A NOTE ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY RIESZ TYPICAL MEANS OF THEIR FOURIER SERIES

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(Received 20 April 1967)

In [1], the following theorem is proved:

THEOREM. If $f \in C_{2\pi}$, α is a positive integer,

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) = \sum_{\nu=0}^{\infty} A_\nu(x),$$

and

$$R_\lambda^\alpha(x) = \sum_{\nu \leqq \lambda} \left(1 - \frac{\nu}{\lambda}\right)^\alpha A_\nu(x),$$

then

$$R_\lambda^\alpha(x) - f(x) = \frac{\alpha}{\pi} \int_a^\infty \frac{\phi_x(t/\lambda)}{t^2}\, dt + O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right),$$

where $a > 0$, $\phi_x(t) = f(x+t) + f(x-t) - 2f(x)$,

$$\omega_2(h, f) = \sup_{|t| \leqq h} ||\phi_x(t)|| = \sup_{|t| \leqq h} \max_x |\phi_x(t)|.$$

The aim of this note is to prove that it is true for any $\alpha > 3$. Let $s(u) = \sum_{\nu \leqq u} A_\nu(x)$. Then

$$\begin{aligned} R_\lambda^\alpha(x) &= \frac{\alpha}{\lambda^\alpha} \int_0^\lambda (\lambda - u)^{\alpha-1} s(u)\, du \\ &= \frac{\alpha}{\pi\lambda^\alpha} \int_0^\lambda (\lambda - u)^{\alpha-1} \lim_{N\to\infty} \int_{-N}^{N} f(x+t)\, \frac{\sin ut}{t}\, dt\, du \\ &= \frac{\alpha}{\pi\lambda^\alpha} \int_{-\infty}^{\infty} \frac{f(x+t)}{t} \int_0^\lambda (\lambda - u)^{\alpha-1} \sin ut\, du\, dt. \end{aligned}$$

Hence

$$R_\lambda^\alpha(x)-f(x) = \frac{\alpha}{\pi\lambda^\alpha}\int_0^\infty \frac{\phi_x(t)}{t}\int_0^\lambda (\lambda-u)^{\alpha-1}\sin ut\,du\,dt$$

$$= \frac{\alpha}{\pi\lambda^\alpha}\int_{1/\lambda}^\infty \frac{\phi_x(t)}{t}\int_0^\lambda (\lambda-u)^{\alpha-1}\sin ut\,du\,dt+O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right)$$

$$= \frac{\alpha}{\pi\lambda}\int_{1/\lambda}^\infty \frac{\phi_x(t)}{t^2}\,dt-\frac{\alpha(\alpha-1)}{\pi\lambda^\alpha}\int_{1/\lambda}^\infty \frac{\phi_x(t)}{t^2}\int_0^\lambda (\lambda-u)^{\alpha-2}\cos ut\,du\,dt$$

$$+O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right)$$

$$= \frac{\alpha}{\pi\lambda}\int_{1/\lambda}^\infty \frac{\phi_x(t)}{t^2}\,dt-\frac{\alpha(\alpha-1)(\alpha-2)}{\pi\lambda^3}\int_{1/\lambda}^\infty \frac{\phi_x(t)}{t^4}\,dt$$

$$+\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{\pi\lambda^\alpha}\int_{1/\lambda}^\infty \frac{\phi_x(t)}{t^4}\int_0^\lambda (\lambda-u)^{\alpha-4}\cos ut\,du\,dt$$

$$+O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right).$$

The integral in the third term may be written as

$$\int_{1/\lambda}^{2\pi} \frac{\phi_x(t)}{t^4}\int_0^\lambda (\lambda-u)^{\alpha-4}\cos ut\,du\,dt$$

$$+\int_0^{2\pi} \phi_x(t)\left[\sum_{p=1}^\infty \frac{1}{(t+2p\pi)^4}\right]\int_0^\lambda (\lambda-u)^{\alpha-4}\cos ut\,du\,dt.$$

Since $\omega_2(t, f) = O(\lambda^2 t^2\omega_2(1/\lambda, f))$, the third term is $O(\omega_2(1/\lambda, f))$.

Similarly, the second term is $O(\omega_2(1/\lambda, f))$. Hence

$$R_\lambda^\alpha(x)-f(x) = \frac{\alpha}{\pi}\int_1^\infty \frac{\phi_x(t/\lambda)}{t^2}\,dt+O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right)$$

$$= \frac{\alpha}{\pi}\int_a^\infty \frac{\phi_x(t/\lambda)}{t^2}\,dt+O\left(\omega_2\left(\frac{1}{\lambda}, f\right)\right).$$

Reference

[1] B. Kwee, 'The approximation of continuous functions by Riesz typical means of their Fourier Series', *J. Australian Math. Soc.*, 7 (1967), 539–544.

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https://doi.org/10.1017/S1446788700005759 Published online by Cambridge University Press