal{F}$-centric and weakly $\mathcal{F}$-closed. Moreover,* $\mathrm{Out}_S(T) = S/T \cong C_2 \times D_8$, $\mathrm{Out}_{\mathcal{F}}(T) \cong C_2 \times \mathrm{GL}_3(2)$, *and* $\mathrm{Out}_{\mathcal{H}}(T) \cong C_2 \times S_4$ *is the maximal parabolic in* $\mathrm{Out}_{\mathcal{F}}(T)$*, lying over the Borel subgroup* $\mathrm{Out}_S(T)$*, given by the stabilizer of Z in the action of* $\mathrm{Out}_{\mathcal{F}}(T)$ *on* $\Omega_1(T)$.

Proof. By [AC10, Lemma 4.9(c)], there is a unique homocyclic subgroup of S of rank 3 and exponent 4, T is the centralizer in S of that subgroup, and $C_H(T)$ is a split maximal torus of H. Since T is abelian, this shows that T is the unique subgroup of S of its isomorphism type. Then [AC10, Lemmas 4.3 and 4.8] show that $S/T \cong C_2 \times D_8$, and $\mathrm{Out}_{\mathcal{H}}(T) \cong C_2 \times S_4$. The structure of the outer automorphism group $\mathrm{Out}_{\mathcal{F}}(T)$ follows from the construction of the Aschbacher-Chermak amalgam in [AC10, Lemma 5.2]. All other points follow. □

2.6. *The standard elementary abelian chain in S*

We refer to Sections 4 and 7 of [AC10] for more discussion on the following items. Set $z := [-1, -1, 1] = [1, 1, -1] \in S$, so $z \in Z(\mathrm{Spin}_7(q))$ as in Section 2.3. There is a chain of elementary abelian subgroups

$$Z < U < E < A$$

of ranks 1, 2, 3, and 4, respectively, where $Z = Z(S) = \langle z \rangle$, U is the unique normal four subgroup of S of Notation 2.11(f), $E = \Omega_1(T) = \langle [-1, 1, 1], [1, -1, 1], [a^{2^l}, a^{2^l}, a^{2^l}] \rangle$, and $A = E\langle \mathbf{d} \rangle$. For a member X_n of the above chain of rank n, $\mathrm{Aut}_{\mathcal{F}}(X_n) = \mathrm{Out}_{\mathcal{F}}(X_n) \cong \mathrm{GL}_n(2)$ by [LO02, Lemma 3.1]. Also, $\mathcal{H} = C_{\mathcal{F}}(Z)$ and $\mathcal{K} = N_{\mathcal{F}}(U)$ by [AC10, Proposition 9.2].

2.7. Centric radicals containing the torus

In the next five lemmas, we identify, using Lemma 2.7, the outer automorphism groups of the centric radical subgroups that contain the 2-torus T.

Lemma 2.14. *The subgroup $C_S(E)$ of S is $\mathcal{F}$-centric and weakly $\mathcal{F}$-closed, $C_S(E) = T\langle \mathbf{d} \rangle$, and $C_H(C_S(E)) = Z(C_S(E))) = E$. Moreover, $\mathrm{Out}_S(C_S(E)) = S/C_S(E) \cong D_8$, $\mathrm{Out}_{\mathcal{F}}(C_S(E)) \cong \mathrm{GL}_3(2)$, and $\mathrm{Out}_{\mathcal{H}}(C_S(E)) \cong S_4$ is the maximal parabolic in $\mathrm{Out}_{\mathcal{F}}(C_S(E))$ given by the stabilizer of Z under the natural action of $\mathrm{Out}_{\mathcal{F}}(C_S(E))$ on E.*

Proof. Since $E = \Omega_1(T)$, we have $C_S(E) \geqslant T$. As T is $\mathcal{F}$-centric, so is $C_S(E)$. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(C_S(E), S)$. By Lemma 2.13, $T^\varphi = T$, so also $E^\varphi = E$. Hence, $C_S(E)^\varphi \leqslant C_S(E^\varphi) = C_S(E)$, and so $C_S(E)$ is weakly $\mathcal{F}$-closed. From the description of $\mathrm{Out}_{\mathcal{F}}(T)$ in Lemma 2.13, the kernel of the action of S/T on E is of order 2. Now $\mathbf{d} \in S$ inverts T, so centralizes $E = \Omega_1(T)$. Hence, $\mathbf{d}$ represents the lone nontrivial coset of $C_H(T)$ in $C_H(E)$, whose elements invert the maximal torus $C_H(T)$ of H containing T (see [AC10, Lemma 4.3(a,d)]). So $C_S(E) = T\langle \mathbf{d} \rangle$, and $C_H(C_S(E)) \leqslant T$ from Lemma 2.13. Hence, the center $Z(C_S(E))$ is $C_H(C_S(E)) = C_T(\mathbf{d}) = E$.

As $O_2(\mathrm{Out}_{\mathcal{F}}(T)) = \mathrm{Out}_{C_S(E)}(T)$, the descriptions of the outer automorphism groups in $\mathcal{F}$ and $\mathcal{H}$ follow from Lemmas 2.7 and 2.13. □

Lemma 2.15. *$N_H(S) = S$ and $\mathrm{Out}_{\mathcal{H}}(S) = \mathrm{Out}_{\mathcal{F}}(S) = \mathrm{Out}_{\mathcal{K}}(S) = 1$.*

Proof. Since $C_S(E)$ contains its centralizer in H from Lemma 2.14, so does S. Then as the Sylow 2-subgroups of S_4 and $\mathrm{GL}_3(2)$ are self-normalizing, the lemma now follows from Lemmas 2.7 and 2.14. □

Lemma 2.16. *The subgroup $C_S(U)$ of S is $\mathcal{F}$-centric and weakly $\mathcal{F}$-closed, and $Z(C_S(U)) = U$. The quotient $C_S(U)/C_S(E)$ is the unipotent radical of the stabilizer in $\mathrm{Out}_{\mathcal{F}}(C_S(E))$ of U. Thus, $\mathrm{Out}_S(C_S(U)) = \mathrm{Out}_{\mathcal{H}}(C_S(U)) \cong C_2$ is induced by $\langle \tau \rangle$, and $\mathrm{Out}_{\mathcal{F}}(C_S(U)) \cong S_3$ is induced by X.*

Proof. From the structure of $\mathrm{Out}_{\mathcal{F}}(C_S(E))$ in Lemma 2.14, $\mathrm{Out}_{C_S(U)}(C_S(E)) = C_S(U)/C_S(E)$ is the unipotent radical of the stabilizer of U in the action of $\mathrm{Out}_{\mathcal{F}}(C_S(E))$ on E, so, in particular, $Z(C_S(U)) = C_E(C_S(U)) = U$. The descriptions of the outer automorphism groups now follow from Lemmas 2.7 and 2.14 and the structure of $\mathrm{GL}_3(2)$. □

Lemma 2.17. *The subgroup $C_S(E/Z) = \{s \in S \mid [E, s] \leqslant Z\}$ is $\mathcal{F}$-centric and weakly $\mathcal{F}$-closed, and $Z(C_S(E/Z)) = Z$. The quotient $C_S(E/Z)/C_S(E)$ is the unipotent radical of the stabilizer in $\mathrm{Out}_{\mathcal{F}}(C_S(E))$ of Z in the natural action on E. Thus, $\mathrm{Out}_{\mathcal{H}}(C_S(E/Z)) = \mathrm{Out}_{\mathcal{F}}(C_S(E/Z)) \cong S_3$.*

Proof. Observe that, $C_S(E) \leqslant C_S(E/Z)$ and that $C_S(E/Z)/C_S(E)$ is the group of transvections in $\mathrm{Out}_{\mathcal{F}}(C_S(E))$ on E with center Z. So $C_S(E/Z)/C_S(E)$ is the unipotent radical of the stabilizer of Z. Also, as $Z(C_S(E)) = Z$ from Lemma 2.14, we have $Z(C_S(E/Z)) = C_E(C_S(E/Z)) = Z$. Since $C_S(E)$ is $\mathcal{F}$-centric, weakly $\mathcal{F}$-closed, and $\mathrm{Aut}_{\mathcal{H}}(C_S(E)) = C_{\mathrm{Aut}_{\mathcal{F}}(C_S(E))}(Z)$, all points follow from Lemmas 2.7 and 2.14 as in the previous lemma. □

Lemma 2.18. *The collection of $\mathcal{F}$-centric radical subgroups containing T is $\{C_S(E), C_S(U), C_S(E/Z), S\}$. The collection of $\mathcal{H}$-centric radical subgroups containing T is $\{C_S(E/Z), S\}$.*

Proof. There are four 2-radical subgroups in $\mathrm{GL}_3(2)$ inside a fixed Sylow 2-subgroup: the identity subgroup and the unipotent radicals of the three associated parabolics. So the lemma follows from the bijection of Lemma 2.7 together with Lemmas 2.14–2.17. □

2.8. The sectional rank of S

Before continuing, we record the sectional rank of S using the later Proposition 3.2, which locates an extraspecial subgroup of order 2^7 in S.

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Lemma 2.19. *The sectional rank of S is* 6.

Proof. By Lemma 3.2(a) below, S contains an extraspecial subgroup with central quotient of rank 6, and hence $s(S) \geqslant 6$. On the other hand, the sectional rank of a group is at most the sum of the sectional ranks of a normal subgroup and corresponding quotient, so Lemma 2.13 shows that $s(S) \leqslant s(T) + s(S/T) = 3 + 3 = 6$. □

3. Centric radicals in $\mathrm{Sol}(q)$

The aim of this section is to refine the description of the centric radical subgroups of a Benson-Solomon system that results from a combination of [AC10, Section 10] and [COS08, Section 2]. A starting point is the next result due to Aschbacher and Chermak, which allows us to work in the groups H and K separately. Adopt the notation from Section 2, and, in particular, from Notations 2.11 and 2.12 and Sections 2.5 and 2.6. Recall that G is the Aschbacher-Chermak free amalgamated product, and that $\mathcal{F} = \mathcal{F}_S(G)$.

Proposition 3.1. *Up to $\mathcal{F}$-conjugacy, a subgroup $P \leqslant S$ is $\mathcal{F}$-centric radical if and only if*

(a) *$P = A$ is elementary abelian of order 2^4 and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{GL}_4(2)$;*
(b) *$P = C_S(E)$ and $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{GL}_3(2)$;*
(c) *Either:*
   (i) *$N_G(P) \leq K$ and $P \in \mathcal{K}^{cr}$; or*
   (ii) *$N_G(P) \leq H$ and $P \in \mathcal{H}^{cr}$.*

Proof. See [AC10, Lemma 10.9]. □

For the smallest Benson-Solomon system, the results of [COS08], when combined with Proposition 3.1, supply sufficiently precise information for our needs, as we make clear in Section 3.1. For the larger systems, Proposition 3.2 below yields a sufficiently detailed description for the centric radicals occurring in Proposition 3.1(c)(ii), whose normalizer in G is not contained in K.

Recall that $(V, \mathfrak{q})$ is the orthogonal space from Section 2.3; $\mathfrak{q}(v)$ is referred to as the *norm* of the vector v. Following [AC10, Section 10], we write $\Lambda(V)$ for the collection of all sets of pairwise orthogonal subspaces whose sum is V. For $\Lambda \in \Lambda(V)$, the *type* of Λ is the nondecreasing list of dimensions of the members of Λ. Write $N_H(\Lambda)$ for the subgroup of H which permutes the members of Λ, and write $C_H(\Lambda)$ for the subgroup of H which acts on each member of Λ. We use exponential notation for the type, writing, for example, 1^7 for $(1, \ldots, 1)$ and $1^5 2$ for $(1, \ldots, 1, 2)$. Also, we write 2_+^{1+2k} and 2_-^{1+2k} for the extraspecial 2-groups of width k and plus and minus type, respectively. Finally, if Y is a finite group and π is a set of primes, we write (as usual) $O_\pi(Y)$ for the unique maximal normal π-subgroup of Y, $O_{\pi,\pi'}(Y)$ for the preimage in Y of $O_{\pi'}(Y/O_\pi(Y))$, and $O_{\pi,\pi',\pi}(Y)$ for the preimage in Y of $O_\pi(Y/O_{\pi,\pi'}(Y))$.

Proposition 3.2. *Suppose that $P \in \mathcal{H}^{cr}$ with $N_H(P) \not\leq K$. Then, using $*$ to denote a central product, one of the following holds.*

(a) *$P = C_H(\Lambda)$ for some $\Lambda \in \Lambda(V)$ of type 1^7 with each member of Λ spanned by a vector of square norm. Moreover, $P \cong D_8 * D_8 * D_8 \cong 2_+^{1+6}$, and $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathcal{H}}(P) \cong \begin{cases} A_7 \text{ if } l = 0 \\ S_7 \text{ if } l > 0; \end{cases}$*
(b) *$P = C_H(\Lambda)$ for some $\Lambda \in \Lambda(V)$ of type 1^7 with exactly six 1-spaces spanned by a vector of nonsquare norm. Moreover, $P \cong C_4 * D_8 * Q_8 = C_4 * 2_-^{1+4} \cong C_4 * 2_+^{1+4}$ and $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathcal{H}}(P) \cong S_6$;*
(c) *$l > 0$, and $P = O_2(N_H(\Lambda))\langle t\rangle$ for some $\Lambda \in \Lambda(V)$ of type $1^5 2$ with each 1-space spanned by a vector of square norm and with the 2-space a hyperbolic line. Moreover, t acts as -1 on the 1-spaces and as a reflection on the line, $P \cong D_8 * Q_8 * Q_{2^{l+3}}$, and $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathcal{H}}(P) \cong S_5$;*
(d) *$P = C_S(E/Z)$, $|S : P| = 2$, and $\mathrm{Out}_{\mathcal{F}}(P) \cong \mathrm{Out}_{\mathcal{H}}(P) \cong S_3$.*

Moreover, there is exactly one $\mathcal{H}$-conjugacy class of subgroups of S of each of the given types.

