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Separated spin evolution quantum hydrodynamics of degenerate electrons with spin–orbit interaction and extraordinary wave spectrum

Pavel A. Andreev[1,†] **and Mariya Iv. Trukhanova**[1,†]

[1]Department of General Physics, Faculty of Physics, Lomonosov Moscow State University, Moscow, Russian Federation

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To consider the contribution of the spin–orbit interaction in the extraordinary wave spectrum we derive a generalization of the separate spin evolution quantum hydrodynamics. Applying the corresponding nonlinear Pauli equation we include the Fermi spin current contribution in the spin evolution. We find that the spectrum of extraordinary waves consists of three branches: two of them are well-known extraordinary waves and the third one is the spin-electron acoustic wave. A change of the extraordinary wave spectrum due to the spin–orbit interaction is also obtained.

Key words: plasma dynamics, plasma waves

1. Introduction

It is demonstrated that the separate spin evolution quantum hydrodynamics (SSE-QHD) leads to existence of the spin-electron acoustic waves (SEAWs) (Andreev 2015*b*; Andreev & Kuz'menkov 2015*b*; Andreev & Iqbal 2016) and allows us to derive an equation of state for the thermal part of the spin current in the regime of degenerate electrons (Andreev & Kuz'menkov 2015*c*), where the distribution of electrons in the quantum states is caused by Pauli blocking.

The SSE-QHD is generalized to consider the spin current evolution (Trukhanova 2015). An equation of state for the spin current flux (Andreev & Kuz'menkov 2015*a*; Trukhanova 2015) is also derived via SSE-QHD (Andreev & Kuz'menkov 2015*c*). The Coulomb exchange interaction is included in SSE-QHD in Andreev (2016*a*). In this paper, we continue the development and generalization of SSE-QHD and include the spin–orbit interaction (described in §§ 33 and 83 in Berestetskii, Lifshitz & Pitaevskii 1982).

The resent progress described above is based on a long study of quantum plasmas. Let us describe some major results related to this paper. The interest in semi-relativistic interactions increased after the construction of many-particle quantum hydrodynamics for the charged spin-1/2 particles with spin–spin interactions, as accomplished in 2001 by Kuz'menkov and coauthors (Kuz'menkov, Maksimov & Fedoseev 2001*a*). First of all, the analysis of the spin–current interaction was

† Email addresses for correspondence: andreevpa@physics.msu.ru, mar-tiv@yandex.ru

performed in 2007 (Andreev & Kuz'menkov 2007). The spin–current interaction gives the force field $qn\boldsymbol{v}\times\boldsymbol{B}_s/c+M^{\beta}\nabla B_c^{\beta}$, where $\boldsymbol{B}_s$ and $\boldsymbol{B}_c$ are the magnetic fields caused by the spins (magnetic moments) and currents, correspondingly (q is the charge of particles, n is the concentration, $\boldsymbol{v}$ is the velocity field, M is the magnetization, c is the speed of light, and β is the vector index, and the Einstein summation convention is assumed). However, more interesting results come from the account of the spin–orbit interaction considered in Andreev & Kuz'menkov (2009) in 2009 and Andreev & Kuzmenkov (2011) in 2011. Applications of the hydrodynamic model derived in Andreev & Kuzmenkov (2011) are presented in Andreev & Kuz'menkov (2012), Trukhanova (2013*b*). The spin–orbit interaction is also considered in Asenjo *et al.* (2012), along with the Darwin term. However, the Darwin term is considered in reduced form corresponding to the single-particle motion only, but for a system of identical particles it makes a different contribution (see the discussion in Ivanov, Andreev & Kuz'menkov 2015). The complete result for the Darwin term or the Darwin interaction contribution is presented in Ivanov *et al.* (2015). There is another quantum-relativistic force which appears via the simultaneous consideration of the Coulomb interaction and the semi-relativistic part of the kinetic energy (Ivanov, Andreev & Kuz'menkov 2014; Ivanov *et al.* 2015). Let us call it the CI-SPKE. The Darwin interaction force and the CI-SPKE force have structures similar to each other. The Darwin interaction force and the CI-SPKE force give contributions which are equal to each other in the linear approximation of hydrodynamic equations (Ivanov *et al.* 2015).

The semi-relativistic part of the quantum Bohm potential is derived in Ivanov *et al.* (2014, 2015) along with the current–current interaction (CCI). The CCI leads to three kinds of terms in the Euler equation: the first of them is $qn\boldsymbol{v}\times\boldsymbol{B}_c/c$, the classic semi-relativistic forces, and the quantum semi-relativistic forces (Ivanov *et al.* 2014). Some works (see for instance Asenjo *et al.* 2012; Hussain, Stefan & Brodin 2014) are dedicated to the kinetic model of semi-relativistic effects in plasmas.

Fully relativistic effects are also considered in the literature. The 'fluidization' of the Dirac equation, via the construction of observables from bilinear covariants, was done by Takabayasi (1955*b*, 1956*a*,*b*, 1957) and is nicely reviewed in Asenjo *et al.* (2011). A quantum-relativistic Vlasov equation is derived from the single-particle Klein–Fock–Gordon (Mendonca 2011) equation and the single-particle Dirac (Zhu & Ji 2012) equation. It is applied to the calculation of the spectrum of collective excitations (the Langmuir and electromagnetic waves) in quantum-relativistic unmagnetized plasmas. Some relativistic effects in quantum plasmas are reviewed in Uzdensky & Rightley (2014).

One of the most important implications of the SSE-QHD known to date is the spin-electron acoustic waves (SEAWs). So, let us describe the results obtained during the SEAW study.

SEAWs are longitudinal waves in spin-1/2 plasmas similar to the Langmuir and Trivelpiece–Gould waves, which are well-known examples of longitudinal waves. The SEAWs can be described if we consider the spin-up and spin-down electrons as two different fluids. Moreover, it is necessary to have different equilibrium concentrations of the spin-up and spin-down electrons. Hence, the contributions of their pressures are also different. All of these differences are caused by the partial spin polarization of electrons. The SEAW was predicted for waves propagating parallel to the external magnetic field (Andreev 2015*b*). In this regime, there is one branch of the dispersion curve of SEAWs. However, there are two branches of the bulk SEAWs in the oblique propagation regime (Andreev & Kuz'menkov 2015*b*). Influence of the spin polarization on the traditional electrostatic waves is considered in Margulis & Marguli (1987).

The lower branch of SEAWs has zero frequency at the zero wave vector $k=0$. It becomes a single branch when waves propagate parallel to the external magnetic field $\theta=0$, where θ is the angle between the propagation vector and the external magnetic field. The upper branch is located above the Trivelpiece–Gould wave spectrum. Its frequency tends to Ω_e at $k\rightarrow 0$, where $\Omega_e=eB_0/mc$ is the cyclotron frequency of the electrons. It becomes the single branch of the SEAWs if waves propagate in perpendicular direction $\theta=\pi/2$ (Andreev & Kuz'menkov 2015*b*). A small Landau damping of SEAWs propagating parallel to the external magnetic field is demonstrated in Andreev (2016*b*), where the SSE quantum kinetics is derived. Properties of the bulk SEAWs in the electron–positron plasmas are similar to SEAWs in electron–ion plasmas, but the frequencies are modified due to the equality of the masses of electrons and positrons (Andreev & Iqbal 2016). The SEAWs in electron–positron–ion plasmas, where the number of electrons and positrons are different, are also described in the regime of bulk waves. Plasma shows the existence of three longitudinal waves with a linear spectrum if waves propagate parallel to the external field: the SEAW, the positron-acoustic wave, and the spin-electron–positron acoustic wave. Each of these waves has a sibling in the regime of oblique propagation. These siblings have dispersion curves located above the Trivelpiece–Gould wave spectrum (Andreev & Iqbal 2016). In this paper we shift our attention to the SSE's role in the longitudinally transverse waves such as extraordinary waves since waves with a transverse field are important for different applications (see for instance Giannini *et al.* 2011).

Collective behaviour of quantum-relativistic plasmas interacting with an intense circularly polarized electromagnetic wave is considered in Mahajan & Asenjo (2016). It is pointed out that the spin-up–down degeneracy is removed by the electromagnetic field. A modified dispersion relation for an ordinary electromagnetic wave is found (see equation (75) of Mahajan & Asenjo 2016). Effects of the spin separation modify the contribution of the medium in the dispersion relation. Instead of the traditional plasma frequency square, they found a general term containing the SSE effects.

The nonlinear evolution of the bulk SEAWs propagating parallel to the external field leads to soliton formation (Andreev 2016*a*). This soliton is described in terms of a generalized SSE-QHD containing the Coulomb exchange interaction (Andreev 2016*a*). Surface SEAWs are described for half-spaced plasma-like media (Andreev & Kuz'menkov 2016*b*). In this regime, they appear as longitudinal waves with linear spectra. If the spin polarization is relatively small, the surface SEAW can linearly interact with the plasmon branch. This interaction leads to the generation of SEAWs. The SEAWs in two-dimensional structures are described in Andreev & Kuz'menkov (2016*a*). They appear to be similar to the spin plasmons described in Ryan (1991), Perez (2009), Agarwal *et al.* (2011), Agarwal *et al.* (2014).

The hydrodynamic description of spin-up and spin-down electrons was addressed in Kuzmenkov & Harabadze (2004). However, the difference of pressures for the spin-up and spin-down electrons was not considered there. Effects caused by non-conservation of electron number in each subspecies due to the spin–spin interactions were not included either. A similar model was suggested in Brodin, Misra & Marklund (2010) and it suffers the same limitations. A complete model is presented in Andreev (2015*b*).

There is a well-known interpretation of the spin–orbit interaction as a force acting on a moving magnetic moment in a constant electric field (Berestetskii *et al.* 1982). However, the same mechanism can exist in classic electrodynamics, where it is known as the hidden momentum (see discussion in Griffiths (2012), particularly p. 6).

The quantum weakly relativistic Hamiltonian for a single particle in an external electromagnetic field is derived and discussed in Foldy & Wouthuysen (1950) and Foldy (1952). It is shown (Newton & Wigner 1949) that there is the position operator with commuting components in Dirac theory. This position operator has localized eigenfunctions in the manifold of positive energy wave functions. This position operator is also called the mean-position operator in Foldy & Wouthuysen (1950).

A specific Lagrangian-based method is introduced in Dixit *et al.* (2013) for the analysis of weakly relativistic effects in plasma-like media. The suggested Lagrangian approach leads to generalized forms of the density and current which contain high in $1/c$ terms. The continuity equation directly found from the Noether's theorem considers with the short form of the continuity equation mentioned above (equation (41) in Dixit *et al.* 2013).

The generalized density and the current can be used for advanced hydrodynamic or kinetic equations. However, this approach requires more application. Hence, we restrict our analysis to the traditional means of direct application of the Pauli equation.

Some papers on weakly relativistic effects neglect the relativistic correction to the kinetic energy in favour of the Darwin interaction (Asenjo *et al.* 2012; Dixit *et al.* 2013). However, the relativistic correction to the kinetic energy makes several contributions to the equations of motion. One of them is similar to the contribution of the Darwin interaction (Ivanov *et al.* 2015). Moreover, the Darwin interaction between two electrons differs from the Darwin interaction between an electron and the external field by a factor two, as follows from the well-known Breit Hamiltonian (Berestetskii *et al.* 1982; Ivanov *et al.* 2015).

A fully relativistic spin-1/2 kinetic model is developed in the large scale regime giving first-order quantum corrections to descriptions of strongly magnetized plasmas (Ekman, Asenjo & Zamanian 2017). The model is based on separating positive and negative energy states of the Dirac equation. Hence, it has restriction on the amplitude of the electromagnetic field generated in plasmas during plasma dynamics.

The Wigner equation including the Zeeman effect and the spin–orbit coupling and the corresponding extended set of hydrodynamic equations are found in Hurst, Hervieux & Manfredi (2017).

Overall, existing works on the weakly relativistic models of plasmas deal with the Foldy–Wouthuysen transformation and corresponding Breit Hamiltonian for the Pauli-like equation. This paper takes the same path, but the independent evolution of the spin-up and spin-down electrons is included.

The experimental realization of a spin polarized electron gas of a high degree $\eta = 0.1$–0.7 is achieved by means of high intensity laser pulses on a femtosecond time scale (Del Sorbo *et al.* 2017). Theoretical analysis of the spin current generation by short laser pulses is presented in Hurst, Hervieux & Manfredi (2018).

This paper is organized as follows. In § 2 the basic model is described. It contain the nonlinear Pauli equation with the spinor Fermi pressure contribution and the spin–orbit interaction. In § 3 the general form of SSE-QHD equations with the spin–orbit interaction is derived from the nonlinear Pauli equation (NLPE). In § 4 we discuss the structure of the spin–orbit interaction force field. In § 5 we present the closed set of SSE-QHD equations arising after introduction of the velocity field. In § 6 we present a linearized set of hydrodynamic equations. In § 7 we study the contribution of the SSE and the spin–orbit interaction in the spectrum of extraordinary waves. In § 8 a brief summary of obtained results is presented.

2. Model: NLPE with spin–orbit interaction

The spin-1/2 QHD of plasmas can be directly derived from the many-particle Pauli equation (Kuz'menkov *et al.* 2001*a*; Kuz'menkov, Maksimov & Fedoseev 2001*b*). This method allows us to include the spin–orbit interaction (Andreev & Kuzmenkov 2011; Andreev & Kuz'menkov 2012) and other relativistic effects (Andreev & Kuz'menkov 2007; Ivanov *et al.* 2014; Andreev 2015*a*; Ivanov *et al.* 2015). Next, the QHD equations can be represented in the form of the nonlinear Schrödinger equation or the nonlinear Pauli equation (Kuz'menkov & Maksimov 1999; Kuz'menkov *et al.* 2001*a*; Andreev & Kuz'menkov 2012; Andreev 2014, 2015*a*).

The set of QHD equations can be presented as the nonlinear Schrödinger equation (NLSE) (Kuz'menkov & Maksimov 1999). This can be done for spinless particles (Kuz'menkov & Maksimov 1999) and spinning particles (Kuz'menkov *et al.* 2001*a*). In the latter case we have the nonlinear Pauli equation (Kuz'menkov *et al.* 2001*a*). NLSEs for three-dimensional and two-dimensional degenerate electron gases with Coulomb exchange interactions are derived in Andreev (2014). The nonlinear Pauli equation for an electron gas with spin–orbit interaction is presented in Andreev & Kuz'menkov (2012). The set of two nonlinear Pauli equations for spin-1/2 electron–positron plasmas with the annihilation interaction is derived in Andreev (2015*a*). NLSEs, or more exactly, nonlinear Klein–Fock–Gordon and nonlinear Feynman–Gell-Mann equations are presented in Mahajan & Asenjo (2014). A discussion of the nonlinear Pauli equation can also be found in Mahajan & Asenjo (2014).

We want to present a new model and thus we apply a simplified method of derivation. Hence, readers can focus on new effects instead of rather large equations existing in the many-particle derivation. A derivation of the SSE model from the many-particle Pauli equations is presented in Andreev (2016*b*), where it is done for the SSE quantum kinetics. We use the nonlinear Pauli equation (NLPE) derived in Andreev & Kuz'menkov (2012) for spin-1/2 quantum plasmas with spin–orbit interaction. Following Andreev & Kuz'menkov (2015*c*) we generalize the NLPE to include the spinor pressure. We apply the generalized NLPE to derive the SSE-QHD with the spin–orbit interaction and the Fermi spin current.

The NLPE equation appears as follows

$$\imath\hbar\partial_t\Phi(\boldsymbol{r},t)=\left(\frac{1}{2m}\widehat{\boldsymbol{D}}^2+q_e\varphi+\widehat{\pi}-\gamma_e\widehat{\boldsymbol{\sigma}}\boldsymbol{B}-\frac{\gamma_e}{mc}(\widehat{\boldsymbol{\sigma}}\cdot(\boldsymbol{E}\times\hat{\boldsymbol{D}}))\right)\Phi(\boldsymbol{r},t), \tag{2.1}$$

where $\boldsymbol{D}=\boldsymbol{p}-q_e\boldsymbol{A}/c$ and γ_e is the magnetic moment of the particles. Equation (2.1) is coupled to the electro-magneto-static Maxwell equations.

In this equation, the wave function is the spinor function and we present its explicit form (see § 56 in Landau & Lifshitz 1977): $\Phi=\binom{\Phi_u}{\Phi_d}$. Each of the functions Φ_u and Φ_d can be presented as $\Phi_s=a_s e^{\imath\phi_s}$. The last three terms in the NLPE (2.1) contain the Pauli matrices $\widehat{\sigma}^\alpha$. The third term on the right-hand side describes the spinor pressure contribution $\widehat{\pi}=\begin{pmatrix}\pi_u & 0\\ 0 & \pi_d\end{pmatrix}$, which is a diagonal second rank spinor. It can be represented in term of the Pauli matrixes $\widehat{\pi}=\pi_u(\hat{\text{I}}+\widehat{\sigma}_z)/2+\pi_d(\hat{\text{I}}-\widehat{\sigma}_z)/2$, where $\hat{\text{I}}$ is the unit second rank spinor $\hat{\text{I}}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. The explicit form of π_s is determined by the equation of state. In this paper, we consider the degenerate electrons. Hence, π_s is determined by the Fermi pressure $\pi_s=(6\pi^2 n_s)^{2/3}\hbar^2/2m$. Here, we consider one particle in a quantum state, instead of two particles with different spin directions in each state (see § 57 in Landau & Lifshitz 1980).

Equation (2.1) is coupled with the following equations of the field $\nabla\cdot\boldsymbol{B}=0$, $\nabla\times\boldsymbol{E}=0$,

$$\nabla\cdot\boldsymbol{E}=4\pi(en_i-e\Phi^\dagger\Phi), \tag{2.2}$$

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and

$$\nabla\times\boldsymbol{B}=\frac{2\pi q_e}{mc}(\Phi^{\dagger}\boldsymbol{D}\Phi+(\boldsymbol{D}\Phi)^{\dagger}\Phi)+4\pi\gamma_e\nabla\times(\Phi^{\dagger}\boldsymbol{\sigma}\Phi), \tag{2.3}$$

where $\Phi^{\dagger}$ is the Hermitian conjugate of the spinor wave function. We consider the field acting on the particle to be caused by the weakly relativistic motion of other particles. The assumed structure of the electromagnetic field is discussed in appendix A.

The spin–orbit interaction in the model described by (2.1)–(2.3) is presented in accordance with the semi-relativistic limit of the Dirac equation and the Breit Hamiltonian (Berestetskii *et al.* 1982). The Breit Hamiltonian (Berestetskii *et al.* 1982) or the Darwin Lagrangian in the classic regime (Landau & Lifshitz 1975) describes the semi-relativistic interaction of particles. Therefore, model (2.1)–(2.3) is constructed for the semi-relativistic description of the medium and the corresponding interaction of the fluid elements.

Considering quantum plasmas with spin–orbit interaction is to work in the weakly relativistic regime. This regime requires relatively large values of the parameters. If a degenerate electron gas is under consideration, then concentration is one of major parameters. The equilibrium concentration of electrons determines the Fermi velocity $v_{Fe}=\sqrt[3]{3\pi^2 n_{0e}}\hbar/m_e$. If the Fermi velocity is close to the speed of light c, the system of electrons is in the relativistic regime. If $v_{Fe}\approx 0.1c$ we get the weakly relativistic regime. If the Fermi velocity is even smaller $v_{Fe}\leqslant 0.01c$ then the weakly relativistic corrections might be negligibly small. However, this depends on the required accuracy. The relation between the Coulomb interaction and the spin–orbit interaction is governed by the concentration also. It requires concentrations to be of the same order as follows from the estimation of the Fermi velocity. Estimation $v_{Fe}\approx 0.1c$ gives the following value for the concentration $n_e\sim 10^{27}$ cm^{-3}.

Condition $\nabla\times\boldsymbol{E}=0$ presented above is not an obvious condition for the weakly relativistic approach if the Hamiltonian is derived from the single-particle Dirac equation. However, the properties of weakly relativistic interparticle interaction follow in more detail from the derivation of the Breit Hamiltonian (in the quantum case) or the Darwin Lagrangian (in the classic case). They explicitly show that the electric field is a potential field.