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Proof. Except for the last statement and the alternative descriptions of the groups P in (c) and (d), this is proved in [AC10, Lemma 10.7] (note each subgroup in (a)–(d) has center Z, so $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{H}}(P)$ in all cases). To see that $P \cong D_8 * Q_8 * Q_{2^{l+3}}$ in (c), we recall the setup of Aschbacher and Chermak as follows. Set $\overline{H} := H/Z \cong \Omega_7(q)$. The description of the subgroup in part (c) is discussed at and around [AC10, p. 937, l.5]. For such a subgroup P as in (c), P preserves a decomposition $V = V_1 \perp \cdots \perp V_5 \perp W$, where $\dim V_i = 1$ and where W is a hyperbolic line. Set $V_0 = V_1 + \cdots + V_5$, $H_1 = C_H(W)$, $H_2 = C_H(V_0)$, and let t be an element which acts as -1 on the V_i and which induces a reflection on W. The subgroup of H preserving the above decomposition is of the form $H_1H_2\langle t\rangle$, where $O_2(H_1) \cong D_8 * Q_8$ has center Z, $H_1/O_2(H_1) \cong S_5$, and H_2 is cyclic of order $2(q-1)$. Further, $P = P_1P_2\langle t\rangle$, where $P_1 = O_2(H_1)$, $P_2 = O_2(H_2)$, and $[P_1, P_2\langle t\rangle] = 1$. The image of t in $\overline{H}$ has -1-eigenspace of dimension 6, so t squares to z in H by Lemma 2.10. Likewise, an element $s \in P_2$ acting as -1 on W and as the identity on V_0 squares to z. This shows $P_2\langle t\rangle = Q_{2^{l+3}}$ and $Z(P_1) = Z = Z(P_2\langle t\rangle)$, so that P has the structure as claimed in (c). The subgroup in (d) appears in the proof of 10.7 as the only subgroup P satisfying the conditions that contains an elementary abelian normal subgroup P_0 of rank at least 3. Having such P_0 of rank $\geqslant 4$ is ruled out on [AC10, p. 957, lines 19–22]. Let $P \in \mathcal{H}^{cr}$ with $N_H(P) \nleq K$, and assume that there is an elementary abelian normal subgroup P_0 of P of 2-rank 3. Then [AC10, p. 973, lines 22–29] shows that $P_0 = E$ and $C_S(E) \leqslant P$. As $N_H(S) = S \leqslant K$ from Lemma 2.15, we have $P = C_S(E/Z)$ by Lemma 2.18. Lemma 2.17 then gives $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{H}}(P) \cong S_3$.

Finally, we must verify the last statement. For P in (a)–(b), this follows from a slight extension of Witt's lemma, as stated in [GLS98, Lemma 2.7.2], and induction on dimension. Consider a subgroup satisfying the conditions in (c). From the description of P in the first paragraph, we see that P is a Sylow 2-subgroup of $O_{2',2}(N_H(\Lambda))$. By [GLS98, Lemma 2.7.2] again, $O_{2',2}(N_H(\Lambda))$ is uniquely determined up to H-conjugacy, so P is uniquely determined up to H-conjugacy by Sylow's theorem in $O_{2',2}(N_H(\Lambda))$. For uniqueness of the subgroup in (d), there is nothing to do. This completes the proof of the proposition. □

Notation 3.3. We denote a member of the $\mathcal{F}$-conjugacy class of a subgroup appearing in Proposition 3.2 parts (a), (b), and (c) by R_{1^7}, R'_{1^7}, and R_{1^52}, respectively, to best indicate their origins. The reader should not confuse these with the generalized quaternion groups R_1, R_2, and R_3. When $l = 0$, the subgroups R_{1^7} and R'_{1^7} correspond with the subgroups R and R^* of Section 2 of [COS08].

We next describe the centric radical subgroups arising in case (c)(i) of Proposition 3.1. Recall Notations 2.11 and 2.12. In addition, for any subgroup Y of K, we set $Y_0 = Y \cap L_0$, and let Y_i be the *projection* of Y_0 in L_i for $1 \leqslant i \leqslant 3$. That is, Y_i is the image in L_i of the projection of the preimage of Y_0 in $\widehat{L}_i$ (cf. Notation 2.11(a)) under the quotient map $\widehat{K} \to K$.

Proposition 3.4. *Fix $P \leqslant S$. Then $P \in \mathcal{K}^{cr}$ if and only if*

(a) *$P \cap L_0 = P_1P_2P_3$, and for each $i \in \{1, 2, 3\}$, either $P_i \in \mathcal{Q}_i$ or $P_i = R_i$; and*
(b) *one of the following holds. Either,*
  (i) *$P \in \{C_S(U), S\}$,*
  (ii) *$P = P_1P_2P_3 \leqslant R_0$ with $P_i \in \mathcal{Q}_i$ for at least two indices i, or*
  (iii) *$P = P_0\langle s\rangle$ for some $s \in P\backslash C_P(U)$, such that*
    (1) *$s^2 \in P_0$,*
    (2) *either $P_3 \in \mathcal{Q}_3$ or $P_i \in \mathcal{Q}_i$ for both $i = 1$ and 2, and*
    (3) *if $P_3 \in \mathcal{Q}_3$, then $\mathrm{Out}_{L_3}(P)$ is not a 2-group.*

Proof. This is part of [AC10, Lemma 10.2], namely, (c) and (d) of that lemma together with the statement beginning "Conversely". The requirement here in (b)(iii)(1) that s square into P_0 does not appear in [AC10], but it is needed for the "if" part of the proposition to hold in general. A patch for the proof of the "if" part in [AC10, Lemma 10.2] is given later in Remark 3.10. □

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

3.1. The case $l = 0$

An important distinguishing feature of the smallest Benson-Solomon system is that R_0 is normal in the fusion system $\mathcal{K}$. When $l = 0$, this is most naturally seen over $\mathbb{F}_3$, where a Q_8 Sylow 2-subgroup is normal in $\mathrm{SL}_2(3)$. Over $\mathbb{F}_5$, the normalizer of a quaternion Sylow 2-subgroup of $\mathrm{SL}_2(5)$ is $\mathrm{SL}_2(3)$, which still controls 2-fusion in $\mathrm{SL}_2(5)$ (c.f. Lemma 2.9). It will therefore be convenient to treat the cases $l = 0$ and $l > 0$ separately. So assume here that $l = 0$. We adopt the previous notation, except that we set

$$Q := R_0 = R_1R_2R_3 = Q_1Q_2Q_3 \tag{3.1}$$

in this smallest case so that Q, R_{1^7}, and R'_{1^7} correspond with the groups "Q," "R," and "R^*" considered in [COS08, Section 2].

The next proposition lists the $\mathcal{K}$-centric radicals when $l = 0$, and does not require Proposition 3.4.

Proposition 3.5. *Let $l = 0$ and $P \in \mathcal{K}^{cr}$. Then exactly one of the following holds.*

(a) *$P = S$, and $\mathrm{Out}_{\mathcal{K}}(P) = 1$;*

(b) *$P = Q$, and $\mathrm{Out}_{\mathcal{K}}(P) \cong (C_3)^3 \overset{-1\times\wr}{\rtimes} (C_2 \times S_3)$, where here, and in Table 1, the notation $-1\times\wr$ indicates that the C_2 factor acts by inversion while S_3 acts by wreathing;*

(c) *$P = Q\langle\tau\rangle$, and $\mathrm{Out}_{\mathcal{K}}(P) \cong (C_3 \times C_3) \overset{-1}{\rtimes} C_2$;*

(d) *$P = Q\langle\tau'\rangle$, and $\mathrm{Out}_{\mathcal{K}}(P) \cong S_3$; or*

(e) *$P = C_S(U) = Q\langle\mathbf{d}\rangle$, and $\mathrm{Out}_{\mathcal{K}}(P) \cong S_3$.*

Proof. As Q is a centric normal 2-subgroup of $\mathcal{K}$, it is contained in every member of $\mathcal{K}^{cr}$ by Lemma 2.5. Now S/Q is a four group (the four group $\langle\mathbf{d}, \tau\rangle$ is a complement to Q in S), so there are only five possible centric radical subgroups. Since $O^2(K) \cap S = Q$, if two distinct subgroups of S containing Q were $\mathcal{K}$-conjugate, then two distinct subgroups of the abelian group S/Q would be $K/O^2(K) \cong S/Q$-conjugate. Since this is not the case, no two distinct subgroups of S containing Q are $\mathcal{K}$-conjugate. Next, from the definition of U, both Q and $\mathbf{d}$ centralize U while τ does not, so we must have $Q\langle\mathbf{d}\rangle = C_S(U)$. This shows the equality in (e).

The structure of the outer automorphism groups are computable from knowledge of $\mathrm{Out}_{\mathcal{K}}(Q)$: note that from the structure of K (cf. Lemma 2.9),

$$\mathrm{Out}_{\mathcal{K}}(Q) \cong (C_3)^3 \rtimes (C_2 \times S_3)$$

is a split extension of the wreath product $C_3 \wr S_3$ by the group generated by the class $[c_{\mathbf{d}}] \in \mathrm{Out}_{\mathcal{K}}(Q)$ of conjugation by $\mathbf{d}$ acting by inversion on the base. As Q is weakly $\mathcal{K}$-closed and centric, $\mathrm{Out}_{\mathcal{K}}(P) \cong N_{\mathrm{Out}_{\mathcal{K}}(Q)}(\mathrm{Out}_P(Q))/\mathrm{Out}_P(Q)$ for each overgroup P of Q in S by Lemma 2.7. From a computation in the group $(C_3)^3 \overset{-1\times\wr}{\rtimes} (C_2 \times S_3)$, one sees, for example, that

$$N_{\mathrm{Out}_{\mathcal{K}}(Q)}(\langle[c_\tau]\rangle) = C_{\mathrm{Out}_{\mathcal{K}}(Q)}(\langle[c_\tau]\rangle) \cong (C_3)^2 \overset{-1,1}{\rtimes} (C_2 \times C_2),$$

where the acting group is given by $\langle[c_{\mathbf{d}}]\rangle \times \langle[c_\tau]\rangle$. This shows that $\mathrm{Out}_{\mathcal{K}}(Q\langle\tau\rangle) = N_{\mathrm{Out}_{\mathcal{K}}(Q)}(\langle[c_\tau]\rangle)/\langle[c_\tau]\rangle \cong (C_3 \times C_3) \overset{-1}{\rtimes} C_2$ as claimed. Cases (d) and (e) are handled similarly. Visibly, no resulting outer automorphism group has a nontrivial normal 2-subgroup, so all the candidate subgroups are $\mathcal{K}$-centric radical. □

Proposition 3.6. *Let $l = 0$. Then, up to conjugacy, the $\mathcal{K}$-, $\mathcal{H}$-, and $\mathcal{F}$-centric radical subgroups of S together with their orders and automorphism groups appear in Table 1, where a "$-$" indicates that the subgroup is not centric radical in that fusion system.*

Proof. By Proposition 3.5, the column for $\mathcal{K}$ is correct. By [COS08, Lemma 2.1] and [LO02, Lemma A.11(e,f)], the column for $\mathcal{H}$ is correct. We work up to $\mathcal{F}$-conjugacy in what follows. Let $P \in \mathcal{F}^{cr}$. By Proposition 3.1, either P is listed in the last two rows of Table 1, or one of the following holds: (1)

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Table 1. $\mathrm{Sol}(5)$*-conjugacy classes of* $\mathrm{Sol}(5)$*-centric radical subgroups.*

P	$\lvert P\rvert$	$\mathrm{Out}_{\mathcal{H}}(P)$	$\mathrm{Out}_{\mathcal{K}}(P)$	$\mathrm{Out}_{\mathcal{F}}(P)$
S	2^{10}	1	1	1
Q	2^8	$(C_3)^3 \rtimes (C_2 \times C_2)$	$(C_3)^3 \rtimes (C_2 \times S_3)$	$(C_3)^3 \rtimes (C_2 \times S_3)$
$Q\langle\tau\rangle$	2^9	$(C_3 \times C_3) \overset{-1}{\rtimes} C_2$	$(C_3 \times C_3) \overset{-1}{\rtimes} C_2$	$(C_3 \times C_3) \overset{-1}{\rtimes} C_2$
$Q\langle\tau'\rangle$	2^9	S_3	S_3	S_3
$C_S(U)$	2^9	–	S_3	S_3
R_{17}	2^7	A_7	–	A_7
R'_{17}	2^6	S_6	–	S_6
$C_S(E/Z)$	2^9	S_3	–	S_3
$C_S(E)$	2^7	–	–	$\mathrm{GL}_3(2)$
A	2^4	–	–	$\mathrm{GL}_4(2)$

$P \in \mathcal{K}^{cr}$ and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{K}}(P)$, or (2) $N_G(P) \nleq K$, $P \in \mathcal{H}^{cr}$, and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{H}}(P)$. If (1) holds, then P is listed in the first five rows of the table by Proposition 3.5. If (1) does not hold, then (2) holds, P is listed in the next three rows of the table, where the entries follow from Proposition 3.2(a,b,d).

That no additional $\mathcal{F}$-conjugacy can occur between these subgroups can be seen in several ways, one of which as follows. Only three subgroups have pairwise equal orders and isomorphic outer automorphism groups in $\mathcal{F}$, namely, $Q\langle\tau'\rangle$, $C_S(U)$, and $C_S(E/Z)$.

By Lemma 2.14, $C_S(E/Z)$ has center Z. Likewise, since $Z(Q) = U$, we have $Z(Q\langle\tau'\rangle) = C_U(\tau') = Z$. So as $U \leqslant Z(C_S(U))$, it follows that $C_S(U)$ is not $\mathcal{F}$-conjugate to either of the other two subgroups.

Finally, note that $C_S(E/Z)$ contains the torus T. On the other hand, from the description of T in Section 2.5, we see that $Q \cap T = \langle [a, 1, 1], [1, a, 1], [c^2, c^2, c^2]\rangle$ is of index 2 in T. As each element in the coset $Q\tau'$ is nontrivial on $Q \cap T$ and T is abelian, it follows that $Q\langle\tau'\rangle \cap T$ is still of index 2 in T. So $C_S(E/Z)$ contains T, but $Q\langle\tau'\rangle$ does not. Since T is weakly $\mathcal{F}$-closed (Lemma 2.13), the subgroups $C_S(E/Z)$ and $Q\langle\tau'\rangle$ are not $\mathcal{F}$-conjugate. □

We end this subsection with two lemmas in the case $l = 0$, which will be needed later.