We are going to derive the SSE-QHD from the NLPE (2.1). To this end, it is useful to present the explicit form of the NLPE via the wave functions of spin-up and spin-down electrons:

$$\imath\hbar\partial_t\Phi_u=\left(\frac{\left(\frac{\hbar}{\imath}\nabla-\frac{q_e}{c}\boldsymbol{A}\right)^2}{2m}+q_e\varphi-\gamma_e B_z+\pi_u-\frac{\gamma_e}{mc}(E_xD_y-E_yD_x)\right)\Phi_u$$
$$-\gamma_e(B_x-\imath B_y)\Phi_d-\frac{\gamma_e}{mc}((E_yD_z-E_zD_y)-\imath(E_zD_x-E_xD_z))\Phi_d, \tag{2.4}$$

and

$$\imath\hbar\partial_t\Phi_d=\left(\frac{\left(\frac{\hbar}{\imath}\nabla-\frac{q_e}{c}\boldsymbol{A}\right)^2}{2m}+q_e\varphi+\gamma_e B_z+\pi_d+\frac{\gamma_e}{mc}(E_xD_y-E_yD_x)\right)\Phi_d$$
$$-\gamma_e(B_x+\imath B_y)\Phi_u-\frac{\gamma_e}{mc}((E_yD_z-E_zD_y)+\imath(E_zD_x-E_xD_z))\Phi_u. \tag{2.5}$$

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These equations describe the evolution of the spin-up electrons and the spin-down electrons independently. We can represent this evolution in terms of observables. This representation leads to the hydrodynamic equations.

The semi-relativistic or weakly relativistic approximation assumes that the Coulomb interaction dominates over the spin–orbit (SO) interaction. If we reach conditions where the SO interaction is comparable with the Coulomb interaction, we have reached a relativistic regime. This means that extra effects will occur. These extra effects do not have any trace in the weakly relativistic regime, but they are comparable with the SO interaction and the Coulomb interaction in the fully relativistic regime.

3. Hydrodynamic equations: general form

3.1. *Continuity equations*

To take the first step in the derivation of the SSE-QHDs we define the concentrations of spin-up and spin-down electrons $n_u = \Phi_u^* \Phi_u$ and $n_d = \Phi_d^* \Phi_d$ and differentiate them with respect to time. Applying (2.4) and (2.5) for the time derivatives of the partial wave functions Φ_u and Φ_d, we derive the continuity equations

$$\partial_t n_s + \nabla \boldsymbol{j}_s = \pm \frac{\gamma_e}{\hbar} \varepsilon^{z\beta\gamma} S_\beta B_\gamma \mp \frac{2\gamma_e}{\hbar c} \varepsilon^{z\mu\nu} \varepsilon^{\mu\alpha\beta} E^\alpha [\Phi^*_{s' \neq s} r_s^\nu D^\beta \Phi_s + \text{c.c.}]/2m, \tag{3.1}$$

where $\boldsymbol{r}_u = \{1, \imath, 1\}$, $\boldsymbol{r}_d = \{1, -\imath, -1\}$, $q_e = -e$ for electrons and c.c. means complex conjugate.

The spin densities presented in the continuity equations have the following definitions: $S_x = \Phi^* \widehat{\sigma}_x \Phi = \Phi_d^* \Phi_u + \Phi_u^* \Phi_d = 2a_u a_d \cos \Delta\phi$, $S_y = \Phi^* \widehat{\sigma}_y \Phi = \imath(\Phi_d^* \Phi_u - \Phi_u^* \Phi_d) = -2a_u a_d \sin \Delta\phi$, where $\Delta\phi = \phi_u - \phi_d$.

The sum of $[\Phi_d^* r_u^\alpha D^\beta \Phi_u + \text{c.c.}]/2m$ and $[\Phi_u^* r_d^\alpha D^\beta \Phi_d + \text{c.c.}]/2m$ gives the non-relativistic part of the spin current tensor $J^{\alpha\beta}_{\text{n.r.}} = (\Phi^+ \sigma^\alpha D^\beta \Phi + \text{h.c.})/2m$, where n.r. means non-relativistic and h.c. means Hermitian conjugation.

During the derivation of continuity equation (3.1), we find the explicit forms of the particle current: $j_s^\alpha = (1/2m)(\Phi_s^* D^\alpha \Phi_s + \text{c.c.}) + (-1)^{i_s} \gamma_e \varepsilon^{z\alpha\beta} n_s E_\beta / mc$, where $s = u$ or d and i_s is a number that is different for spin-up and spin-down electrons: $i_u = 2$, $i_d = 1$. Below, we use the non-relativistic part of the particle current $j_{0s}^\alpha = (1/2m)(\Phi_s^* D^\alpha \Phi_s + \text{c.c.})$ along with j_s^α. We introduce the velocity fields $\boldsymbol{v}_s$ via the particle currents $\boldsymbol{j}_s \equiv n_s \boldsymbol{v}_s$, with the following explicit form of the velocities $\boldsymbol{v}_s = (\hbar/m)\nabla\phi_s - (q_e/mc)\boldsymbol{A} + (-1)^{i_s} \gamma_e \varepsilon^{z\alpha\beta} E_\beta / mc$, where ϕ_s is the phase of the partial wave function $\Phi_s = a_s \mathrm{e}^{\imath \phi_s}$.

Below, we show that the right-hand sides of the continuity equations contain traditional hydrodynamic variables after the introduction of the velocity fields.

Summing up the partial concentrations n_s, we obtain the full concentration of electrons $n_e = n_u + n_d$, which should satisfy the continuity equation with zero right-hand side. However, directly summing the continuity equations (3.1), we find a non-zero right-hand side of the continuity equation for n_e:

$$\partial_t n_e + \nabla(\boldsymbol{j}_u + \boldsymbol{j}_d) = \frac{2\gamma_e}{\hbar c} E^\alpha \varepsilon^{z\mu\nu} \varepsilon^{\mu\alpha\beta} [(\Phi_d^* r_u^\nu D^\beta \Phi_u + \text{c.c.})/2m - (\Phi_u^* r_d^\nu D^\beta \Phi_d + \text{c.c.})/2m], \tag{3.2}$$

where the right-hand side is caused by the spin–orbit interaction. Considering the right-hand side of (3.2), we find that it can be presented as the divergence of a vector $-\Delta\boldsymbol{j}$. During this calculation, we have

$$[\Phi_d^* r_u^\nu D^\beta \Phi_u + \text{c.c.}] - [\Phi_u^* r_d^\nu D^\beta \Phi_d + \text{c.c.}] = \varepsilon^{\nu z \lambda} \hbar \partial_\beta S_\lambda. \tag{3.3}$$

Using $\nabla \times \boldsymbol{E} = 0$, the right-hand side of (3.3) can be written as the divergence of a vector $\Delta\boldsymbol{j}$. This vector Δj^{β} is a part of the particle current $\Delta j^{\beta} = -(2\gamma_e/\hbar c)E^{\gamma}\varepsilon^{z\mu\nu}\varepsilon^{\mu\gamma\beta}\varepsilon^{\nu z\lambda}\hbar S_{\lambda}$ which can be rewritten as

$$\Delta j^{\beta} = \frac{\gamma_e}{mc}E^{\gamma}(\varepsilon^{\beta\gamma\mu}S_{\mu} - \varepsilon^{\beta\gamma z}S_z). \tag{3.4}$$

Consequently, the full concentration n_e satisfies the continuity equation

$$\partial_t n_e + \nabla \boldsymbol{j} = 0, \tag{3.5}$$

where the right-hand side is equal to zero, $\boldsymbol{j} = \boldsymbol{j}_u + \boldsymbol{j}_d + \Delta\boldsymbol{j}$, with $\boldsymbol{j}$ is the flux of all electrons.

The current of all electrons contains the non-relativistic part and the spin–orbit part. The SSE exists such that a part of the flux of all electrons $\boldsymbol{j}$ becomes the flux for each subspecies of electrons $\boldsymbol{j}_s$ and a part of the full flux $\Delta\boldsymbol{j}$ separates to give the z projection of spin–orbit torque. So, we have a linear decomposition of the current. This separation is not made on purpose, but it automatically appears in the derivation.

The difference of the partial concentrations n_s is the z-projection of the spin density $S_z = n_u - n_d$. Applying the continuity equations (3.1), we find an equation for S_z

$$\partial_t S_z + \nabla(\boldsymbol{j}_u - \boldsymbol{j}_d) = \frac{2\gamma_e}{\hbar}\varepsilon^{z\beta\gamma}S_{\beta}B_{\gamma} - \frac{2\gamma_e}{\hbar c}\varepsilon^{z\mu\nu}\varepsilon^{\mu\alpha\beta}E^{\alpha}J^{\nu\beta}, \tag{3.6}$$

where

$$J^{\alpha\beta} = \frac{1}{2m}(\Phi^{+}\sigma^{\alpha}D^{\beta}\Phi + \text{h.c.}) + \frac{\gamma_e}{2mc}\varepsilon^{\alpha\beta\gamma}n_e E_{\gamma} \tag{3.7}$$

is the full spin current. The last term in (3.6) presents the z-projection of the spin torque caused by the spin–orbit interaction. The full expression on the right-hand side of (3.6) is the z-projection of the full spin torque.