Lemma 3.7. *Each member of $\mathcal{F}^{cr} - \{A\}$ is weakly $\mathcal{F}$-closed when $l = 0$.*

Proof. The subgroup S is clearly weakly closed, and $C_S(E)$, $C_S(U)$, and $C_S(E/Z)$ were shown to be weakly $\mathcal{F}$-closed in Lemmas 2.14, 2.16, and 2.17. Let P be one of the remaining subgroups, but not A. By Proposition 3.6, P is centric and radical in $\mathcal{H}$, and either $P = Q$ or $Z(P) = Z$. The quotient P/Z is centric and radical in $\mathcal{H}/Z$ by [LO02, Lemma A.11(e)]. Hence, P/Z is weakly $\mathcal{H}/Z$-closed by [COS08, Lemma 2.1]. It follows that P is weakly $\mathcal{H}$-closed. Since Q is normal in $\mathcal{K}$, it is weakly $\mathcal{K}$-closed. Hence, Q is weakly $\mathcal{F}$-closed since $\mathcal{H}$ and $\mathcal{K}$ are fusion systems over S which generate $\mathcal{F}$ (end of Section 2.4).

We are reduced to the case in which $Z(P) = Z$. Assume on the contrary that P is not weakly $\mathcal{F}$-closed. By Alperin's Fusion theorem [BLO03, Theorem A.10], there is an overgroup $Y \leqslant S$ of P and an automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Y)$, such that $P^\alpha \neq P$. Then $Z(Y) \leqslant Z(P) = Z$, as P is centric, so that $Z(Y) = Z$ is centralized by α. That is, $\alpha \in \mathcal{H}$. But then $P^\alpha = P$ by the previous paragraph, a contradiction. □

Lemma 3.8. *$QR_{17} = Q\langle\tau\rangle$ and $QR'_{17} = Q\langle\tau'\rangle$ when $l = 0$.*

Proof. This is a statement depending on H only. Since $l = 0$, $q = 5$. Write $\overline{H}$ for H/Z. Fix a decomposition

$$V = \ell_1 \perp \ell_2 \perp \ell_3 \perp \langle x_7\rangle$$

with the following properties ([AC10, cf. Lemmas 4.4, 4.6]):

1. each $\ell_i = \langle x_{2i-1}, x_{2i}\rangle$ is a hyperbolic line (i.e., $\mathfrak{q}(x_{2i-1}) = 0 = \mathfrak{q}(x_{2i}), \mathfrak{b}(x_{2i-1}, x_{2i}) = 1$), and $\mathfrak{q}(x_7) = 1$.
2. $\ell_1 \perp \ell_2 = \langle x_1, x_4\rangle \oplus \langle x_3, x_2\rangle$, with each summand on the right side a natural $\mathbb{F}_5 L_1$-module; in particular, $[a, 1, 1]$ and $[b, 1, 1]$ act via the matrices
$$[a,1,1] \mapsto \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \qquad [b,1,1] \mapsto \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$
with respect to the basis $\{x_1, x_2, x_3, x_4\}$.
3. $\ell_1 \perp \ell_2 = \langle x_1, x_3\rangle \oplus \langle x_4, x_2\rangle$, with each summand on the right side a natural $\mathbb{F}_5 L_2$-module; in particular, $[1, a, 1]$ and $[1, b, 1]$ act via the matrices
$$[1,a,1] \mapsto \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad [1,b,1] \mapsto \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$
with respect to the basis $\{x_1, x_2, x_3, x_4\}$.
4. $\ell_3 \perp \langle x_7\rangle$ is the three-dimensional orthogonal module for $L_3 \cong \mathrm{Spin}_3(5)$. We may view it as the module in which L_3 acts by conjugation on 2×2 trace zero matrices $M_2^0(\mathbb{F}_5)$ with quadratic form given by the determinant, via the isometry $M_2^0(\mathbb{F}_5) \longrightarrow \ell_3 \perp \{x_7\}$ defined by
$$\left\{\left[\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 & 0 \\ 0 & -2 \end{smallmatrix}\right]\right\} \longmapsto \{x_5, x_6, x_7\}.$$
Under this identification, $[1, 1, a]$ acts via the matrix $\mathrm{diag}(-1, -1, 1)$ with respect to the ordered basis $\{x_5, x_6, x_7\}$, and $[1, 1, b]$ acts via the matrix $\left[\begin{smallmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right]$.

Next, define u_j and v_j via
$$\begin{aligned} u_{2i-1} &= x_{2i-1} - 2x_{2i}, & v_{2i-1} &= u_{2i-1} + u_{2i}, \\ u_{2i} &= -2x_{2i-1} + x_{2i}, & v_{2i} &= u_{2i-1} - u_{2i}, \\ u_7 &= x_7, & v_7 &= u_7. \end{aligned}$$

Thus, $\{u_1, \ldots, u_7\}$ is an orthonormal basis for V, and $\{v_1, \ldots, v_7\}$ is an orthogonal basis, such that $\mathfrak{q}(v_i) = 2 \notin \mathbb{F}_5^{\times 2}$ for each $i = 1, \ldots, 6$. The decompositions
$$\Lambda = \{\langle u_i\rangle \mid i \in \{1, \ldots, 7\}\}, \qquad \text{and} \qquad \Lambda' = \{\langle v_i\rangle \mid i \in \{1, \ldots, 7\}\}$$
of V are therefore of the type appearing in Proposition 3.2(a) and (b), respectively. The centralizers of the decompositions are
$$\begin{aligned} C_H(\Lambda) &= \{e \in \mathrm{Spin}_7(5) \mid (u_i)\overline{e} = \pm u_i, i = 1, \ldots, 7\}, \text{ and} \\ C_H(\Lambda') &= \{f \in \mathrm{Spin}_7(5) \mid (v_i)\overline{f} = \pm v_i, i = 1, \ldots, 7\}. \end{aligned}$$

Observe from the definition of the v_i that $C_H(\Lambda) \leqslant N_H(\Lambda')$. Similarly, it is a straightforward computation to see using (2)–(4) that Q acts on the sets Λ and Λ', that is, $Q \leqslant N_H(\Lambda)$ and $Q \leqslant N_H(\Lambda')$. It follows that $QC_H(\Lambda)C_H(\Lambda)'$ is a 2-subgroup of H. Hence, we may choose $h \in H$ with $(QC_H(\Lambda)C_H(\Lambda'))^h \leqslant S$. But $Q \leqslant S$, and so $Q^h = Q$ by Lemma 3.7. Likewise, it follows from Lemma 3.7 that $C_H(\Lambda)^h = R_{1^7}$ and $C_H(\Lambda')^h = R'_{1^7}$. Replacing S with $S^{h^{-1}}$ if necessary, we may assume that $R_{1^7} = C_H(\Lambda)$ and $R'_{1^7} = C_H(\Lambda')$.

For a subset $I \subseteq \{1, \ldots, 7\}$, write e_I for a fixed element of $C_H(\Lambda)$ which maps $u_i \mapsto -u_i$ if $i \in I$, and which fixes u_i otherwise. When $I \subseteq \{1, \ldots, 6\}$, denote by f_I an analogous element of $C_H(\Lambda')$ with respect to the v_i's. A computation of the action of Q with respect to the bases $\{u_i \mid 1 \leqslant i \leqslant 7\}$ and

$\{v_i \mid 1 \leqslant i \leqslant 7\}$ using (2)–(4) yields

$$\begin{aligned} Q \cap R_{17} &= \langle [-1,1,1], [b,ab,1], [ab,b,1], [1,1,a], [1,1,b] \rangle \\ &= \langle e_{1234}, e_{13}, e_{14}, e_{56}, e_{57} \rangle \\ &\cong C_2 \times (Q_8 * Q_8), \end{aligned}$$

and

$$\begin{aligned} Q \cap R'_{17} &= \langle [-1,1,1], [b,b,1], [ab,ab,1], [1,1,a] \rangle \\ &= \langle f_{1234}, f_{23}, f_{13}, f_{56} \rangle \\ &\cong C_2 \times (Q_8 * C_4), \end{aligned}$$

where here we have used Lemma 2.10 and the identity $[e,f] = (ef)^2$ to determine the isomorphism types. The order $|Q \cap R_{17}| = 2^6$, and so $|QR_{17}| = \frac{|Q||R_{17}|}{|Q \cap R_{17}|} = 2^9$. Similarly, $|Q \cap R'_{17}| = 2^5$, so also $|QR'_{17}| = 2^9$.

We have shown that $\{QR_{17}, QR'_{17}\} \subset \{Q\langle \mathbf{d} \rangle, Q\langle \tau \rangle, Q\langle \tau' \rangle\}$. The involution $e_{4567} \in R_{17} - Q$ (Lemma 2.10) acts as -1 on $\ell_3 \perp \langle x_7 \rangle$, so centralizes L_3. It also interchanges the one-dimensional subspaces $\langle x_3 \rangle$ and $\langle x_4 \rangle$ while centralizing the line ℓ_1, and hence from (2)–(3), it interchanges L_1 and L_2 by conjugation. It follows that $QR_{17} = Q\langle \tau \rangle$, since neither $Q\langle \mathbf{d} \rangle$ nor $Q\langle \tau' \rangle$ have such an element.

Finally, we show that $QR'_{17} = Q\langle \tau' \rangle$. First, since $f_{1234} \in U - Z$ does not commute with f_{45} by Lemma 2.10, it follows that QR'_{17} is not contained in $C_S(U) = Q\langle \mathbf{d} \rangle$. Next, observe that in contrast to the previous case, $C_{R'_{17}}(L_3) = C_{R'_{17}}(\ell_3 + \langle x_7 \rangle)$ (for example, note that "f_{4567}" has nontrivial spinor norm). The group $C_{R'_{17}}(L_3) = \langle f_{12}, f_{13}, f_{14} \rangle$ induces the permutation group $\langle (1,2)(3,4) \rangle$ on $\{\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle, \langle x_4 \rangle\}$, and hence $C_{R'_{17}}(L_3)$ acts on L_1 and L_2 by (2)–(3). Therefore, QR'_{17} has no element centralizing L_3 and interchanging L_1 and L_2, and so $QR'_{17} = Q\langle \tau' \rangle$. □

3.2. *The case* $l > 0$

In this subsection, we determine a set of representatives for the $\mathcal{D}$-conjugacy classes of elements in $\mathcal{D}^{cr}$ for $\mathcal{D} \in \{\mathcal{K}, \mathcal{H}, \mathcal{F}\}$, in the case when $l > 0$. First, we treat the case $\mathcal{D} = \mathcal{K}$.

Proposition 3.9. *Suppose that $l > 0$. There are 11 $\mathcal{K}$-conjugacy classes of elements of $\mathcal{K}^{cr}$. Representatives of these classes together with their outer automorphism groups in $\mathcal{K}$ are listed in Table 2.*

Proof. Let $P \leqslant S$ be a centric radical subgroup of $\mathcal{K}$, taken up to $\mathcal{K}$-conjugacy. We proceed through the possibilities in the description of $\mathcal{K}^{cr}$ given by Proposition 3.4 and refer to the labelings of the three cases given there. If P occurs in (b)(i), then P is listed in the first two rows of the table. By Lemma 2.15, $\mathrm{Aut}_{\mathcal{K}}(S) = \mathrm{Inn}(S)$, so that $\mathrm{Out}_{\mathcal{K}}(S) = 1$. Also, $C_S(U) = R_0\langle \mathbf{d} \rangle$, so that $\mathrm{Out}_{\mathcal{K}}(C_S(U)) \cong S_3$ is induced by X.

Consider a subgroup P in (b)(ii). First, assume that $P_i \in \mathcal{Q}_i$ for all i. Upon conjugating in L_0, we may assume that $P_i = Q_i$ or Q'_i for each i. Conjugating by $\mathbf{d}$, which interchanges Q_i and Q'_i for each i, we may assume that there is at most one Q'_i among the P_i's. Finally, we may conjugate by elements of X to see that P is one of the subgroups in rows 3 and 4 of the table.

To compute $\mathrm{Out}_{\mathcal{K}}(P)$, observe that if $t \in L_0X$, then P^t and P have the same number of components P_i^t which are L_i-conjugate to Q_i, while $P^{\mathbf{d}}$ has three minus the number for P. This shows that $N_K(P) = N_{L_0X}(P)$. Thus, if $P = Q_1Q_2Q_3$, then $N_K(P) = (N_{L_1}(Q_1)N_{L_2}(Q_2)N_{L_3}(Q_3))X$, and we see that $\mathrm{Out}_{\mathcal{K}}(P) \cong S_3 \wr S_3$ by Lemma 2.9. Likewise, if $P = Q_1Q_2Q'_3$, then $N_K(P) = N_{L_1}(Q_1)N_{L_2}(Q_2)N_{L_3}(Q'_3)\langle \tau \rangle$, so that $\mathrm{Out}_{\mathcal{K}}(P) \cong (S_3 \wr C_2) \times S_3$.

Next, assume that $P_i \in \mathcal{Q}_i$ for exactly two indices i. Then as before, we may conjugate so that $P = Q_1Q_2R_3$ or $Q_1Q'_2R_3$ is on the table. Appealing to Lemma 2.9 again to see that $\mathrm{Out}_{L_3}(R_3) = 1$, we have in the former case that $\mathrm{Out}_{\mathcal{K}}(P) \cong S_3 \wr C_2$ with the class of τ wreathing, while in the latter case, we have a similar situation with the class of τ' wreathing. This concludes the case (b)(ii).

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Table 2. *$\mathcal{K}$-conjugacy classes of $\mathcal{K}$-centric radical subgroups, $l > 0$.*

P	$\lvert P\rvert$	$\mathrm{Out}_{\mathcal{K}}(P)$
S	2^{10+3l}	1
$C_S(U)$	2^{9+3l}	S_3
$Q_1Q_2Q_3$	2^8	$S_3 \wr S_3$
$Q_1Q_2Q_3'$	2^8	$(S_3 \wr C_2) \times S_3$
$Q_1Q_2R_3$	2^{8+l}	$S_3 \wr C_2$
$Q_1Q_2'R_3$	2^{8+l}	$S_3 \wr C_2$
$Q_1Q_2Q_3\langle\tau\rangle$	2^9	$S_3 \times S_3$
$Q_1Q_2Q_3'\langle\tau\rangle$	2^9	$S_3 \times S_3$
$Q_1Q_2R_3\langle\tau\rangle$	2^{9+l}	S_3
$Q_1Q_2'R_3\langle\tau'\rangle$	2^{9+l}	S_3
$R_1R_2Q_3\langle\tau\rangle$	2^{9+2l}	S_3

Consider now a subgroup P in (b)(iii), and recall that $Z(L_0) = U$. Thus, $P = P_0\langle s\rangle$ with $s \in P - C_P(U)$ normalizing P_0. Set $N = N_K(P)$ and $M = N_K(P_0)$. Denote quotients modulo P_0 with bars. We set $M^+ = \overline{M}/O_{2'}(\overline{M})$ and write quotients modulo $O_{2'}(\overline{M})$ with pluses. Thus, for any subgroup $Y \leqslant M$, we write Y^+ for the image of Y modulo the preimage of $O_3(\overline{M})$ in M.

Since $L_0 \trianglelefteq K$, we see that $P_0 = P \cap L_0 \trianglelefteq N$, so that $N \leqslant M$. In particular, $\overline{N}$ is defined. Also, since $\overline{s}$ is of order 2, N is the preimage in M of $C_{\overline{M}}(\overline{s})$. As P is radical, we must have

$$\langle\overline{s}\rangle = O_2(C_{\overline{M}}(\overline{s})). \tag{3.2}$$

We consider separately the cases where $P_0 \notin \mathcal{K}^{cr}$ and where $P_0 \in \mathcal{K}^{cr}$. Assume first that $P_0 \notin \mathcal{K}^{cr}$, the easier case. Upon comparing the conditions in (b)(ii) and (b)(iii), we have by our assumption that $P_3 \in \mathcal{Q}_3$ and $P_i = R_i$ for $i = 1, 2$. Thus, $\overline{M} = \langle\overline{\tau}\rangle \times \overline{N_{L_3}(P_3)} \cong C_2 \times S_3$, and so $P = P_0\langle\tau\rangle$ by (3.2). Hence, $\mathrm{Out}_{\mathcal{F}}(P) \cong S_3$, and P appears in the last row of the table.

Assume next that $P_0 \in \mathcal{K}^{cr}$, so that P_0 is conjugate to a subgroup considered in (b)(ii), rows 3–6 of the table. First, assume that P_0 itself appears in rows 3–6. Our description of the normalizer in K of P_0 in a previous paragraph together with order considerations imply that $N_{R_0}(P_0)\langle\tau\rangle/P_0$ is a Sylow 2-subgroup of $\mathrm{Out}_{\mathcal{K}}(P_0)$ if P_0 appears in rows 3–5, and that $N_{R_0}(P_0)\langle\tau'\rangle/P_0$ is a Sylow 2-subgroup of $\mathrm{Out}_{\mathcal{K}}(P_0)$ if P_0 appears in row 6. This shows that $\mathrm{Aut}_S(P_0)$ is a Sylow 2-subgroup of $\mathrm{Aut}_{\mathcal{K}}(P_0)$, that is, P_0 is fully $\mathcal{K}$-automized [AKO11, I.2.2]. Since P_0 is $\mathcal{K}$-centric, it is fully $\mathcal{K}$-centralized ([AKO11, I.3.1]). Hence, P_0 is fully $\mathcal{K}$-normalized by [AKO11, I.2.6(c)]. Thus, by Lemma 2.6 (and since $P_0 \trianglelefteq P$), we may in any case replace P by a $\mathcal{K}$-conjugate and assume that P_0 is in rows 3–6 of the table.

Case 1. Assume $P_0 = Q_1Q_2Q_3$, and recall Lemma 2.8(e).

Here, $\overline{M} = \overline{N_{L_0}(P_0)X} \cong S_3 \wr S_3$, $\overline{N_{R_0}(P_0)}^+ = O_2(M^+)$, and $\overline{N_S(P_0)} = \overline{N_{R_0}(P_0)}\langle\overline{\tau}\rangle$. By assumption, s is not in $C_P(U)$, so it is not in R_0. Hence, $\overline{s}$ is not in $\overline{N_{R_0}(P_0)}$. Thus,

$$\overline{s} \in \overline{N_{R_0}(P_0)}\overline{\tau}. \tag{3.3}$$

Write $\overline{s} = \overline{[t_1, t_2, t_3]}\overline{\tau}$, where each $t_i \in R_i$, and where we take $t_i = 1$ if $t_i \in P_i$ and $t_i = a^{2^{l-1}}$ if $t_i \notin P_i$. As τ acts by swapping R_1 and R_2 and centralizing R_3, it follows from (3.3) that $C_{\overline{N_{R_0}(P_0)}}(\overline{s})$ is of order 4 generated by $\overline{[a^{2^{l-1}}, a^{2^{l-1}}, 1]}$ and $\overline{[1, 1, a^{2^{l-1}}]}$.

Note that $t_1 = t_2$ since $\overline{s}$ is of order 2. We claim that (3.2) and (3.3) imply $t_3 = 1$. Assume on the contrary that $t_3 = a^{2^{l-1}}$. Then $O_{2'}(C_{\overline{M}}(\overline{s})) = C_{O_{2'}(\overline{M})}(\overline{s}) \leqslant \overline{N_{L_1}(Q_1)N_{L_2}(Q_2)}$. By (3.3), $C_{O_{2',2,2'}(\overline{M})}(\overline{s}) = C_{O_{2',2}(\overline{M})}(\overline{s})$. So since $\langle\overline{[1, 1, a^{2^{l-1}}]}\rangle$ is a normal 2-subgroup of $\overline{N_S(P_0)}$, it follows that $\langle\overline{[1, 1, a^{2^{l-1}}]}\rangle$ is a normal 2-subgroup of $C_{\overline{M}}(\overline{s})$. By (3.2), $\overline{s} = \overline{[1, 1, a^{2^{l-1}}]}$. This contradicts (3.3).

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

We conclude that $\overline{s} = \overline{\tau}$ or $\overline{s} = \overline{[a^{2^{l-1}}, a^{2^{l-1}}, 1]\tau}$, and hence

$$C_{\overline{M}}(\overline{s}) = \langle \overline{s} \rangle \times C_{\overline{N_{L_1}(Q_1)N_{L_2}(Q_2)}}(\overline{s}) \times C_{\overline{N_{L_3}(Q_3)}}(\overline{s}) \cong C_2 \times S_3 \times S_3.$$

However, since the two possibilities for $\overline{s}$ are conjugate under $\overline{N_{R_1}(Q_1)}$, we may take $\overline{s} = \overline{\tau}$, as desired.

Case 2. Assume $P_0 = Q_1Q_2Q_3'$. In this case, $\overline{M} \cong S_3 \wr \langle \overline{\tau} \rangle \times S_3$. Replacing all occurrences of Q_3 by Q_3', we argue verbatim as in Case 1, except that in the present situation, we have $O_{2',2,2'}(\overline{M}) = O_{2',2}(\overline{M}) = \overline{M}$, obviating the need to observe that $C_{O_{2',2,2'}(\overline{M})}(\overline{s}) = C_{O_{2',2}(\overline{M})}(\overline{s})$. Again, we may take $\overline{s} = \overline{\tau}$.

Case 3. Assume $P_0 = Q_1Q_2R_3$. We have $\overline{M} = \overline{N_{L_0}(P_0)X} \cong S_3 \wr \langle \overline{\tau} \rangle$, $\overline{N_{R_0}(P_0)}^+ = O_2(M^+)$, and $\overline{N_S(P_0)} = \overline{N_{R_0}(P_0)}\langle \overline{\tau} \rangle$. By assumption, s is not in $C_P(U)$, so it is not in R_0. Hence, $\overline{s}$ is not in $\overline{N_{R_0}(P_0)}$. Thus, $\overline{s} \in \overline{N_{R_0}(P_0)}\overline{\tau}$. As in the previous cases, (3.2) forces $\overline{s} = \overline{\tau}$ or $\overline{[a^{2^{l-1}}, a^{2^{l-1}}, 1]}\overline{\tau}$, so that

$$C_{\overline{M}}(\overline{s}) = \langle \overline{s} \rangle \times C_{\overline{N_{L_1}(Q_1)N_{L_2}(Q_2)}}(\overline{s}) = C_2 \times S_3.$$

Again, these possibilities for $\overline{s}$ are conjugate under $\overline{N_{R_1}(Q_1)}$, and we see that we may take $\overline{s} = \overline{\tau}$, as needed.

Case 4. Assume $P_0 = Q_1Q_2'R_3$. This time, replace τ by τ', and repeat the argument from the previous case. □

Remark 3.10. The minor omission in the proof of [AC10, Lemma 10.2] alluded to in the proof of Proposition 3.4 occurs in the middle of page 953 with the claim "$|O_3(\overline{N})| = 9$". It is possible that $|O_3(\overline{N})| = 3$ under the hypotheses there. More precisely, consider the subgroup $P = P_0\langle s \rangle$, where $P_0 = Q_1Q_2Q_3$ and $s = [a^{2^{l-1}}, 1, 1]\tau$. Then $\overline{s}$ centralizes $O_3(\overline{N_{L_3}(Q_3)})$, $\overline{s}$ has order 4, and $\overline{s}$ squares to $\overline{[a^{2^{l-1}}, a^{2^{l-1}}, 1]}$. As $\overline{s}$ centralizes no nontrivial element in $O_3(\overline{N_{L_1L_2}(P_0)}) \cong C_3 \times C_3$, we have $O_3(C_{\overline{M}}(\overline{s})) = O_3(\overline{N_{L_3}(Q_3)})$ is of order 3. But in this case, $O_2(N_{\overline{M}}(\langle \overline{s} \rangle)) \cong D_8$ while $\langle \overline{s} \rangle$ is cyclic of index 2 in this subgroup, and so $|O_2(\mathrm{Out}_{\mathcal{K}}(P))| = 2$ is generated by the image of $\overline{[a^{2^{l-1}}, 1, 1]}$. Thus, $P_0\langle s \rangle$ satisfies Proposition 3.4(b)(iii)(2-3), but is not $\mathcal{K}$-radical.

This example also indicates how to patch the proof of [AC10, Lemma 10.2]. Paragraph 3 of page 953 gives an argument for the statement that if (b)(iii)(2–3) holds (in our numbering), then P is centric radical. Follow it until line -2 of that paragraph. In particular, one is reduced to the case in which $P_0 = Q_1Q_2Q_3$, and $\overline{M} \cong S_3 \wr S_3$. The Sylow 2-subgroup $\overline{N_S(P_0)}$ of $\overline{M}$ has the structure $D_8 \times C_2$, and it acts on $O_3(\overline{M})$ decomposably with nontrivial summands $O_3(\overline{N_{L_1L_2}(P_0)})$ and $O_3(\overline{N_{L_3}(P_0)})$. Fix any element $s \in N_S(P_0) - C_S(U)$, such that $\overline{s}$ is of order 4 in $\overline{M}$. Indeed, there are exactly two possibilities for $\langle \overline{s} \rangle$ and hence for $P = P_0\langle s \rangle$, namely, $\langle \overline{[a^{2^{l-1}}, 1, 1]\tau} \rangle$ and $\langle \overline{[a^{2^{l-1}}, 1, a^{2^{l-1}}]\tau} \rangle$. The latter determines a subgroup $P = P_0\langle s \rangle$ that is not $\mathcal{K}$-radical because it does not satisfy Proposition 3.4(b)(iii)(3), while the first determines a subgroup $P = P_0\langle s \rangle$ that is also not $\mathcal{K}$-radical (from the previous paragraph). Hence, we must have $\overline{s}$ is of order 2, that is, it is necessary that (b)(iii)(1) also holds.

We next determine the set $\mathcal{H}^{cr}$ up to $\mathcal{H}$-conjugacy in the case $l > 0$.

Proposition 3.11. *Suppose that $l > 0$. There are 18 $\mathcal{H}$-conjugacy classes of elements of $\mathcal{H}^{cr}$. Representatives for these classes together with their outer automorphism groups in $\mathcal{H}$ are listed in Table 3.*

Proof. Let $P \in \mathcal{H}^{cr}$. If $N_H(P) \nleq K$, then using Proposition 3.2(a)–(d), we obtain the groups in the last four rows of Table 3. Hence, for the remainder of the proof, we may assume that

$$N_H(P) \leqslant K. \tag{3.4}$$

By [LO02, Lemma 3.3(a)], P is $\mathcal{F}$-centric, so that P is also $\mathcal{K}$-centric.

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Table 3. *$\mathcal{H}$-conjugacy classes of $\mathcal{H}$-centric radical subgroups, $l > 0$.*

P	$\lvert P \rvert$	$\mathrm{Out}_{\mathcal{H}}(P)$
S	2^{10+3l}	1
$Q_1Q_2Q_3$	2^8	$S_3 \times S_3 \wr C_2$
$Q_1Q_2Q_3'$	2^8	$S_3 \wr C_2 \times S_3$
$Q_1'Q_2Q_3$	2^8	S_3^3
$Q_1Q_2R_3$	2^{8+l}	$S_3 \wr C_2$
$Q_1R_2Q_3$	2^{8+l}	$S_3 \times S_3$
$Q_1Q_2'R_3$	2^{8+l}	$S_3 \wr C_2$
$Q_1R_2Q_3'$	2^{8+l}	$S_3 \times S_3$
$Q_1Q_2Q_3\langle\tau\rangle$	2^9	$S_3 \times S_3$
$Q_1Q_2Q_3'\langle\tau\rangle$	2^9	$S_3 \times S_3$
$Q_1Q_2R_3\langle\tau\rangle$	2^{9+l}	S_3
$Q_1Q_2'R_3\langle\tau'\rangle$	2^{9+l}	S_3
$R_1R_2Q_3\langle\tau\rangle$	2^{9+2l}	S_3
$Q_1R_2R_3$	2^{8+2l}	S_3
R_{1^7}	2^7	S_7
R_{1^7}'	2^6	S_6
R_{1^52}	2^{7+l}	S_5
$C_S(E/Z)$	2^{9+3l}	S_3

Suppose first that P is $\mathcal{K}$ radical, so that $P \in \mathcal{K}^{cr}$. In this case, we appeal to Proposition 3.9 to obtain the first 13 entries in Table 3, as follows. A case-by-case check shows that for each $\mathcal{K}$-conjugacy class $\mathcal{C} = Y^{\mathcal{K}}$ of a subgroup Y listed in Table 2, one of the following holds: either no member of $\mathcal{C}$ is $\mathcal{H}$-radical ($Y = C_S(U)$), $\mathcal{C}$ meets exactly one $\mathcal{H}$-radical conjugacy class ($Y = S, Q_1Q_2Q_3, Q_1Q_2Q_3\langle\tau\rangle, Q_1Q_2Q_3'\langle\tau\rangle, Q_1Q_2R_3\langle\tau\rangle, Q_1Q_2'R_3\langle\tau'\rangle$, $R_1R_2Q_3\langle\tau\rangle$), or $\mathcal{C}$ is the class of one of the entries in rows 4 through 6 of Table 2 ($Y = Q_1Q_2Q_3', Q_1Q_2R_3$, or $Q_1Q_2'R_3$). In this last case, $\mathcal{C}$ meets one of two $\mathcal{H}$-classes of $\mathcal{H}$-radical subgroups, and corresponding representatives of these $\mathcal{H}$-classes appear in rows 3 through 8 of Table 3. In each of the three cases, $\mathrm{Out}_{\mathcal{H}}(P)$ is computed using (3.4) and the descriptions $K = C_H(U)X$ and $H \cap K = C_K(Z) = N_H(U) = C_H(U)\langle\tau\rangle = C_H(U)\langle\tau'\rangle$.