3.2. *Euler equations*

Application of the NLPE (2.1) to the time evolution of the momentum density of the spin-up electrons j_u^{α} gives the following Euler equations for the spin-s electrons:

$$\begin{aligned} m\partial_t j_s^{\alpha} + \partial_{\beta}\Pi_s^{\alpha\beta} &= q_e n_s E^{\alpha} + \frac{q_e}{c}\varepsilon^{\alpha\beta\gamma}j_{0s}^{\beta}B^{\gamma} + F_{\text{SO}s}^{\alpha} \\ &\quad \pm \gamma_e n_s \partial^{\alpha}B_z + \frac{\gamma_e}{2}(S_x\partial^{\alpha}B_x + S_y\partial^{\alpha}B_y) \pm \frac{m\gamma_e}{\hbar}\varepsilon^{z\beta\gamma}J^{\beta\alpha}B^{\gamma}, \end{aligned} \tag{3.8}$$

where the force field of the spin–orbit interaction $F_{\text{SO}u}^{\alpha}$ has the following form:

$$\begin{aligned} F_{\text{SO}s}^{\alpha} &= \pm\frac{\gamma_e}{mc}\varepsilon^{z\alpha\beta}\partial_t(n_s E^{\beta}) \mp \frac{2\gamma_e}{\hbar c}\varepsilon^{z\mu\nu}\varepsilon^{\mu\beta\gamma}E^{\beta}j^{\nu\alpha\gamma} \\ &\quad - \frac{q_e}{2mc}\frac{\gamma_e}{mc}E^{\beta}B^{\delta}\varepsilon^{\alpha\gamma\delta}(S_x\varepsilon^{x\beta\gamma} + S_y\varepsilon^{y\beta\gamma} \pm 2n_s\varepsilon^{z\beta\gamma}) + \frac{\gamma_e}{2m^2c}(\partial^{\alpha}E^{\beta}) \\ &\quad \times [\pm\varepsilon^{z\beta\gamma}\Phi_s^{*}D^{\gamma}\Phi_s + \varepsilon^{x\beta\gamma}\Phi_s^{*}D^{\gamma}\Phi_{s'\neq s} \mp \varepsilon^{y\beta\gamma}\Phi_s^{*}\imath D^{\gamma}\Phi_{s'\neq s} + \text{c.c.}], \end{aligned} \tag{3.9}$$

where

$$j^{\alpha\beta\gamma} = \frac{1}{4m^2}(\Phi^{+}D^{\gamma}D^{\beta}\sigma^{\alpha}\Phi + \text{h.c.}) \tag{3.10}$$

is a part of the spin current flux. The full expression for the non-relativistic spin current flux is

$$J^{\alpha\beta\gamma}=\frac{1}{4m^2}(\Phi^+D^\gamma D^\beta\sigma^\alpha\Phi+(D^\gamma\Phi)^+D^\beta\sigma^\alpha\Phi+\text{h.c.}). \tag{3.11}$$

In (3.9), we have dropped the term proportional to $\nabla\times\boldsymbol{E}$, since it is equal to zero in the semi-relativistic approach.

The momentum current for the spin-s electrons appears during our derivation of (3.8) in the following form:

$$\begin{aligned}\Pi_s^{\alpha\beta} &= \frac{1}{4m}(\Phi_s^*D^\alpha D^\beta\Phi_s+(D^\alpha\Phi_s)^*D^\beta\Phi_s+\text{c.c.})\\ &\quad-\frac{\gamma_e}{mc}\varepsilon^{\mu\gamma\beta}E^\gamma[\Phi_{s'\neq s}^*r_s^\mu D^\alpha\Phi_s+\text{c.c.}]/2m+n_s\nabla\pi_s.\end{aligned} \tag{3.12}$$

The last term in the Euler equation (3.8) describes the spin flipping contribution in the momentum evolution.

3.3. *Spin evolution equation*

In the SSE-QHD we need to have equations for the evolution of the x and y projections of the spin density. If we include the contribution of the spin–orbit interaction they are obtained as follows:

$$\partial_t S_j+\Im_j+\partial_\beta J^{j\beta}=\frac{2\gamma_e}{\hbar}\varepsilon^{j\beta\gamma}S^\beta B^\gamma+T_{\text{SO}j}, \tag{3.13}$$

where $j=x, y,$ and

$$T_{\text{SO}}^\alpha=-\frac{\gamma_e}{\hbar c}\varepsilon^{\alpha\mu\nu}\varepsilon^{\mu\gamma\beta}E_\gamma J^{\nu\beta} \tag{3.14}$$

is the spin torque caused by the SO interaction corresponding to the earlier works on the single-fluid model of electrons (Andreev & Kuz'menkov 2009, 2012).

The spin current is an important characteristic of a medium in spintronics (Sun & Xie 2005; An *et al.* 2012; Sinova *et al.* 2015). In our model, the many-particle spin current naturally appears in the spin evolution equation. It also appears in the Euler equation in terms describing non-conservation of the particle number when we account for the spin–spin interaction. If we account for the spin–orbit interaction, the spin current appears in the force field of the spin–orbit interaction and in the spin torque caused by the spin–orbit interaction. Parts of the spin current also exist in the continuity equation.

Modification of the distribution of electrons in momentum space and the influence of this on the equation of state under the influence of a strong magnetic field are described in the literature (see for instance Strickland, Dexheimer & Menezes 2012). In contrast with these works, we focus on the different occupation of quantum states by the spin-up and spin-down electrons.

3.4. *Discussion of the Euler equations*

Let us compare the obtained results with the Euler equation found for the single-fluid model of electrons considered in Kuz'menkov *et al.* (2001*a*), Brodin & Marklund (2007), Marklund & Brodin (2007), Shukla & Eliasson (2011), Koide (2013) and especially Andreev & Kuzmenkov (2011), Andreev & Kuz'menkov (2012) concerning the spin–orbit interaction. To this end, we need to consider the sum of two Euler equations to find the Euler equation in terms of the single-fluid model.

Combining Euler equations (3.8), we find the following equation

$$\partial_t(j_u^\alpha + j_d^\alpha) + \partial_\beta(\Pi_u^{\alpha\beta} + \Pi_d^{\alpha\beta}) = q_e n_e E^\alpha + \frac{q_e}{c}\varepsilon^{\alpha\beta\gamma}(j_{0u}^\beta + j_{0d}^\beta)B^\gamma + \gamma_e S^\beta \partial^\alpha B^\beta + F_{\text{SO}}^\alpha, \quad (3.15)$$

where

$$F_{\text{SO}}^\alpha = \frac{\gamma_e}{mc}\varepsilon^{z\alpha\beta}\partial_t(S_z E^\beta) + \frac{\gamma_e}{mc}\partial^\alpha E^\beta \cdot \varepsilon^{\beta\gamma\delta}J^{\delta\gamma} - \frac{q_e}{mc}\frac{\gamma_e}{mc}\varepsilon^{\alpha\gamma\delta}E^\beta B^\delta \varepsilon^{\beta\gamma\mu}S^\mu. \quad (3.16)$$

Equation (3.16) is not the complete spin–orbit interaction force for the single-fluid model. Above we showed that the full particle current consists of three terms (3.5). If we consider the time evolution of the full particle current $\boldsymbol{j} = \boldsymbol{j}_u + \boldsymbol{j}_d + \Delta\boldsymbol{j}$ we need to include the time evolution of vector $\Delta\boldsymbol{j}$. Its explicit form is given by (3.4). Differentiating (3.4) with respect to time and combining the result with the Euler equation (3.16) we obtain the time derivative of the full particle current $\partial_t j^\alpha$ on the left-hand side. On the right-hand side we have the change in the force field of the spin–orbit interaction. The second term in $\partial_t \Delta j^\alpha$ cancels the first term in the force field (3.16). Thus, we have that the first term in $\partial_t \Delta j^\alpha$ replaces the first term in (3.16) and we obtain the final form for the spin–orbit interaction as:

$$\bar{F}_{\text{SO}}^\alpha = \frac{\gamma_e}{mc}\varepsilon^{\alpha\beta\gamma}\partial_t(E^\beta S^\gamma) + \frac{\gamma_e}{mc}\partial^\alpha E^\beta \cdot \varepsilon^{\beta\gamma\delta}J^{\delta\gamma} - \frac{q_e}{mc}\frac{\gamma_e}{mc}\varepsilon^{\alpha\gamma\delta}E^\beta B^\delta \varepsilon^{\beta\gamma\mu}S^\mu. \quad (3.17)$$

Before we present a comparison of our result with the previously obtained equations in the single-fluid model (Andreev & Kuz'menkov 2012; Ivanov *et al.* 2015), we need to mention that the first term in (3.17) can be represented in the following way. If we take the time derivative we find two terms $(\gamma_e/mc)\varepsilon^{\alpha\beta\gamma}(\partial_t E^\beta)S^\gamma + (\gamma_e/mc)\varepsilon^{\alpha\beta\gamma}E^\beta\partial_t S^\gamma = (\gamma_e/mc)\varepsilon^{\alpha\beta\gamma}(\partial_t E^\beta)S^\gamma + (\gamma_e/mc)\varepsilon^{\alpha\beta\gamma}E^\beta((2\gamma_e/\hbar)\varepsilon^{\gamma\mu\nu}S^\mu B^\nu - \mathfrak{I}^\gamma - \partial_\mu J^{\gamma\mu})$. Other terms can be found in Andreev & Kuzmenkov (2011), Andreev & Kuz'menkov (2012) except for the last term in (3.17). It can be shown that the last term in (3.17) can be found in the single-fluid model. However, it was not reported in the mentioned papers since it was lost during the derivation.

4. Spin–orbit interaction contribution in the spectrum of longitudinal waves

It has been recently demonstrated that longitudinal spin waves, called spin-electron acoustic waves, can exist in a degenerate electron gas (Andreev 2015*b*). The oblique propagation of longitudinal waves reveals the existence of the second or upper SEAW (Andreev & Kuz'menkov 2015*b*). Partial spin polarization is required for the existence of the SEAWs. It is also necessary to apply the SSE-QHD. The SSE-QHD is generalized in this paper. The explicit analytical contribution of the spin–orbit interaction in the spectrum of the Langmuir waves propagating perpendicular to the external magnetic field (the upper hybrid wave) was found in Ivanov *et al.* (2015). It was shown that the spin–orbit interaction gives a contribution to the spectrum of the longitudinal waves. This contribution arises from the following term: $F_1^\alpha = -(\gamma_e/mc)S_{0z}\varepsilon^{\alpha\beta z}\partial_t \delta E^\beta$. In this paper, we generalize the force field of the spin–orbit interaction including the following term: $F_2^\alpha = -(q_e/mc)(\gamma_e/mc)S_{0z}B_0\varepsilon^{\alpha\gamma z}\varepsilon^{\beta\gamma z}\delta E^\beta$. These do not equal to each other. However, in the linear approximation they cancel the contribution of each other in the dispersion curves of the longitudinal waves. Therefore, we do not expect any contribution of the spin–orbit interaction in the spectra of oblique propagating longitudinal waves: Langmuir, Trivelpiece–Gould, lower and upper SEAWs. However, the spin–orbit interaction affects these waves if we do not apply the restriction of considering longitudinal waves. In the general case they are longitudinal–transverse waves and the spin–orbit interaction gives a contribution via the transverse part.