We illustrate the argument of the previous paragraph with four examples. First, consider $Y = C_S(U)$. As $C_H(U)$ is normal in K with Sylow $C_S(U)$, we have $C_S(U)$ is strongly $\mathcal{K}$-closed. In particular, $C_S(U)^{\mathcal{K}} = \{C_S(U)\}$. Appealing to Lemma 2.16, we see that $\mathrm{Out}_{\mathcal{H}}(C_S(U)) \cong C_2$, so $C_S(U)$ is not $\mathcal{H}$-radical.

Next, consider $Y = Q_1Q_2Q_3$. Since X normalizes Y, we have that $Y^{\mathcal{K}} = Y^{\mathcal{H}}$, and $N_H(Y)$ is of index 3 in $N_K(Y)$. Since $N_H(U) = C_H(U)\langle\tau\rangle = C_H(U)\langle\tau'\rangle$, it follows that $\mathrm{Out}_{\mathcal{H}}(Y) \cong S_3 \wr C_2 \times S_3$ is of index 3 in $\mathrm{Out}_{\mathcal{K}}(Y) \cong S_3 \wr S_3$.

Third, consider $Y = Q_1Q_2'R_3\langle\tau'\rangle$. This time, no element of $K - C_K(Z)$ normalizes Y, so $N_H(Y) = N_K(Y)$ by (3.4). However, we claim that $Y^{\mathcal{K}} = Y^{\mathcal{H}}$. For the proof, assume on the contrary that there is $k \in K$ with $Y^k \leqslant S$ and Y^k not H-conjugate to Y. Then $k \notin K - N_H(U)$, and k normalizes $R_0 = L_0 \cap S$, so $\langle k\rangle$ permutes the set $\{R_1, R_2, R_3\}$ transitively. It follows that the element $\tau'^k \in S$ interchanges R_3 and some other R_i by conjugation. This is a contradiction, as R_3 is a fixed point in the action of S on $\{R_1, R_2, R_3\}$ (see Section 2.4). Hence, $Y^{\mathcal{K}}$ meets exactly one $\mathcal{H}$-conjugacy class as claimed, and $\mathrm{Out}_{\mathcal{H}}(Y) = \mathrm{Out}_{\mathcal{K}}(Y)$, so Y is $\mathcal{H}$-radical.

Finally, consider $Y = Q_1Q_2R_3$. Then $N_X(Y) = \langle\tau\rangle$ is of order 2. As L_3 is $C_H(U)$-invariant, R_3 is strongly closed in $C_S(U)$ with respect to $C_H(U)$, so no element of $C_H(U)O_3(X) - C_H(U)$ normalizes Y. It follows that $N_H(Y) = N_K(Y)$ from (3.4). This also shows that if we fix a nontrivial element $x \in O_3(X)$, then representatives for the $\mathcal{H}$-conjugacy classes in $Y^{\mathcal{K}}$ may be taken as a subset of $\{Y, Y^x, Y^{x^2}\} = \{Q_1Q_2R_3, Q_1R_2Q_3, R_1Q_2Q_3\}$. As $\tau \in S$ interchanges $R_1Q_2Q_3$ and $Q_1R_2Q_3$, it follows that $Y^{\mathcal{K}}$ meets two $\mathcal{H}$-conjugacy classes (at most), with representatives Y and $Q_1R_2Q_3$. From $N_H(Y) = N_K(Y)$, we have $\mathrm{Out}_{\mathcal{H}}(Y) = \mathrm{Out}_{\mathcal{K}}(Y) \cong S_3 \wr C_2$, while $\mathrm{Out}_{\mathcal{H}}(Q_1R_2Q_3) \cong S_3 \times S_3$ is induced by $N_{L_1}(Q_1)N_{L_3}(Q_3)$. So indeed $Q_1R_2Q_3$ is $\mathcal{H}$-radical and not $\mathcal{H}$-conjugate to Y.

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Table 4. *$\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric radical subgroups, $l > 0$.*

P	$\lvert P\rvert$	$\mathrm{Out}_{\mathcal{K}}(P)$	$\mathrm{Out}_{\mathcal{H}}(P)$	$\mathrm{Out}_{\mathcal{F}}(P)$
S	2^{10+3l}	1	1	1
$C_S(U)$	2^{9+3l}	S_3	–	S_3
$Q_1Q_2Q_3$	2^8	$S_3 \wr S_3$	$S_3 \times S_3 \wr C_2$	$S_3 \wr S_3$
$Q_1Q_2Q_3'$	2^8	$S_3 \times S_3 \wr C_2$	S_3^3	$S_3 \times S_3 \wr C_2$
$Q_1Q_2R_3$	2^{8+l}	$S_3 \wr C_2$	$S_3 \wr C_2$	$S_3 \wr C_2$
$Q_1Q_2'R_3$	2^{8+l}	$S_3 \wr C_2$	$S_3 \wr C_2$	$S_3 \wr C_2$
$Q_1Q_2Q_3\langle\tau\rangle$	2^9	$S_3 \times S_3$	$S_3 \times S_3$	$S_3 \times S_3$
$Q_1Q_2Q_3'\langle\tau\rangle$	2^9	$S_3 \times S_3$	$S_3 \times S_3$	$S_3 \times S_3$
$Q_1Q_2R_3\langle\tau\rangle$	2^{9+l}	S_3	S_3	S_3
$Q_1Q_2'R_3\langle\tau'\rangle$	2^{9+l}	S_3	S_3	S_3
$R_1R_2Q_3\langle\tau\rangle$	2^{9+2l}	S_3	S_3	S_3
R_{1^7}	2^7	–	S_7	S_7
R_{1^7}'	2^6	–	S_6	S_6
R_{1^52}	2^{7+l}	–	S_5	S_5
$C_S(E/Z)$	2^{9+3l}	–	S_3	S_3
$C_S(E)$	2^{7+3l}	–	–	$\mathrm{GL}_3(2)$
A	2^4	–	–	$\mathrm{GL}_4(2)$

It remains to consider the case in which P is not $\mathcal{K}$-radical. We claim here that P is $\mathcal{H}$-conjugate to $Q_1R_2R_3$, the last remaining entry of Table 3. Observe first that $P \leqslant C_S(U) = R_0\langle \mathbf{d}\rangle$. Indeed, otherwise $Z(P) \cap U = Z$ would be $N_K(P)$-invariant, and so as $N_H(P) = N_K(P)$ by (3.4), this would yield that $\mathrm{Out}_{\mathcal{H}}(P) = \mathrm{Out}_{\mathcal{K}}(P)$ has no nontrivial normal 2-subgroups, contradicting the assumption that P is not $\mathcal{K}$-radical. Hence, $P \leqslant C_S(U) = R_0\langle \mathbf{d}\rangle$ as claimed, and so $U \leqslant Z(P)$ since P is centric.

Set $P_0 = P \cap L_0$, and for each $i = 1, 2, 3$, let P_i be the projection of P_0 in R_i as before; see the remarks just before Proposition 3.4. A reading of the first three paragraphs of the proof of [AC10, Lemma 10.2] reveals that the given argument applies to an $\mathcal{H}$-centric radical $P \leqslant S$ whose normalizer $N_H(P)$ is contained in K, our current situation (3.4). We conclude that $P_i = P \cap R_i$ and that $P_i \in \mathcal{Q}_i$ or $P_i = R_i$ for each $i = 1, 2, 3$. In particular, $P_0 = P_1P_2P_3$.

We next claim that $P = P_0$. Suppose on the contrary that $P_0 < P = C_P(U)$, and choose $d \in P - C_P(U)$. Then $d \in R_0\langle \mathbf{d}\rangle - R_0$, and since $\mathbf{d}$ interchanges the R_0-conjugacy classes of subgroups in $\mathcal{Q}_i$ for each i (c.f. Notation 2.12(e,f)), d has the same property. On the other hand, as d normalizes R_i and P, it normalizes $P_i = P \cap R_i$. We conclude that $P_i = R_i$ for each i. But then $P = R_0\langle \mathbf{d}\rangle = C_S(U)$ and $\mathrm{Out}_{\mathcal{H}}(P)$ is of order 2. Thus, P is not $\mathcal{H}$-radical, contrary to the original choice of P.

Finally, conjugating in $L_0\langle \mathbf{d}\rangle = C_H(U) \leqslant H$ if necessary, we have $P = Q_1R_2R_3$ or $R_1R_2Q_3$. But in the latter case, $\mathrm{Out}_{\mathcal{H}}(P) \cong C_2 \times S_3$ is induced by $\langle\tau\rangle \times N_{L_3}(Q_3)$, so again, P is not $\mathcal{H}$-radical, a contradiction. Thus, up to $\mathcal{H}$-conjugacy, $P = Q_1R_2R_3$ and $\mathrm{Out}_{\mathcal{H}}(P) \cong S_3$ is induced by $N_{L_1}(Q_1)$, and this is the only remaining entry in Table 3. □

Finally, we are able to describe the set of $\mathcal{F}$-centric radical subgroups, up to $\mathcal{F}$-conjugacy:

Theorem 3.12. *Let $\mathcal{F} = \mathrm{Sol}(5^{2^l})$ with $l > 0$. Representatives for the $\mathcal{F}$-conjugacy classes of $\mathcal{F}$-centric radical subgroups, together with their orders and automorphism groups, are listed in Table 4, where "–" indicates that the subgroup is not centric radical in that fusion system.*

Proof. This follows upon combining Propositions 3.1, 3.9, and 3.11. Note that $Q_1R_2R_3$ appears in Table 3, but it does not appear in Table 4 because it does not satisfy the hypotheses of Proposition 3.1(c)(ii): there is an involution in $K \leqslant G$ of the form τ^x for a nonidentity $x \in O_3(X)$ which normalizes $Q_1R_2R_3$ but is not contained in H. Indeed, $O_2(\mathrm{Out}_{\mathcal{F}}(Q_1R_2R_3))$ is of order 2 and induced by conjugation by this element, so $Q_1R_2R_3$ is not $\mathcal{F}$-radical. □

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

4. Number of projective simple modules

In this section, we calculate the number of projective simple modules for the outer automorphism groups in the various tables of the previous section. Let G be a finite group, and let $\mathrm{Irr}(G)$ be the set of ordinary irreducible characters of G, that is, those over a splitting field of characteristic 0. A character $\chi \in \mathrm{Irr}(G)$ is said to be of *p-defect d* if $|G|_p/\chi(1)_p = p^d$, where n_p denotes the p-part of the integer n.

Write $z(kG)$ for the number of projective simple kG-modules. Any projective simple module is the unique indecomposable module in the block of kG in which it lies. Blocks of kG of defect 0 are exactly those which contain such a module. Equivalently, blocks of defect 0 are exactly those which contain a unique ordinary irreducible character of defect 0. Thus, one can count projective simple kG-modules by looking at the list of irreducible character degrees for G.

Theorem 4.1. *Let G be a finite group and p be a prime. Then $z(kG)$ is the number of ordinary irreducible characters of defect* 0.

Proof. See [Nav98, Theorem 3.18]. □

Along with Theorem 4.1, the following result will be used in order to compute the character degrees of various solvable groups.

Theorem 4.2. *Let G be a finite group with normal subgroup A, and let $\Theta = [\mathrm{Irr}(A)/G]$ denote a set of representatives for the G-orbits on $\mathrm{Irr}(A)$. Assume that each irreducible character θ of A extends to an irreducible character $\widehat{\theta}$ of its inertia subgroup $I_G(\theta)$. Then there is a bijection*

$$\{\ (\theta,\beta)\ \mid\ \theta\in\Theta,\ \beta\in\mathrm{Irr}(I_G(\theta)/A)\ \}\longrightarrow\mathrm{Irr}(G) \tag{4.1}$$

given by sending (θ,β) to the induced character $(\widehat{\theta}\beta)\uparrow^G_{I_G(\theta)}$, where $\widehat{\theta}$ is any extension of θ, and where β is regarded also as an irreducible character of $I_G(\theta)$ with A in its kernel.

Proof. See [NT89, Chapter 3, Theorem 5.8 and Corollary 5.9]. □

Proposition 4.3. *Each of the groups listed in Table 5 has the stated number of blocks of defect zero.*

Proof. Let G be one of the groups listed in Table 5. It suffices by Theorem 4.1 to compute the number of characters having degree divisible by the 2-part of the group order. The character tables of $G = \mathrm{GL}_3(2), S_5, S_6, A_7, S_7, \mathrm{GL}_4(2) \cong A_8$ can be found in the ATLAS [CCN+85]. For those G which split as a direct product $G = G_1 \times G_2$, we use the fact that the irreducible characters of G are the pairwise tensor products of the irreducible characters of G_1 and G_2.

In all remaining cases, the character degrees are computed using Theorem 4.2 by taking A to be a normal elementary abelian 3-group which is complemented in G. Each irreducible character of A extends to its stabilizer in G in this case by [CR90, 11.8(ii)], so Theorem 4.2 applies. For a representative θ of an orbit of G/A on $\mathrm{Irr}(A)$, we compute the 2-parts of the index in G and of the irreducible character degrees $\beta(1)$ of the inertia subgroup $I_G(\theta)$. The pairs (θ,β) with $\theta(1)_2 \cdot [G : I_G(\theta)]_2 \cdot \beta(1)_2$ equal to the 2-part of the group order are recorded in Table 6. For example, suppose that $G = (C_3)^3 \rtimes (C_2 \times S_3)$ and set $A = (C_3)^3 \leqslant G$. For each $\theta \in \mathrm{Irr}(A)$, we have $\theta = \theta_{i_1} \otimes \theta_{i_2} \otimes \theta_{i_3}$ for some $1 \le i_j \le 3$. An S_3 factor

Table 5. *The number of projective simple modules.*

G	$z(kG)$	G	$z(kG)$	G	$z(kG)$
S_3	1	$(C_3 \times C_3) \overset{-1}{\rtimes} C_2$	4	S_6	1
$S_3 \times S_3$	1	$(C_3)^3 \rtimes (C_2 \times S_3)$	1	$S_3 \wr S_3$	1
$S_3 \times S_3 \times S_3$	1	$(C_3)^3 \rtimes (C_2 \times C_2)$	4	S_5	0
$S_3 \wr C_2$	0	$\mathrm{GL}_3(2)$	1	A_7	0
$S_3 \wr C_2 \times S_3$	0	$\mathrm{GL}_4(2)$	1	S_7	0

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Table 6. *Characters* $\chi \in \mathrm{Irr}(G)$ *of defect* 0.

G	A	θ	$I_G(\theta)/A$	$\beta(1)$	$\chi(1)$
S_3	$O_3(G)$	$[2]$	1	1	2
$S_3 \wr C_2$	$O_3(G)$	–	–	–	–
$S_3 \wr S_3$	$O_3(G)$	$[2, 2, 2]$	S_3	2	16
$(C_3)^2 \overset{-1}{\rtimes} C_2$	$O_3(G)$	$[1, 1], [1, 2], [2, 1], [2, 2]$	1, 1, 1, 1	1, 1, 1, 1	2, 2, 2, 2
$(C_3)^3 \overset{-1,(1,2)}{\rtimes} (C_2 \times C_2)$	$O_3(G)$	$[1, 2, 1], [1, 2, 2], [1, 2, 3], [2, 3, 2]$	1, 1, 1, 1	1, 1, 1, 1	4, 4, 4, 4
$(C_3)^3 \overset{-1,\wr}{\rtimes} (C_2 \times S_3)$	$(C_3)^3$	$[2, 2, 2]$	S_3	2	4

The dashes mean the group G listed on the line has no irreducible character of defect 0.

in $G/A \cong S_3 \times C_2$ acts on $\mathrm{Irr}(A)$ by permuting the θ_{i_j}, while the C_2 factor fixes θ_1 and interchanges θ_2 and θ_3 in each coordinate. One computes that there are six orbits on irreducible characters. The pair (θ, β), where $\theta = \theta_2 \otimes \theta_2 \otimes \theta_2$ and β is the degree 2 irreducible character of $I_G(\theta)/A \cong S_3$, gives rise to the only irreducible character of G of 2-defect 0. The remaining cases are summarized in Table 6, where a representative $\theta_{i_1} \otimes \theta_{i_2} \otimes \theta_{i_3}$ is abbreviated to $[i_1, i_2, i_3]$, for example. □

Using Tables 1, 3, and 4, we can give a count of the number of weights.

Corollary 4.4. *For* $\mathcal{D} \in \{\mathcal{H}, \mathcal{F}\}$ *and all* $l \geq 0$*, the number of weights associated with the Külshammer-Puig pair* $(\mathcal{D}, 0)$ *is*

$$\mathbf{w}(\mathcal{D}, 0) = 12.$$

Note that $\mathbf{w}(\mathcal{H}, 0) = 12$ is known as a consequence of results in [An93].

5. Külshammer-Puig classes

We give here a proof of Theorem 1.1 essentially by direct computation. Throughout this section, we fix an arbitrary nonnegative integer l and set $q = 5^{2^l}$. We adopt the notation $\mathcal{F}, \mathcal{H}, \mathcal{K}$ from Section 2. These systems depend implicitly on q.

Recall that the Schur multiplier of a finite group G is the cohomology group $M(G) := H^2(G, \mathbb{C}^\times)$. It is a finite abelian group. Given any algebraically closed field k of characteristic 2, the $2'$-primary part of $M(G)$ is isomorphic to $H^2(G, k^\times)$. The approach taken to showing Theorem 1.1 requires the explicit computation of $H^2(G, k^\times)$ (the values of the functor $\mathcal{A}^2$) for each group G appearing as the outer automorphism group of a centric radical in Section 3. The computation of Schur multipliers of finite groups is typically a delicate task. In our case, the task is simpler for two reasons. First, the outer automorphism groups are relatively small finite groups. Second, the task is simpler because of the following lemma, which allows us in many cases to reduce the computation of the odd part of the Schur multiplier to computations of $H^2(G, \mathbb{F}_p)$ for odd primes p. A finite group is said to be *p-perfect* if it has no nontrivial p-group as a quotient.

Lemma 5.1. *Let* $\mathcal{D} \in \{\mathcal{F}, \mathcal{H}, \mathcal{K}\}$*. Then for each subgroup* $P \in \mathcal{D}^{rc}$ *and each odd prime p, the outer automorphism group* $\mathrm{Out}_{\mathcal{D}}(P)$ *is p-perfect.*

Proof. Direct inspection of the outer automorphism groups in Tables 1 and 4. □

The next lemma collects various standard results on group cohomology stated in the special cases in which they will be used. We thank the referee for several simplifications in our original arguments.

Lemma 5.2. *Let G and H be finite groups, and let k be an algebraically closed field of characteristic* 2*. Write* $|G| = 2^r w$*, where w is odd. The following hold.*

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

(a) *For any abelian group A with trivial G-action, $H^1(G, A) = \mathrm{Hom}(G, A)$.*
(b) *There is a surjective map*

$$H^2(G, \mathbb{Z}/w\mathbb{Z}) \to H^2(G, k^\times),$$

which is an isomorphism if G is p-perfect for every odd prime p dividing $|G|$.
(c) *If $H^2(G, \mathbb{F}_p) = 0$ for all odd primes p, then $H^2(G, k^\times) = 0$.*
(d) *If p is odd and G is a p-perfect group with cyclic Sylow p-subgroups, then $H^2(G, \mathbb{F}_p) = 0$.*
(e) *If G is a p-perfect group with an elementary abelian Sylow p-subgroup V of order p^2, then*

$$H^2(G, \mathbb{F}_p) = \begin{cases} \mathbb{F}_p & \textit{if } \mathrm{Aut}_G(V) \subseteq \mathrm{SL}(V), \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

(f) *If G and H are p-perfect, then*

$$H^2(G \times H, \mathbb{F}_p) \cong H^2(G, \mathbb{F}_p) \oplus H^2(H, \mathbb{F}_p).$$

(g) *Let p be an odd prime. If G is p-perfect and the p-part $M(G)_p$ of the Schur multiplier of G is of exponent at most p, then*

$$M(G)_p = H^2(G, \mathbb{C}^\times) \otimes \mathbb{Z}_{(p)} \cong H^2(G, \mathbb{F}_p) \cong H^2(G, k^\times) \otimes \mathbb{Z}_{(p)}.$$

Here, $\mathbb{Z}_{(p)}$ denotes the p-local integers.
(h) *(Schur) If $M(G)$ has exponent e, then e^2 divides the order of G.*

Proof.

(a) This follows from the description $H^1(G, A)$ as the group of derivations $G \to A$ [Wei94, Corollary 6.4.6].
(b) Fix a Sylow 2-subgroup S of G. Since $k^\times$ has all odd roots of unity, powering by w is a surjective endomorphism with kernel $\mathbb{Z}/w\mathbb{Z}$. Thus, there is an exact sequence

$$H^1(G, k^\times) \to H^2(G, \mathbb{Z}/w\mathbb{Z}) \to H^2(G, k^\times) \to H^2(G, k^\times).$$

The last map is multiplication by $w = |G : S|$, and so it factors as

$$H^2(G, k^\times) \xrightarrow{\mathrm{res}} H^2(S, k^\times) \xrightarrow{\mathrm{tr}} H^2(G, k^\times)$$

by [Ben98b, Proposition 3.6.17] applied with $M = M' = k^\times$. Since $H^2(S, k^\times) = 0$, we conclude that the last map is 0. The middle map is therefore a surjection, and since $H^1(G, k^\times) = \mathrm{Hom}(G, k^\times)$ by (a), we see that it is an isomorphism if G is p-perfect for every odd prime p dividing $|G|$.
(c) This follows upon filtering $\mathbb{Z}/w\mathbb{Z}$ by subgroups of prime order, considering the corresponding long exact sequences in cohomology, and applying (b).
(d) Let P be a Sylow p-subgroup with p odd. Restriction induces an isomorphism $H^*(G, \mathbb{F}_p) \to H^*(N_G(P), \mathbb{F}_p) \cong H^*(P, \mathbb{F}_p)^{\mathrm{Aut}_G(P)}$ by [Ben98b, Corollary 3.6.19] applied with $M = M' = \mathbb{F}_p$. Now $H^*(P, \mathbb{F}_p) \cong \mathbb{F}_p[x, y]/(x^2)$ with $\deg x = 1$ and $\deg y = 2$ [Ben98b, Proposition 3.5.5], and the Bockstein $H^1(P, \mathbb{F}_p) \to H^2(P, \mathbb{F}_p)$ is an isomorphism of $N_G(P)$-modules (cf. [Ben98b, p. 132, Example]). As $N_G(P)$ has no invariants in $H^1(P, \mathbb{F}_p)$ by assumption, it also has no invariants on $H^2(P, \mathbb{F}_p)$.
(e) Restriction to V again identifies $H^*(G, \mathbb{F}_p)$ with the invariants $H^*(V, \mathbb{F}_p)^{\mathrm{Aut}_G(V)}$. Now

$$H^*(V, \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(x_1, x_2) \otimes \mathbb{F}_p[y_1, y_2],$$

with $\deg x_i = 1$ and $\deg y_i = 2$ by [Ben98b, Corollary 3.5.7(ii)], so that

$$H^1(V,\mathbb{F}_p) = \langle x_1, x_2\rangle_{\mathbb{F}_p} \quad \text{and} \quad H^2(V,\mathbb{F}_p) = \langle y_1, y_2, x_1x_2\rangle_{\mathbb{F}_p}.$$

Here, $H^1(V,\mathbb{F}_p)$ is the natural module for $\mathrm{Aut}(V) \cong \mathrm{GL}(V)$, while $H^2(V,\mathbb{F}_p)$ is the direct sum of the natural module $\langle y_1, y_2\rangle_{\mathbb{F}_p}$ and $\langle x_1x_2\rangle_{\mathbb{F}_p}$ on which $\mathrm{GL}(V)$ acts via the determinant map. By assumption, $\mathrm{Aut}_G(V) \leqslant \mathrm{GL}(V)$ has no fixed points on the natural module, so $H^2(V,\mathbb{F}_p)$ is nontrivial generated by x_1x_2 if and only if every element of $\mathrm{Aut}_G(V)$ has determinant 1.

(f) This follows from the Künneth theorem [Ben98a, Theorem 2.7.1] and the assumption.

(g) Powering by p on $\mathbb{C}^\times$ gives the exact sequence

$$H^1(G,\mathbb{C}^\times) \to H^2(G,\mathbb{F}_p) \to H^2(G,\mathbb{C}^\times) \xrightarrow{p} H^2(G,\mathbb{C}^\times).$$

The assumptions imply that tensoring with $\mathbb{Z}_{(p)}$ kills $H^1(G,\mathbb{C}^\times)$ and the last map. Hence, $H^2(G,\mathbb{F}_p) \cong H^2(G,\mathbb{C}^\times) \otimes \mathbb{Z}_{(p)}$. Since $M_{2'}(G)$ is isomorphic to $H^2(G,k^\times)$ ([Kar87, Proposition 2.1.14]) and p is odd, the p-primary part of $H^2(G,k^\times)$ is of exponent at most p by assumption. Thus, the exact same argument with $k^\times$ in place of $\mathbb{C}^\times$ shows that $H^2(G,\mathbb{F}_p) \cong H^2(G,k^\times) \otimes \mathbb{Z}_{(p)}$.

(h) We refer to [Kar87, Theorem 2.1.5] for a proof. □

We are interested in computing the cohomology of the functor $H^2(-,k^\times)$ defined on the subdivision category of the full subcategory of the fusion systems $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{K}$, respectively, on the collection of centric subgroups. Let $\mathcal{C}$ be any full subcategory of a saturated fusion system. Recall from [Lin19, Definition 8.13.2] that the *subdivision category* $S(\mathcal{C})$ of proper inclusions is the category with objects of the chains $\sigma = (X_0 < X_1 < \cdots < X_m)$ (of proper inclusions) in $\mathcal{C}$; here, m is the *length* $|\sigma|$ of σ. Given another object $\tau = (Y_0 < Y_1 < \cdots < Y_n) \in S(\mathcal{C})$, a morphism from σ to τ consists of an order preserving function $\beta\colon \{0,1\ldots,m\} \to \{0,1,\ldots,n\}$ together with isomorphisms $\varphi_i\colon X_i \to Y_{\beta(i)}$ in $\mathcal{C}$ for each $i \in \{0,1,\ldots,m\}$, which make the evident skewed ladder commute. In particular, the automorphism group of the chain σ in $\mathcal{C}$ may be identified with the group of automorphisms of X_m which preserve X_i for all $0 \leqslant i \leqslant m$. We write $[S(\mathcal{C})]$ for the partially ordered set of isomorphism classes of objects in $S(\mathcal{C})$, where $[\sigma] \leqslant [\tau]$ if there are representatives $\sigma' \in [\sigma]$ and $\tau' \in [\tau]$ and a morphism $\sigma' \to \tau'$ in $S(\mathcal{C})$.

There is a simpler resolution than the standard bar resolution for computing cohomology of a functor defined on the subdivision category of any EI-category, which was given in [Lin05].

Lemma 5.3. *Let $\mathcal{C}$ be any full subcategory of a saturated fusion system, and let $F\colon [S(\mathcal{C})] \to \mathcal{A}b$ be a covariant functor. The cohomology groups $H^n([S(\mathcal{C})], F)$, and thus the derived functors of* $\lim F$*, can be computed via the cochain complex $C^*(F)$ defined as follows:*

$$C^n(F) = \bigoplus_{|\sigma|=n} F([\sigma]),$$

whose elements are viewed as functions α from isomorphism classes of chains of length n, and where $|\sigma|$ denotes the length of σ. The coboundary map $\delta^n\colon C^n(F) \to C^{n+1}(F)$ is defined by

$$\delta^n(\alpha)([\sigma]) = \sum_{i=0}^{n} (-1)^i F(\iota_{[\sigma(i)],[\sigma]})(\alpha([\sigma(i)])),$$

where $\sigma(i)$ denotes the chain σ with its ith term removed and $\iota_{[\sigma(i)],[\sigma]}$ denotes the unique morphism from $[\sigma(i)]$ to $[\sigma]$.

Proof. This is [Lin05, Proposition 3.2], applied as in [Par10, Lemma 3.1]. □

Finally, the aim of the following, highly specialized lemma is to orient the reader to the way in which Lemma 5.3 will be used later in the proof of Theorem 5.6.