5. Hydrodynamic equations with the velocity field

After the introduction of the velocity field in the equations derived above for SSE-QHDs with the spin–orbit interaction we find a closed set of hydrodynamic equations. The parts of the spin currents found in the terms describing the spin–orbit interaction are found via the concentrations and velocity fields of the spin-up and spin-down electrons. Next, we obtain the following forms of the continuity equations:

$$\partial_t n_s + \nabla(n_s \boldsymbol{v}_s) = (-1)^{i_s}\frac{\gamma_e}{\hbar}\varepsilon^{\alpha\beta z}S^\alpha B^\beta - \frac{2\gamma_e}{\hbar c}\varepsilon^{z\mu\nu}\varepsilon^{\mu\alpha\beta}E^\alpha\left(\frac{1}{2}v_s^\beta S^\nu - (-1)^{i_s}\frac{\hbar}{4m}\frac{\partial^\beta n_s}{n_s}\varepsilon^{z\nu\delta}S^\delta\right). \tag{5.1}$$

The last term in the continuity equation is caused by the spin–orbit interaction. The introduction of the velocity field modifies the right-hand side of the continuity equation, so it contains the hydrodynamic variables.

The introduction of the velocity field transforms the Euler equations to a form closer to the traditional form:

$$\begin{aligned} mn_s(\partial_t + \boldsymbol{v}_s\nabla)\boldsymbol{v}_s + \nabla p_s - \frac{\hbar^2}{4m}n_s\nabla\left(\frac{\triangle n_s}{n_s} - \frac{(\nabla n_s)^2}{2n_s^2}\right) \\ = q_e n_s\left(\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}_s, \boldsymbol{B}]\right) + \boldsymbol{F}_{SSs} + \tilde{\boldsymbol{F}}_{SOs}, \end{aligned} \tag{5.2}$$

with the thermal or Fermi pressure p_s, the force field of spin–spin interaction

$$\boldsymbol{F}_{SSs} = (-1)^{i_s}\gamma_e n_u \nabla B_z + \frac{\gamma_e}{2}(S_x\nabla B_x + S_y\nabla B_y) + (-1)^{i_s}\frac{m\gamma_e}{\hbar}\varepsilon^{\beta\gamma z}(\boldsymbol{J}_{(M)\beta}B^\gamma - \boldsymbol{v}_s S^\beta B^\gamma), \tag{5.3}$$

and the force field of spin–orbit interaction acting on spin-s electrons

$$\begin{aligned} \tilde{F}^\alpha_{SOs} = {} & \pm\frac{\gamma_e}{mc}\varepsilon^{z\alpha\beta}\partial_t(n_s E^\beta) \mp \frac{2\gamma_e}{\hbar c}\varepsilon^{z\mu\nu}\varepsilon^{\mu\beta\gamma}E^\beta j^{\nu\alpha\gamma} \\ & - \frac{q_e}{2mc}\frac{\gamma_e}{mc}E^\beta B^\delta\varepsilon^{\alpha\gamma\delta}(S_x\varepsilon^{x\beta\gamma} + S_y\varepsilon^{y\beta\gamma} \pm 2n_s\varepsilon^{z\beta\gamma}) \\ & + \frac{1}{2}\frac{\gamma_e}{mc}(\partial^\alpha E^\beta)[\varepsilon^{x\beta\gamma}v^\gamma_{s'\neq s}S^x + \varepsilon^{y\beta\gamma}v^\gamma_{s'\neq s}S^y + \varepsilon^{z\beta\gamma}v^\gamma_s S^z] \\ & \pm\frac{\hbar}{4m}\frac{\partial^\gamma n_{s'\neq s}}{n_{s'\neq s}}\frac{\gamma_e}{mc}(\partial^\alpha E^\beta)\varepsilon^{\mu\beta\gamma}\varepsilon^{z\mu\delta}S^\delta. \end{aligned} \tag{5.4}$$

A presentation of tensor $j^{\nu\alpha\gamma}$ in terms of the velocity fields is discussed in appendix C.

In terms of the velocity field the spin current tensor is found as follows

$$J_{j\alpha} = \frac{1}{2}(v_u^\alpha + v_d^\alpha)S_j - \frac{\hbar}{4m}\varepsilon^{j\beta z}\left(\frac{\partial^\alpha n_u}{n_u} - \frac{\partial^\alpha n_d}{n_d}\right)S_\beta. \tag{5.5}$$

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The relativistic part of the spin current tensor is hidden in the definition of velocity fields.

In the Euler equations (5.2) we have used a reduced form of the spin current $\boldsymbol{J}_{(M)x}$ and $\boldsymbol{J}_{(M)y}$ which means $J^{x\alpha}$ and $J^{y\alpha}$ correspondingly. Here, the bold symbols indicate a vector related to the second index in $J^{\alpha\beta}$.

The existence of the quantum part of the spin current or, in other words, the quantum Bohm potential contribution in the spin evolution equation was demonstrated by Takabayasi in Takabayasi (1955*a*). The contribution of this effect to the wave properties of quantum plasmas is considered in Trukhanova (2013*a*), where it is applied for the spin-plasma waves.

After the introduction of the velocity field, the spin evolution equations have the following form:

$$\partial_t S_j + \frac{1}{2}\nabla[S_j(\boldsymbol{v}_u + \boldsymbol{v}_d)] - \frac{\hbar}{4m}\varepsilon^{j\beta z}\nabla\left(S^\beta\left(\frac{\nabla n_u}{n_u} - \frac{\nabla n_d}{n_d}\right)\right) + \Im_j = \frac{2\gamma_e}{\hbar}\varepsilon^{j\beta\gamma}S^\beta B^\gamma + T_{SOj}, \tag{5.6}$$

where $j = x, y$.

The electric and magnetic fields entering SSE-QHD equations (5.1)–(5.6) satisfy the quasi-static Maxwell equations ($\nabla \times \boldsymbol{E} = 0$ and (2.3)), where sources of fields are presented in terms of hydrodynamic variables: $\Phi^\dagger\Phi = n_{eu} + n_{ed}$, $(1/2m)(\Phi^\dagger\boldsymbol{D}\Phi + (\boldsymbol{D}\Phi)^\dagger\Phi) = n_{eu}\boldsymbol{v}_{eu} + n_{ed}\boldsymbol{v}_{ed}$, and $\Phi^\dagger\boldsymbol{\sigma}\Phi = \{S_{ex}, S_{ey}, (n_{eu} - n_{ed})\}$. We consider motionless ions.

The SSE-QHD equations (5.1)–(5.6) are derived from model (2.1)–(2.3) for the motion of a medium with semi-relativistic interactions of particles. This model does not describe the propagation of the electromagnetic radiation in the medium and the action of the radiation on the medium since we have incomplete Maxwell equations. Next, we promote our model to include the radiation propagation. To this end, we restore $\partial_t\boldsymbol{E}$ and $\partial_t\boldsymbol{B}$ in the Maxwell equations. So, the SSE-QHD equations (5.1)–(5.6) are coupled with the following Maxwell equations $\nabla\cdot\boldsymbol{B} = 0$, $\nabla\times\boldsymbol{E} = -(1/c)\partial_t\boldsymbol{B}$,

$$\nabla\cdot\boldsymbol{E} = 4\pi(en_i - en_{eu} - en_{ed}), \tag{5.7}$$

$$\nabla\times\boldsymbol{B} = \frac{1}{c}\partial_t\boldsymbol{E} + \frac{4\pi q_e}{c}(n_{eu}\boldsymbol{v}_{eu} + n_{ed}\boldsymbol{v}_{ed}) + 4\pi\nabla\times\boldsymbol{M}_e. \tag{5.8}$$

Since we consider degenerate electrons we use the Fermi pressures for spin-up and spin-down electrons as equations of state: $p_s = (6\pi^2)^{2/3}\hbar^2 n_s^{5/3}/5m$.

Vector $\Im$ in the spin evolution equations (5.6) is the divergence of the thermal part of the spin current tensor $\Im^\alpha = \partial_\beta J^{\alpha\beta}_{\text{th}}$. In accordance with the NLPE equation, as it was demonstrated in Andreev & Kuz'menkov (2015*c*), we have

$$\Im = (\pi_u - \pi_d)\gamma_e[\boldsymbol{S}, \boldsymbol{e}_z]/\hbar. \tag{5.9}$$

Due to its nature we can also call this the Fermi spin current.

Our model is bound to an independent hydrodynamic description of spin-up and spin-down electrons which was originally introduced in Kuzmenkov & Harabadze (2004). However, this model appears to be incomplete (Andreev 2015*b*). Before, the final version of the model had been presented in 2015 (Andreev 2015*b*) some works based on Kuzmenkov & Harabadze (2004) had been presented. Let us discussed some of them for a better perspective in the understanding of our results. In § II A of Zamanian *et al.* (2010) the authors discussed a two-fluid form of the quantum hydrodynamic equations. The spin–orbit interaction was not discussed. Equations were not demonstrated explicitly, but the authors suggested that 'However, since the kinetic equation is linear in f, it is straightforward to divide the electron fluid into two parts depending on the initial spin state, e.g., up or down relative to the magnetic field'. However, our analysis shows that the division of hydrodynamic equations into two fluids is not straightforward even if the spin–orbit interaction is dropped. Otherwise, the Euler equation presented in Zamanian *et al.* (2010) for the single-fluid description of the electrons is the traditional Euler equation for the spin-1/2 quantum plasmas (Kuz'menkov *et al.* 2001*a*).