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Lemma 5.4. *Let $\mathcal{C}$ be a saturated fusion system, and let $F\colon [S(\mathcal{C}^{cr})] \to \mathcal{A}b$ be a covariant functor. Then $\lim_{[S(\mathcal{C}^{cr})]} F = 0$ under either of the following conditions.*

(a) *$F([X]) = 0$ for all subgroups $X \in \mathcal{C}^{cr}$;*
(b) *F is zero on all but two distinct chains $[X_0]$ and $[X_1]$ of length zero, and there exists a subgroup $Y \in \mathcal{C}^{cr}$, such that $F([Y]) = 0$,*
  (i) *$X_0 < X_1 > Y$, and*
  (ii) *the maps $F([X_0]) \to F([X_0 < X_1])$ and $F([X_1]) \to F([Y < X_1])$ are injective.*

Proof. We view $\lim_{[S(\mathcal{C}^{cr})]} F$ as the degree 0 cohomology of the functor F. As such, it can be computed by using the cochain complex of Lemma 5.3. The coboundary map $\delta^0\colon C^0(F) \to C^1(F)$ on 0-cochains is obtained by extending linearly from

$$\delta^0(\alpha)([X < X']) = F(\iota_{[X'],[X<X']})(\alpha([X'])) - F(\iota_{[X],[X<X']})(\alpha([X])).$$

With this in mind, the two parts of the lemma are simply ways of saying that the kernel of δ^0, and thus $\lim_{[S(\mathcal{C}^{cr})]} F$, is 0. This is trivial in the case of part (a). The assumption in (b) implies that $C^0(F) = F([X_0]) \oplus F([X_1])$, and then (i) and (ii) ensure that the composite

$$C^0(F) \xrightarrow{\delta^0} C^1(F) \xrightarrow{\text{proj}} F([X_0 < X_1]) \oplus F([Y < X_1])$$

is injective. □

We now begin the computation of the higher limits of $H^2(-, k^\times)$ in the cases of interest.

Lemma 5.5. *Fix an algebraically closed field k of characteristic 2, and let $\mathcal{D} \in \{\mathcal{F}, \mathcal{H}, \mathcal{K}\}$. For each $P \in \mathcal{D}^{cr}$, one of the following holds.*

(a) *$H^2(\mathrm{Out}_{\mathcal{D}}(P), k^\times) = 0$, or*
(b) *$l = 0$, $H^2(\mathrm{Out}_{\mathcal{D}}(P), k^\times) \cong H^2(\mathrm{Out}_{\mathcal{D}}(P), \mathbb{F}_3) \cong C_3$, and either*
  (i) *$P = QR_{17}$,*
  (ii) *$P = Q$ and $\mathcal{D} = \mathcal{H}$, or*
  (iii) *$P = R_{17}$ and $\mathcal{D} = \mathcal{H}$ or $\mathcal{F}$.*

Proof. We first prove the lemma for $l > 0$. Fix $P \in \mathcal{D}^{cr}$ appearing in Tables 2, 3, or 4, and let $G = \mathrm{Out}_{\mathcal{D}}(P)$ be its outer automorphism group in $\mathcal{D}$, for short. In order to show that (a) holds in this case ($l > 0$), it suffices to show that $H^2(G, \mathbb{F}_p) = 0$ for all odd primes p by Lemma 5.2(c). Now G is p-perfect for all odd primes p by Lemma 5.1. An inspection of the tables shows that one of three cases holds: (1) G has cyclic Sylow p-subgroups for all odd primes p, (2) G has cyclic Sylow p-subgroups for all $p \geqslant 5$ and elementary Sylow 3-subgroups of order 3^2, or (3) $G \cong S_3 \wr S_3$. By Lemma 5.2(d), we have $H^2(G, \mathbb{F}_p) = 0$ for all odd primes p in Case (1). Assume (2). Then $G \cong S_6$, S_7, $GL_4(2)$, $S_3 \times S_3$, or $S_3 \wr C_2$. In all cases, $H^2(G, \mathbb{F}_p) = 0$ for all $p \geqslant 5$, again by Lemma 5.2(d). For a Sylow 3-subgroup V of G, we have $\mathrm{Aut}_G(V) \nleq \mathrm{SL}(V)$ by direct computation, and so $H^2(G, \mathbb{F}_3) = 0$ in Case (2) as well, by Lemma 5.2(e). Finally, assume Case (3), so that $G \cong S_3 \wr S_3$. Again, we just need to show that $H^2(G, \mathbb{F}_3) = 0$. In this case, one can apply the Lyndon-Hochschild-Serre spectral sequence [Wei94, Lyndon-Hochschild-Serre Spectral Sequence 6.8.2] with respect to the base B of the wreath product. The relevant parts of the E_2-page are

- $H^0(S_3, H^2(B, \mathbb{F}_3)) = 0$ (the coefficients are 0);
- $H^1(S_3, H^1(B, \mathbb{F}_3)) = 0$ (since the base is 3-perfect); and
- $H^2(S_3, H^0(B, \mathbb{F}_3)) = 0$ (trivial invariants).

Hence, $H^2(G, \mathbb{F}_3) = 0$. This completes the proof in the case $l > 0$. We now turn to the case $l = 0$. By inspection of Table 1, either it was shown in the previous case that $H^2(\mathrm{Out}_{\mathcal{D}}(P), k^\times) = 0$, or else the subgroup P is listed in (b)(i)–(b)(iii) of the lemma. We go through these three cases in turn, and we set $G = \mathrm{Out}_{\mathcal{D}}(P)$ again for short.

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Case 1. $P = QR_{1^7}$ and $G \cong (C_3 \times C_3) \overset{-1}{\rtimes} \langle \mathbf{d} \rangle$:

Recall that $M_{2'}(G) \cong H^2(G, k^\times)$ is an abelian group of odd order, so it must be $M_3(G)$. Let e be its exponent. From Lemma 5.2(h), e^2 divides $|G| = 3^2 \cdot 2$. Hence, $e = 1$ or 3. By Lemma 5.2(g), we see that

$$H^2(G, \mathbb{F}_3) \cong H^2(G, k^\times) \otimes \mathbb{Z}_{(3)} = H^2(G, k^\times),$$

and $H^2(G, \mathbb{F}_3) \cong \mathbb{F}_3$ by Lemma 5.2(e) ($\mathbf{d}$ acts by minus the identity). This completes the proof of Case 1.

Case 2. $P = Q$: Suppose first that $\mathcal{D} = \mathcal{H}$. Then $G := \mathrm{Out}_{\mathcal{D}}(P) \cong C_3^3 \rtimes (C_2 \times C_2)$ with one factor inverting C_3^3 and the other swapping the first two C_3 factors. We first claim that the exponent of $H^2(G, k^\times)$ is not divisible by 3^2. Indeed, this follows directly from Lemma 5.2(h), as otherwise $|G|$ would be divisible by 3^4, which is not the case. It follows that $H^2(G, k^\times) \cong H^2(G, \mathbb{F}_3)$ by Lemma 5.2(g). Now

$$H^2(C_3^3, \mathbb{F}_3) = \langle x_1x_2, x_1x_3, x_2x_3, y_1, y_2, y_3 \rangle_{\mathbb{F}_3}, \tag{5.1}$$

where the y_i are polynomial generators and the x_i are exterior generators. Further, $H^2(G, \mathbb{F}_3)$ is the invariants under $\langle \mathbf{d}, \tau \rangle$ here. We compute directly that the invariants are spanned by $x_1x_3 + x_2x_3$, and so have dimension 1. Thus, $H^2(G, k^\times) \cong H^2(G, \mathbb{F}_3) \cong C_3$.

Now suppose that $\mathcal{D} = \mathcal{K}$ or $\mathcal{F}$. Then $G = \mathrm{Out}_{\mathcal{D}}(P) \cong C_3^3 \rtimes (\langle \mathbf{d} \rangle \times S_3)$ with $\mathbf{d}$ inverting, and we have

$$H^2(G, \mathbb{F}_3) = H^2(C_3 \wr C_3, \mathbb{F}_3)^{\langle \mathbf{d}, \tau \rangle},$$

since a Sylow 3-subgroup is normal in G. Let $W \cong C_3 \wr C_3$ be the Sylow 3-subgroup of G, and write W_0 for the base subgroup of W. By a result of Nakaoka [Nak61, Theorem 3.3], we have

$$H^2(C_3 \wr C_3, \mathbb{F}_3) = H^0(C_3, H^2(W_0, \mathbb{F}_3)) \oplus H^1(C_3, H^1(W_0, \mathbb{F}_3)) \oplus H^2(C_3, H^0(W_0, \mathbb{F}_3)).$$

The middle term above vanishes: by Lemma 5.2(a),

$$H^1(W_0, \mathbb{F}_3) \cong \mathrm{Hom}_{\mathbb{F}_3}(\mathbb{F}_3[C_3], \mathbb{F}_3) = \mathrm{Coind}_1^{C_3} \mathbb{F}_3$$

as a W/W_0-module, so that $H^1(C_3, H^1(W_0, \mathbb{F}_3)) \cong H^1(C_3, \mathrm{Coind}_1^{C_3} \mathbb{F}_3) = 0$ by Shapiro's Lemma [Wei94, Lemma 6.3.2]. Hence,

$$H^2(C_3 \wr C_3) = H^2(W_0, \mathbb{F}_3)^{C_3} \oplus H^2(C_3, \mathbb{F}_3).$$

With notation as in (5.1), the first summand is spanned by $y_1 + y_2 + y_3$ and $x_1x_2 + x_2x_3 + x_3x_1$, both being negated by the action of $\mathbf{d}\tau$ (note that τ negates $x_1x_2 + x_2x_3 + x_3x_1$). Similarly, the second summand is also negated by $\mathbf{d}\tau$. Hence, $H^2(G, \mathbb{F}_3) = 0$, and we conclude that $H^2(G, k^\times) = 0$ by Lemma 5.2(c).

Case 3. $P = R_{1^7}$: Then $\mathcal{D} = \mathcal{H}$ or $\mathcal{F}$, and $\mathrm{Out}_{\mathcal{D}}(P) \cong A_7$. The odd part of the Schur multiplier is well known to be C_3. Alternatively, apply Lemma 5.2(h) to see that the exponent of the odd part of the Schur multiplier is 3, and then use Lemma 5.2(e,g). □

Theorem 5.6. *For q an odd prime power and $\mathcal{F} = \mathrm{Sol}(q)$, we have*

$$\lim_{[S(\mathcal{F}^{cr})]} \mathcal{A}^2_{\mathcal{F}} = 0.$$

Proof. Let $(\mathcal{F}, \alpha)$ be a Külshammer-Puig pair. When $l > 0$, all minimal elements of the partially ordered set $[S(\mathcal{F}^{cr})]$, namely, the chains $\sigma = (R)$ of length one, have $\alpha_{[\sigma]} = 0$ by Lemma 5.5. Thus, the theorem holds in this case by Lemma 5.4(a).

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

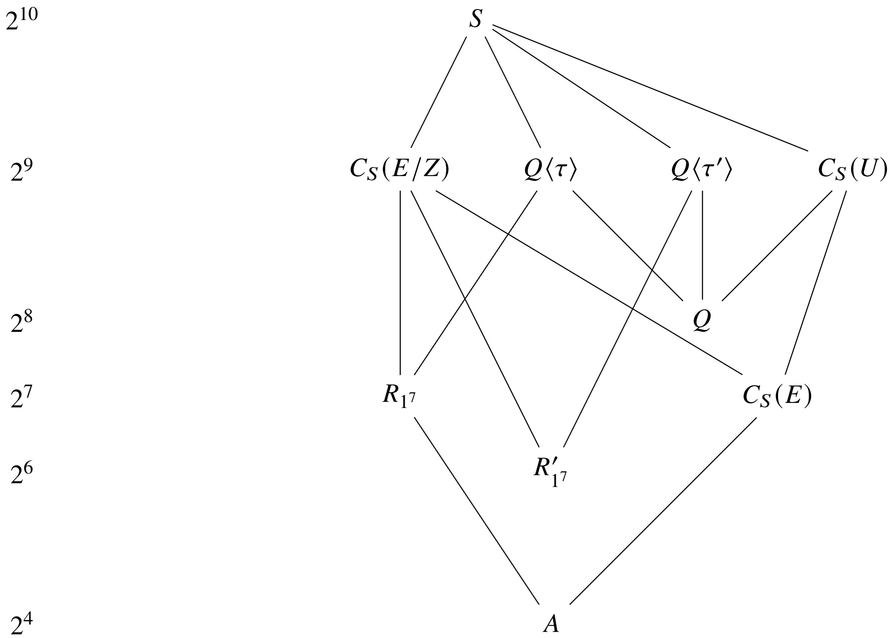

Figure 1. *Hasse diagram for* $[\mathrm{Sol}(q)^{cr}]$, $q \equiv \pm 3 (\mathrm{mod}\ 8)$.

It remains to consider the case $l = 0$. Then $H^2(\mathrm{Out}_{\mathcal{F}}(P), k^\times)$ is nonzero (of order 3) if and only if $P = R_{1^7}$ or QR_{1^7}. For the remainder of the proof, we set $R := R_{1^7}$, for short. Consider the chains $\sigma := (R < QR)$ and $\tau := (Q < QR)$. All three subgroups R, Q, and QR are weakly $\mathcal{F}$-closed by Lemma 3.7; hence, $\mathrm{Aut}_{\mathcal{F}}(\sigma) = \mathrm{Aut}_{\mathcal{F}}(QR) = \mathrm{Aut}_{\mathcal{F}}(\tau)$ and the induced map on $\mathcal{A}^2$ is the identity in each of these cases. We next prove that the induced map

$$H^2(\mathrm{Aut}_{\mathcal{F}}(R), k^\times) \to H^2(\mathrm{Aut}_{\mathcal{F}}(\sigma), k^\times) \tag{5.2}$$

is injective. Once this is done, Lemma 5.4(b) then yields that $\lim_{[S(\mathcal{F}^{cr}])} \mathcal{A}^2 = 0$.