Considering a kinetic model for spin-1/2 quantum particles, Stefan & Brodin (2013) started their description with the single-fluid model of electrons, where the spin was described via an extra argument of the quasi-distribution function. An argument describing spin is a vector with a fixed module, but it can rotate in three-dimensional space and its evolution is reduced to a two-dimensional evolution described by angles in spherical coordinates. It seems that the coherent states for the spin degree of freedom were used. Hence, the values at $\theta=0$ or $\theta=\pi$ do not directly correspond to spin-up and spin-down states. Next, the authors presented an equilibrium distribution function $f_0(s, p)$ and considered the small amplitude transverse perturbations. Waves propagating parallel to the external field directed parallel to Oz axes was considered there. An integral form of the dispersion relation for transverse waves with left and right circular polarization was obtained. An algebraic form of this equation was obtained after integration was performed in the long wavelength limit. The long wavelength limit presented by equation (14) of Stefan & Brodin (2013) does not contain the wave vector. So, there are probably limit values for the frequencies for two waves at $k=0$. Judging by the fact that some terms contain $(\mu/mc^2)^2$, we conclude that the calculation and result are presented beyond the limit required by the weakly relativistic regime mentioned before equation (1) of Stefan & Brodin (2013).

Equation (14) in Stefan & Brodin (2013) is a huge formula with no numerical analysis. Thus comparison of the contribution of the spin–orbit interaction with our result is not possible.

6. Linearized SSE-QHD equations

Analysis of wave propagation can be made by a consideration of the linearized set of SSE-QHD equations:

$$\partial_t\delta n_s + n_{0s}\nabla\delta \boldsymbol{v}_s = 0, \tag{6.1}$$

$$\begin{aligned} mn_{0s}\partial_t\delta v_s^\alpha + \partial^\alpha\delta p_s &= q_e n_{0s}\delta E^\alpha \pm \gamma_e n_{0s}\partial^\alpha\delta B_z \\ &\quad + q_e n_{0s}\frac{1}{c}B_0\varepsilon^{\alpha\beta z}\delta v_s^\beta \mp \frac{q_e}{mc}\frac{\gamma_e}{mc}n_{0s}B_0\varepsilon^{\alpha\gamma z}\varepsilon^{\beta\gamma z}\delta E^\beta \\ &\quad \mp \frac{2\gamma_e}{\hbar c}\varepsilon^{z\mu\nu}\varepsilon^{\beta\gamma\mu}j_0^{\nu\alpha\gamma}\delta E^\beta \pm \frac{\gamma_e}{mc}n_{0s}\varepsilon^{\alpha\beta z}\partial_t\delta E^\beta, \end{aligned} \tag{6.2}$$

with the upper sign for spin-up electrons and the lower sign for spin-down electrons, and

$$\partial_t\delta S_j + \delta\Im_j = \frac{2\gamma_e}{\hbar}(\varepsilon^{j\beta z}B_{0z}\delta S_\beta - \varepsilon^{j\beta z}S_{0z}\delta B_\beta) \tag{6.3}$$

for the spin evolution, and $\nabla\cdot\delta\boldsymbol{B}=0$, $\nabla\times\delta\boldsymbol{E}=-\partial_t\delta\boldsymbol{B}/c$,

$$\nabla\cdot\delta\boldsymbol{E} = -4e\pi(\delta n_{eu}+\delta n_{ed}), \tag{6.4}$$

$$\nabla\times\delta\boldsymbol{B} = \frac{1}{c}\partial_t\delta\boldsymbol{E} - \frac{4\pi e}{c}(n_{0u}\delta\boldsymbol{v}_u + n_{0d}\delta\boldsymbol{v}_d) + 4\pi\gamma_e\nabla\times\delta\boldsymbol{S}_e. \tag{6.5}$$

We neglect the quantum Bohm potential since it gives a noticeable contribution at large wave vectors close to $k_{max} = n_0^{1/3}$, $S_{0z} = n_{0u} - n_{0d}$ and $\delta\mathfrak{I} = (\pi_{0u} - \pi_{0d})\gamma_e[\delta\boldsymbol{S}, \boldsymbol{e}_z]/\hbar$, with $\pi_{0s} = (6\pi^2 n_{0s})^{2/3}\hbar^2/2m$. The quantum Bohm potential becomes noticeable at $kv_{Fe}/\omega_{Le} \geqslant 1$. It corresponds to $\kappa \geqslant 10^3$, but all analyses are made for $\kappa \leqslant 1$, where $\kappa = kc/\omega_{Le}$.

The spin–orbit interaction describes the force acting on the magnetic moment moving in an external electric field. It exists even for static electric fields. Therefore, the spin–orbit interaction affects the evolution of electrostatic waves in spin-1/2 plasmas, since electrons possess the magnetic moments affected by the electric field caused by the charges of the surrounding electrons. We consider high frequency perturbations and assume that the ions are motionless. Hence, on this time scale, ions generate an equilibrium electric field compensated by the equilibrium part of the electric field of the electrons. Relatively small temperatures are considered. Thus, there is a distribution of spins due to the temperature effects, but the deviations from the average value are small.

Traditional derivation of the Vlasov equation performed by N. N. Bogoliubov is made for particles with Coulomb interaction. Hence, the corresponding hydrodynamic equations contain the Coulomb interaction only. This means that the right-hand side of the Euler equation contains $qn\boldsymbol{E}$ plus the force field caused by the external field, where $\boldsymbol{E} = -\nabla\phi$, which satisfies the Poisson equation. Similar analysis in terms of the many-particle quantum hydrodynamics (MPQHD) is given in Kuz'menkov & Maksimov (1999). In spite the fact that the derivations are made for the Coulomb interaction only, the Vlasov equation, the classic hydrodynamic equations and the quantum hydrodynamics are applied for regimes where the full set of Maxwell equations is used. We also need to mention that an averaging of the single-particle equations, which can be obtained in the literature, is not a satisfactory justification. Such models do not consider the interparticle interaction at all. These models avoid existing problem. The same assumption were used in Asenjo *et al.* (2012) and Stefan & Brodin (2013), but without any discussion.

Our goal is to study the structure of the spin–orbit interaction in the hydrodynamic equations with separate spin evolution. This can be done in the weakly relativistic regime. However, we want to study transverse waves. Hence, the Poisson equation is replaced by the full set of Maxwell equations. This can be partially justified by the analysis of external electromagnetic field propagation through plasmas in which the motion of the charges modifies properties of the electromagnetic field.

7. Dispersion dependence of waves propagating perpendicular to the external magnetic field

Let us present the dispersion equation for the waves propagating perpendicular to the external magnetic field

$$\omega^2 - k^2c^2 - \sum_{s=u,d} \frac{\omega_{Ls}^2}{\omega^2 - \Omega_e^2 - k^2U_{Fs}^2} \times (2\omega^2 - \omega_{Le}^2 - k^2c^2 - k^2U_{Fs}^2 + \alpha_e\Omega_e(2\omega_{Le}^2 + k^2U_{Fs}^2)) = 0 \tag{7.1}$$

found in this paper as a solution of (6.1)–(6.5) in the linear approximation for small perturbations of the equilibrium state of magnetized plasmas, where $\alpha_e = \gamma_e/q_ec$. In (7.1) $U_{Fs} = (6\pi^2 n_{0s})^{1/3}\hbar/m$ is the Fermi velocity for spin-s electrons.

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The SSE affects the ordinary electromagnetic wave and spin-plasma wave having an electric field oscillating in the direction of the external magnetic field $\boldsymbol{B}_{ext} = B_0 \boldsymbol{e}_z$ via the Fermi spin current. This effect is described in Andreev & Kuz'menkov (2015*c*). Therefore, we do not discuss the ordinary electromagnetic wave and spin-plasma wave in this paper.

7.1. *Numerical analysis*

For the numerical analysis and presentation of our results we use the following dimensionless parameters $\xi = \omega/\omega_{Le}$, $\kappa = kc/\omega_{Le}$ and the following expression for the equilibrium spin polarization $\eta = \tanh(\mu_B B_0/\varepsilon_{Fe})$, where $\varepsilon_{Fe} = T_{Fe}/k_B = (3\pi^2 n_{0e})^{2/3}\hbar^2/2m$ is the Fermi energy, and k_B is the Boltzmann constant.

The spin–orbit interaction is a relativistic effect modelled in the semi-relativistic approach. To include its contribution we need to consider relatively large equilibrium concentrations $n_{0e} \sim 10^{27}$ cm^{-3}. As was demonstrated and discussed earlier, the exchange interaction plays a considerable role if the concentration is below $\approx 10^{25}$ cm^{-3} (Andreev & Ivanov 2015; Andreev 2016*a*).

Usually, the dispersion equation for longitudinal–transverse waves propagating perpendicular to an external field and having an electric field in the plane perpendicular to the external magnetic field gives two solutions, called extraordinary waves. However, the dispersion equation (7.1) has three solutions due to the SSE. The dispersion equation (7.1) also contains the spin–orbit interaction contribution.

The third solution of (7.1) appears due to the SSE. The SSE lead to the appearance of SEAWs if we consider the regime of longitudinal waves (Andreev 2015*b*; Andreev & Kuz'menkov 2015*b*). Therefore, the third solution is called the extraordinary SEAW.

Next, we need to study the dispersion curves of three extraordinary waves and the influence of the spin–orbit interaction on their dispersion dependencies. Therefore, we consider a degenerate quantum plasma with $n_{0e} = 1.4 \times 10^{27}$ cm^{-3} and $B_0 = 10^{11}$ G. The cyclotron frequency is smaller than the Langmuir frequency in this regime.

The presented regime of parameters corresponds to temperatures below 10^7 K. Required conditions can be found for electrons in neutron stars, where concentrations and magnetic fields are relatively high and temperatures are below the presented regime.

We consider this value of the particle concentration for the following reasons. The spin–orbit interaction is a semi-relativistic effect. Hence, the characteristic velocity $\tilde{v}$ should be comparable to the speed of light $\tilde{v} \approx 0.1c$. The Fermi velocity is the characteristic velocity for a degenerate electron gas. This problem can be addressed more explicitly. The Hamilton function for the spin–orbit interaction is $\gamma Ep/mc = e\hbar Ep/2m^2c^2$. It should be noticeable in comparison with the Coulomb interaction $e\varphi$, but is smaller than the Coulomb interaction $e\hbar Ep/2m^2c^2 \approx 0.1e\varphi$. Estimating the electric field as $\boldsymbol{E} = -\nabla\varphi \approx n^{1/3}\varphi$. So the presented estimation is correct if $\hbar p n^{1/3}/m^2c^2$ is comparable to 0.1. Estimating the momentum p as the Fermi momentum p_{Fe}, we have that $(\hbar n^{1/3}/mc)^2$ is comparable to 0.1. Both estimations lead to the considered concentrations.

To consider the influence of the spin–orbit interaction on the extraordinary waves we present figures 1 and 2. The increase of frequency of the upper extraordinary wave and the decrease of frequency of the lower extraordinary wave in the regime of relatively small wave vectors are demonstrated in figures 1 and 2, respectively.

The contribution of the spin–orbit interaction in the chosen parameter regime $B_0 = 10^{11}$ G $\ll B_{cr} = m_e^2c^3/(e\hbar) = 4.41 \times 10^{13}$ G corresponding to $\hbar\Omega \leqslant mc^2$ and

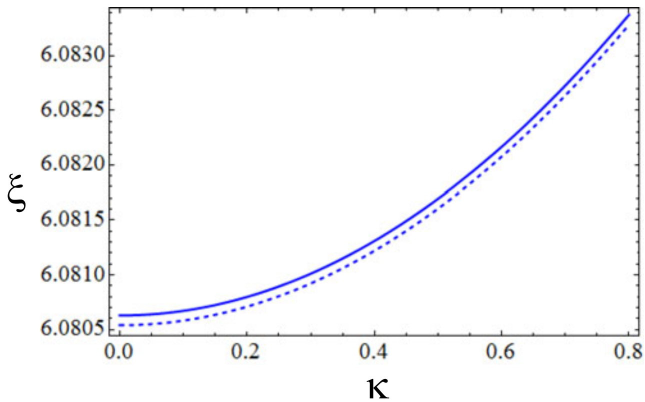

FIGURE 1. The figure shows the upper extraordinary wave. The dashed line presents a wave under the influence of the SSE. The continuous line includes the effect of the spin–orbit interaction in addition to the SSE. This figure is plotted for $n_{0e} = 1.4 \times 10^{27}$ cm^{-3}, $B_0 = 10^{11}$ G, $\eta_e = 0.11$ and temperature $T \ll T_{\rm Fe} = 5.9 \times 10^7$ K. The following notations are used in figures $\xi = \omega/\omega_{\rm Le}$ and $\kappa = kc/\omega_{\rm Le}$.

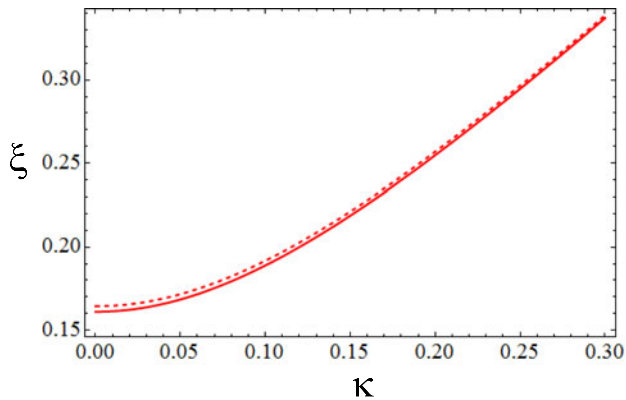

FIGURE 2. The figure shows the lower extraordinary wave. The dashed line presents SSE without spin–orbit interaction. The continuous line presents SSE with spin–orbit interaction. Parameters are as in figure 1.

$\varepsilon_{\rm Fe} = 0.01mc^2$ is relatively small. The contribution of the spin–orbit interaction increases with the increase of concentration. However, we work in the regime of parameters corresponding to the semi-relativistic approach in accordance with the area of applicability of our equations.

8. Conclusions

Generalization of the NLPE containing the spinor pressure contribution has been constructed to include the effect of spin–orbit interaction. Corresponding generalization of the SSE-QHD containing the pair of continuity equations, pair of Euler equations for spin-up and spin-down electrons and equations of the spin evolution for S_x and S_y projections of the spin density has also been constructed.

The spin–orbit interaction is an interesting example of quantum-relativistic effects. Its analysis as presented in this paper is an important step towards the construction of a quantum-relativistic hydrodynamic model. The presented model is a limit case of many-particle QHD located on the SSE in a degenerate electron gas.

We have demonstrated extra shifts of the dispersion dependencies of extraordinary waves under the influence of the spin–orbit interaction.

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Appendix A. Electromagnetic field structure in weakly relativistic plasmas

Considering a system of interacting charged particles, we can focus on the electromagnetic field acting on a particle. This field is caused by the other particles of the system. However, we can consider a part of this field to be caused by a single particle. We have the standard retarding potentials

$$\varphi(\boldsymbol{r}, t) = \int \frac{\rho(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3 r' \tag{A 1}$$

and

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{c} \int \frac{\boldsymbol{j}(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3 r'. \tag{A 2}$$

Let us analyse these potentials in the weakly relativistic regime, meaning that particle/particles creating the field move slowly in comparison with the speed of light. Following, for instance, Landau & Lifshitz (1975), we include that the distribution of charges does not undergo a noticeable change during time $|\boldsymbol{r} - \boldsymbol{r}'|/c$. Therefore, we can expand the density and current into series in $|\boldsymbol{r} - \boldsymbol{r}'|/c$. Making the expansion up to the second order, we find the following form of the scalar potential:

$$\varphi(\boldsymbol{r}, t) = \int \frac{\rho(\boldsymbol{r}', t)}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3 r' - \frac{1}{c} \partial_t \int \rho(\boldsymbol{r}', t) \, \mathrm{d}^3 r' + \frac{1}{2c^2} \partial_t^2 \int |\boldsymbol{r} - \boldsymbol{r}'| \rho(\boldsymbol{r}', t) \, \mathrm{d}^3 r', \tag{A 3}$$

where $\int \rho(\boldsymbol{r}', t) \, \mathrm{d}^3 r'$ is the full charge which does not change. Hence the second term is equal to zero.

Next, we consider the vector potential. This is given by (A 2) which contains $1/c$. Moreover, the Lagrange function governing the evolution of the considered particle (or corresponding equation of motion) contains $\boldsymbol{A}/c$ in term $q\boldsymbol{A}\boldsymbol{v}/c$ (which corresponds to the Lorentz force $q\boldsymbol{v} \times \boldsymbol{B}/c$). This shows that we already have $1/c^2$. Therefore, it is enough to take expansion of the vector potential up to the zeroth order in $|\boldsymbol{r} - \boldsymbol{r}'|/c$. As a result we have

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{c} \int \frac{\boldsymbol{j}(\boldsymbol{r}', t)}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}^3 r'. \tag{A 4}$$

Focusing on the field caused by a single charge and using the explicit forms of the density and current: $\rho = q\delta(\boldsymbol{r} - \boldsymbol{r}(t))$ and $\boldsymbol{A} = q\boldsymbol{v}(t)\delta(\boldsymbol{r} - \boldsymbol{r}(t))$ allows us to perform a

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simple integration in the equations obtained for the scalar and vector potentials and we then find

$$\varphi = \frac{q}{R} + \frac{q}{2c^2}\partial_t^2 R, \quad \boldsymbol{A} = \frac{q\boldsymbol{v}}{cR}, \tag{A 5a,b}$$

where $R = |\boldsymbol{r} - \boldsymbol{r}'|$.

Next, we change these potentials by applying the gauge transformation

$$\varphi' = \varphi - \frac{1}{c}\partial_t f, \quad \boldsymbol{A}' = \boldsymbol{A} + \nabla f. \tag{A 6a,b}$$

For the gauge transformation we choose the gauge function

$$f = \frac{q}{2c}\partial_t R. \tag{A 7}$$

As a result we find the following potentials:

$$\varphi' = \frac{q}{R}, \quad \boldsymbol{A}' = \frac{q\boldsymbol{v}}{cR} + \frac{q}{2c}\nabla\partial_t R. \tag{A 8a,b}$$

Following the representation of the vector potential requires the following calculation: $\nabla\partial_t R = \partial_t \nabla R = \partial_t(\boldsymbol{R}/R) = \dot{\boldsymbol{R}}/R - \boldsymbol{R}\dot{R}/R^2$. Assuming that the point of observation is fixed while the source of the field is moving, we find $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'(t)$ and $\dot{\boldsymbol{R}} = -\boldsymbol{v}$. Differentiating equation $R^2 = \boldsymbol{R}^2$ in time t we find $R\dot{R} = \boldsymbol{R}\dot{\boldsymbol{R}} = -\boldsymbol{R}\boldsymbol{v}$.

It gives the following expressions for the potentials

$$\varphi' = \frac{q}{R}, \quad \boldsymbol{A}' = \frac{q}{2cR}\left(\boldsymbol{v} + \frac{1}{R^2}\boldsymbol{R}(\boldsymbol{v}\boldsymbol{R})\right). \tag{A 9a,b}$$

Considering the electromagnetic field caused by a system of particles gives the field potentials equal to the superposition of the found expressions.

Considering the structure of the electric field following from the found potentials

$$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\partial_t \boldsymbol{A}. \tag{A 10}$$

Substituting the found potentials into the general expression for the electric field we obtain

$$\boldsymbol{E} = -\nabla\frac{q}{R} - \frac{q}{2c^2}\partial_t\left(\frac{\boldsymbol{v}}{R} + \frac{1}{R^3}\boldsymbol{R}(\boldsymbol{v}\boldsymbol{R})\right). \tag{A 11}$$

For the analysis of the Maxwell equations we need to consider the curl of the electric field

$$\nabla \times \boldsymbol{E} = 0 - \frac{q}{2c^2}\partial_t \nabla \times \left(\frac{\boldsymbol{v}}{R} + \frac{1}{R^3}\boldsymbol{R}(\boldsymbol{v}\boldsymbol{R})\right), \tag{A 12}$$

where $\nabla = \partial/\partial_{\boldsymbol{r}}$, $\boldsymbol{v}(t)$ does not depend on $\boldsymbol{r}$. Calculation gives

$$\nabla \times \boldsymbol{E} = 0 - \frac{q}{2c^2}\partial_t \nabla \times \left(\frac{\boldsymbol{v}}{R} + \frac{1}{R^3}\boldsymbol{R}(\boldsymbol{v}\boldsymbol{R})\right). \tag{A 13}$$

Let us represent the last equation in more detail

$$\nabla\times\boldsymbol{E}=-\frac{q}{2c^2}\partial_t\left(\boldsymbol{v}\times\nabla\frac{1}{R}+(\boldsymbol{v}\boldsymbol{R})\boldsymbol{R}\times\nabla\frac{1}{R^3}+\frac{1}{R^3}(\nabla\times\boldsymbol{R})(\boldsymbol{v}\boldsymbol{R})+\frac{1}{R^3}\boldsymbol{R}\times\nabla(\boldsymbol{v}\boldsymbol{R})\right). \tag{A 14}$$

Including the following properties $\nabla(1/R)=-\boldsymbol{R}/R^3$, $\nabla(1/R^3)=-3(\boldsymbol{R}/R^5)$, $\nabla\times\boldsymbol{R}=0$, and $\nabla(\boldsymbol{v}\boldsymbol{R})=\boldsymbol{v}$, we find that the second and third terms are equal to zero and the module of the first term is equal to the module of the last term, but it has opposite sign. Overall, it gives $\nabla\times\boldsymbol{E}=0$ applied in the main part of the paper.