By Lemmas 3.7 and 3.8, QR contains R as a normal subgroup with index 4, and $QR/R \cong C_2 \times C_2$. Hence, Lemma 2.7 yields that the restriction map $\mathrm{Aut}_{\mathcal{F}}(\sigma) = \mathrm{Aut}_{\mathcal{F}}(QR) \to \mathrm{Aut}_{\mathcal{F}}(R)$ induces an isomorphism

$$\mathrm{Aut}_{\mathcal{F}}(QR)/\mathrm{Aut}_R(QR) \longrightarrow N_{\mathrm{Out}_{\mathcal{F}}(R)}(\mathrm{Out}_{QR}(R)).$$

This isomorphism identifies $\mathrm{Aut}_{\mathcal{F}}(QR)/\mathrm{Aut}_R(QR)$ with the normalizer in $\mathrm{Out}_{\mathcal{F}}(R) \cong A_7$ of the four subgroup $QR/R \cong \mathrm{Aut}_{QR}(R)/\mathrm{Aut}_R(R)$.

Since this normalizer contains a Sylow 3-subgroup of A_7, we conclude that the restriction map

$$\rho_3 \colon H^2(\mathrm{Aut}_{\mathcal{F}}(R), \mathbb{F}_3) \longrightarrow H^2(\mathrm{Aut}_{\mathcal{F}}(\sigma), \mathbb{F}_3)$$

in $\mathbb{F}_3$-cohomology is injective by [Ben98b, Corollary 3.6.18]. By [Wei94, Functoriality of H^* and Restriction 6.7.6] on the functoriality of restriction, the diagram

$$\begin{tikzcd}
H^2(\mathrm{Aut}_{\mathcal{F}}(R), \mathbb{F}_3) \arrow[r, "\rho_3"] \arrow[d] & H^2(\mathrm{Aut}_{\mathcal{F}}(\sigma), \mathbb{F}_3) \arrow[d] \\
H^2(\mathrm{Aut}_{\mathcal{F}}(R), k^\times) \otimes \mathbb{Z}_{(3)} \arrow[r, "\rho_{(3)}"] & H^2(\mathrm{Aut}_{\mathcal{F}}(\sigma), k^\times) \otimes \mathbb{Z}_{(3)}
\end{tikzcd}$$

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

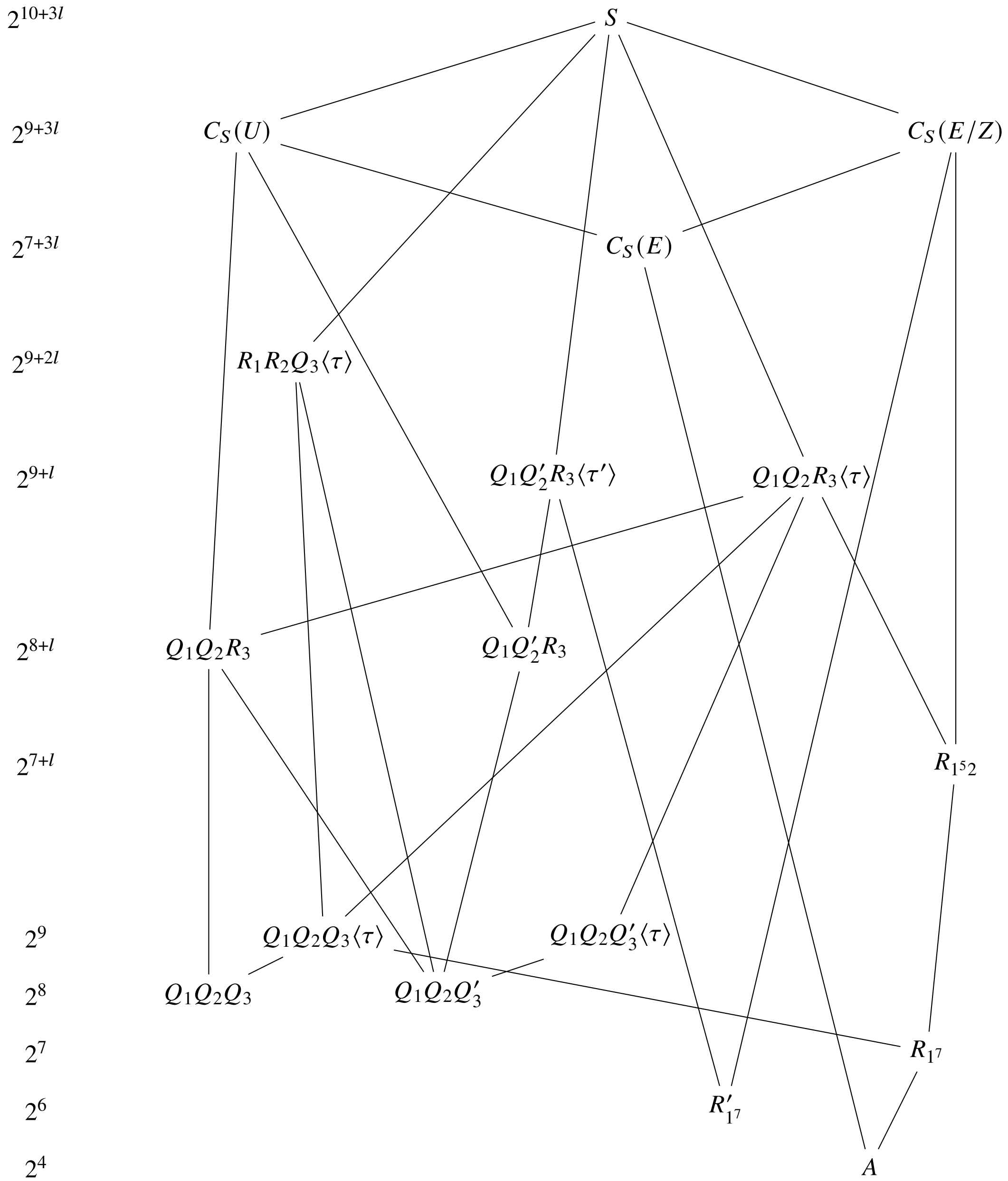

Figure 2. *Hasse diagram for* $[\mathrm{Sol}(q)^{cr}]$, $q \equiv \pm 7 (\mathrm{mod}\ 16)$*, that is, for* $l = 1$.

commutes. Here, the vertical arrows are given by the isomorphisms of Lemma 5.2(g), which applies since $\mathrm{Aut}_{\mathcal{F}}(R)$ and $\mathrm{Aut}_{\mathcal{F}}(\sigma)$ have Sylow 3-subgroups of order 3^2. Therefore, $\rho_{(3)}$ is injective, as claimed. This completes the proof in the case $l = 0$ and of the theorem. ◻

Proof of Theorem 1.1. By [LO02], there exists a centric linking system associated with $\mathcal{F}$. Thus, [Lib11, Theorem 1.2] yields that

$$\lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}} \cong \lim_{[S(\mathcal{F}^{cr})]} \mathcal{A}^2_{\mathcal{F}}.$$

The result now follows from Theorem 5.6. ◻

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press

Appendix: Hasse diagrams

Displayed without proof are Hasse diagrams for the partially ordered set of isomorphism classes of centric radicals in $\mathrm{Sol}(q)$ that were computed with the aid of Magma [BCP97].

Acknowledgements. We thank Markus Linckelmann and Radha Kessar for encouragement, Geoffrey Robinson for comments on an earlier version, and Dave Benson and Ian Leary for helpful conversations concerning Section 5. We express our gratitude to both referees, whose thorough readings and many suggestions resulted in numerous improvements to the paper. JL was supported by Marie Curie Fellowship No. 707778 and National Science Foundation Grant DMS-1902152 during the preparation of this manuscript. JS acknowledges personal support from the Heilbronn Institute for Mathematical Research and the Engineering and Physical Sciences Research Council (grant EP/W028794/1).

Competing interest. The authors have no competing interest to declare.

References

[AC10] M. Aschbacher and A. Chermak, 'A group-theoretic approach to a family of 2-local finite groups constructed by Levi and Oliver', *Ann. of Math. (2)* **171**(2) (2010), 881–978.

[AKO11] M. Aschbacher, R. Kessar and B. Oliver, 'Fusion systems in algebra and topology', in *London Mathematical Society Lecture Note Series* vol. **391** (Cambridge University Press, Cambridge, 2011). MR 2848834

[An93] J. B. An, '2-weights for classical groups', *J. Reine Angew. Math.* **439** (1993), 159–204. MR 1219699

[BCP97] W. Bosma, J. Cannon and C. Playoust, 'The Magma algebra system. I. The user language', *J. Symbolic Comput.* **24**(3–4) (1997), 235–265. *Computational Algebra and Number Theory* (London, 1993). MR 1484478

[Ben94] D. Benson, 'Conway's group Co_3 and the Dickson invariants', *Manuscripta Math.* **85**(2) (1994), 177–193.

[Ben98a] D. J. Benson, 'Representations and cohomology. I', in *Cambridge Studies in Advanced Mathematics* vol. 30, second edn. (Cambridge University Press, Cambridge, 1998). Basic representation theory of finite groups and associative algebras.

[Ben98b] D. J. Benson, 'Representations and cohomology. *II*', in *Cambridge Studies in Advanced Mathematics* vol. 31, second edn. (Cambridge University Press, Cambridge, 1998). Cohomology of groups and modules.

[Ben98c] D. J. Benson, 'Cohomology of sporadic groups, finite loop spaces, and the Dickson invariants', in Geometry and cohomology in group theory (Durham, 1994), *London Mathematical Society Lecture Note Series* vol. 252 (Cambridge University Press, Cambridge, 1998), 10–23. MR 1709949 (2001i:55017)

[BLO03] C. Broto, R. Levi and B. Oliver, 'The homotopy theory of fusion systems', *J. Amer. Math. Soc.* **16**(4) (2003), 779–856 (electronic).

[CCN+85] J. H. Conway , R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of Finite Groups (Oxford University Press, Eynsham , 1985). *Maximal subgroups and ordinary characters for simple groups*. With computational assistance from J. G. Thackray. MR 827219

[COS08] A. Chermak, B. Oliver and S. Shpectorov, 'The linking systems of the Solomon 2-local finite groups are simply connected', *Proc. Lond. Math. Soc. (3)* **97**(1) (2008), 209–238. MR 2434096 (2009g:55018)

[CR90] C. W. Curtis and I. Reiner, 'Methods of representation theory. Vol. I', in *Wiley Classics Library Series* (John Wiley & Sons Inc., New York, 1990). With applications to finite groups and orders. *Reprint of the 1981 original*, A Wiley-Interscience Publication.

[Cra11] D. A. Craven, 'The theory of fusion systems', in *Cambridge Studies in Advanced Mathematics* vol. 131 (Cambridge University Press, Cambridge, 2011). An algebraic approach. *MR* **2808319**

[GLS98] D. Gorenstein, R. Lyons and R. Solomon, 'The classification of the finite simple groups. Number 3. Part I. Chapter A', in *Mathematical Surveys and Monographs* vol. 40 (American Mathematical Society, Providence, RI, 1998). Almost simple K-groups. MR 1490581 (98j:20011)

[HLL23] E. Henke, A. Libman and J. Lynd, 'Punctured groups for exotic fusion systems', Preprint, 2023, arXiv:2201.07160.

[Kar87] G. Karpilovsky, 'The Schur multiplier', in *London Mathematical Society Monographs. New Series* vol. 2 (The Clarendon Press, Oxford University Press, New York, 1987). *MR* 1200015

[Kes06] R. Kessar, 'The Solomon system $\mathcal{F}Sol(3)$ does not occur as fusion system of a 2-block', *J. Algebra* **296**(2) (2006), 409–425.

[KLLS19] R. Kessar, M. Linckelmann, J. Lynd and J. Semeraro, 'Weight conjectures for fusion systems', *Adv. Math.* **357** (2019), 106825, 40. MR 4013803

[KMS20] R. Kessar, G. Malle and J. Semeraro, 'Weight conjectures for ℓ-compact groups and spetses', Preprint, 2020, arXiv:2008.07213.

[Lib11] A. Libman, 'The gluing problem in the fusion systems of the symmetric, alternating and linear groups', *J. Algebra* **341** (2011), 209–245. MR 2824518

[Lin04] M. Linckelmann, 'Fusion category algebras', *J. Algebra* **277**(1) (2004), 222–235.

[Lin05] M. Linckelmann, 'Alperin's weight conjecture in terms of equivariant Bredon cohomology', *Math. Z.* **250**(3) (2005), 495–513.

[Lin09] M. Linckelmann, 'On $H^*(\mathcal{C}; k^\times)$ for fusion systems', *Homology Homotopy Appl*. **11**(1) (2009), 203–218. MR 2506133
[Lin19] M. Linckelmann, 'The block theory of finite group algebras', in *London Mathematical Society Student Texts* vol. 2 (Cambridge University Press, Cambridge, 2019).
[LO02] R. Levi and B. Oliver, 'Construction of 2-local finite groups of a type studied by Solomon and Benson', *Geom. Topol.* **6** (2002), 917–990 (electronic).
[LO05] R. Levi and B. Oliver, 'Correction to: "Construction of 2-local finite groups of a type studied by Solomon and Benson" [Geom. Topol. **6** (2002), 917–990 (electronic); MR 1943386]', *Geom. Topol*. **9** (2005), 2395–2415 (electronic).
[MR17] G. Malle and G. R. Robinson, 'On the number of simple modules in a block of a finite group', *J. Algebra* **475** (2017), 423–438. MR 3612478
[Nak61] M. Nakaoka, 'Homology of the infinite symmetric group', *Ann. of Math. (2)* **73** (1961), 229–257. MR 0131874
[Nav98] G. Navarro, *'Characters and blocks of finite groups', in London Mathematical Society Lecture Note Series* vol. 250 (Cambridge University Press, Cambridge, 1998).
[NT89] H. Nagao and Y. Tsushima, *Representations of Finite Groups* (Academic Press Inc., Boston, MA, 1989). Translated from the Japanese.
[Par10] S. Park, 'Realizing a fusion system by a single finite group', *Arch. Math. (Basel)* **94**(5) (2010), 405–410. MR 2643975
[Sem14] J. Semeraro, 'Trees of fusion systems', *J. Algebra* **399** (2014), 1051–1072. MR 3144626
[Sol74] R. Solomon, 'Finite groups with Sylow 2-subgroups of type .3', *J. Algebra* **28** (1974), 182–198.
[Wei94] C. A. Weibel, 'An introduction to homological algebra', in *Cambridge Studies in Advanced Mathematics* vol. 38 (Cambridge University Press, Cambridge, 1994). MR 1269324
[Wil09] R. A. Wilson, 'The finite simple groups', in *Graduate Texts in Mathematics* vol. 251 (Springer-Verlag London, Ltd., London, 2009). MR 2562037

https://doi.org/10.1017/fms.2023.53 Published online by Cambridge University Press