Appendix B. Spin current

We need to have the explicit form of the spin current tensor components to recognize their parts in the terms caused by the spin–orbit interaction. Therefore, we present their non-relativistic parts here:

$$J^{x\alpha}=\frac{1}{2m}(\Phi_u^*D^\alpha\Phi_d+\Phi_d^*D^\alpha\Phi_u+\text{c.c.}), \tag{B 1}$$

$$J^{y\alpha}=\frac{1}{2m}(\Phi_u^*(-\imath)D^\alpha\Phi_d+\Phi_d^*\imath D^\alpha\Phi_u+\text{c.c.}), \tag{B 2}$$

$$J^{z\alpha}=\frac{1}{2m}(\Phi_u^*D^\alpha\Phi_u-\Phi_d^*D^\alpha\Phi_d+\text{c.c.}). \tag{B 3}$$

Appendix C. Equation of state for $j^{\alpha\beta\gamma}$

An explicit expression for $j^{\alpha\beta\gamma}$ defined by (3.10) is equal to the explicit expression for the spin current flux $J^{\alpha\beta\gamma}$ defined by (3.11) up to the quantum terms similar to the quantum Bohm potential. Qualitatively speaking, the spin current flux is the average value of the following quantity: $J^{\alpha\beta\gamma}=\langle S_i^\alpha v_i^\beta v_i^\gamma\rangle$, where S_i^α and v_i^α are the spin and velocity of the ith particle. Splitting spin and velocity by the value related to the local centre of mass and the value related to the thermal motion $S_i^\alpha=S^\alpha(\boldsymbol{r},t)+s_i^\alpha$ and $v_i^\alpha=v^\alpha(\boldsymbol{r},t)+u_i^\alpha$ we find

$$\begin{aligned}J^{\alpha\beta\gamma}&=S^\alpha v^\beta v^\gamma+S^\alpha\langle u_i^\beta u_i^\gamma\rangle+v^\beta\langle s_i^\alpha u_i^\gamma\rangle+v^\gamma\langle s_i^\alpha u_i^\beta\rangle+\langle s_i^\alpha u_i^\beta u_i^\gamma\rangle\\&=S^\alpha v^\beta v^\gamma+S^\alpha p^{\beta\gamma}+v^\beta J_{\text{th}}^{\alpha\gamma}+v^\gamma J_{\text{th}}^{\alpha\beta}+J_{\text{th}}^{\alpha\beta\gamma},\end{aligned} \tag{C 1}$$

where $J_{\text{th}}^{\alpha\beta}$ and $J_{\text{th}}^{\alpha\beta\gamma}$ are the thermal spin current and the thermal spin current flux.

For our calculations we need the equilibrium value of $j^{\alpha\beta\gamma}$. This means we need the equilibrium spin current flux $J_0^{\alpha\beta\gamma}=\delta^{z\alpha}S^z(\boldsymbol{r},t)p_{\text{Fe}}\delta^{\beta\gamma}+J_{0,\text{th}}^{\alpha\beta\gamma}$. For the calculation of the equilibrium thermal spin current flux $J_{0,\text{th}}^{\alpha\beta\gamma}$ we can use the equilibrium spin distribution functions obtained in Andreev (2016*b*) for SSE quantum kinetics $J^{\alpha\beta\gamma}=\int p^\gamma p^\beta S^\alpha(\boldsymbol{r},\boldsymbol{p},t)\,\mathrm{d}\boldsymbol{p}=\delta^{z\alpha}S^z(\boldsymbol{r},t)p^{\beta\gamma}$. It is already included in $J_0^{\alpha\beta\gamma}$. It means $J_{0,\text{th}}^{\alpha\beta\gamma}=0$. Consequently, we find $j_0^{\alpha\beta\gamma}=\delta^{z\alpha}\delta^{\beta\gamma}S^zp_{\text{Fe}}$ which gives zero contribution in the linear evolution (6.2).

Appendix D. Justification of the structure of the SSE hydrodynamic equations from the SSE kinetic model

There are two regimes for the development of statistical (quantum or classic) physics: the physical system in thermostat (system interacting with its surroundings,

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but kept at a fixed temperature – the temperature of the larger surrounding object), where the grand canonical Gibbs ensemble (distribution) is used, and the isolated physics system (the systems with fixed energy), where the microcanonical Gibbs ensemble is used.

In the first state, the system is located in the mixed quantum state and it is described by the density matrix while our system and the thermostat are located in the pure state and described by the wave function $\Psi(R, t)$ or the equivalent density matrix $\rho(R, R', t) = \Psi^*(R, t)\Psi(R', t)$.

In the second case, the system is located in the pure quantum state and described by the wave function. We consider a regime with a fixed full energy of the system which does not necessarily require the density matrix.

The original paper on the SSE-QHD (Andreev 2015*b*) is based on the single-particle Pauli equation. It has the aim of demonstrating the structure of the hydrodynamic equations which exist for the independent description of the electrons with different spin projections.

The pressure term existing for the collection of particles does not appear there. However, it is included there in accordance with the knowledge of the Euler equation structure.

The appearance of the SSE model from the many-particle wave function is considered in Andreev (2016*b*) for the kinetic model. This kinetic model allows us to present a justification of the SSE-QHD via the calculation of the equations for the evolution of the moments of the scalar and vector distribution functions. Hence, it gives a strong background for the model considered in this paper.

The major hydrodynamic characteristics (the material fields) are introduced in accordance with Andreev (2016*b*) which is made in the traditional way: $n_a(\boldsymbol{r}, t) = \int f_a(\boldsymbol{r}, \boldsymbol{p}, t)\,\mathrm{d}\boldsymbol{p}$, $\boldsymbol{j}_a(\boldsymbol{r}, t) = \int (\boldsymbol{p}/m) f_a(\boldsymbol{r}, \boldsymbol{p}, t)\,\mathrm{d}\boldsymbol{p}$, $\tilde{S}_{ex}(\boldsymbol{r}, t) = \int S_{ex}(\boldsymbol{r}, \boldsymbol{p}, t)\,\mathrm{d}\boldsymbol{p}$ and $\tilde{S}_{ey}(\boldsymbol{r}, t) = \int S_{ey}(\boldsymbol{r}, \boldsymbol{p}, t)\,\mathrm{d}\boldsymbol{p}$.

For the derivation of the evolution equations for the moments of the distribution functions we use the kinetic equations obtained in Andreev (2016*b*): for the spin-up electrons

$$\partial_t f_{e\uparrow} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f_{e\uparrow} + q_e\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}} f_{e\uparrow} + \frac{q_e}{c}[\boldsymbol{v}, \boldsymbol{B}]\cdot\nabla_{\boldsymbol{p}} f_{e\uparrow} + \gamma_e\nabla B^z\cdot\nabla_{\boldsymbol{p}} f_{e\uparrow} + \frac{\gamma_e}{2}(\nabla B_x\cdot\nabla_{\boldsymbol{p}} S_{e,x} + \nabla B_y\cdot\nabla_{\boldsymbol{p}} S_{e,y}) = \frac{\gamma_a}{\hbar}[S_{e,x}B_y - S_{e,y}B_x]. \tag{D 1}$$

For the spin-down electrons it has the same structure as the equation for the spin-up electrons, but with different coefficients:

$$\partial_t f_{e\downarrow} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f_{e\downarrow} + q_e\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}} f_{e\downarrow} + \frac{q_e}{c}[\boldsymbol{v}, \boldsymbol{B}]\cdot\nabla_{\boldsymbol{p}} f_{e\downarrow} - \gamma_e\nabla B^z\cdot\nabla_{\boldsymbol{p}} f_{e\downarrow} + \frac{\gamma_e}{2}(\nabla B_x\cdot\nabla_{\boldsymbol{p}} S_{e,x} + \nabla B_y\cdot\nabla_{\boldsymbol{p}} S_{e,y}) = -\frac{\gamma_a}{\hbar}[S_{e,x}B_y - S_{e,y}B_x], \tag{D 2}$$

and equations for the spin distribution function (the vector distribution function) are

$$\partial_t S_{e,x} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} S_{e,x} + q_e\left(\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}, \boldsymbol{B}]\right)\cdot\nabla_{\boldsymbol{p}} S_{e,x} + \gamma_e\nabla B_x\cdot\nabla_{\boldsymbol{p}}(f_{e\uparrow} + f_{e\downarrow}) - \frac{2\gamma_e}{\hbar}(S_{e,y}B_z - (f_{e\uparrow} - f_{e\downarrow})B_y) = 0, \tag{D 3}$$

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and

$$\partial_t S_y + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} S_{e,y} + q_e\left(\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v},\boldsymbol{B}]\right)\cdot\nabla_{\boldsymbol{p}} S_{e,y} + \gamma_e\nabla B_y\cdot\nabla_{\boldsymbol{p}}(f_{e\uparrow}+f_{e\downarrow}) - \frac{2\gamma_e}{\hbar}((f_{e\uparrow}-f_{e\downarrow})B_x - S_{e,x}B_z) = 0. \tag{D 4}$$

In this subsection $\boldsymbol{v}=\boldsymbol{p}/m$ is an independent variable of the kinetic model. This is in contrast with other parts of this paper, where $\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{r},t)$ is the velocity field and it is a function of space and time.

These equations lead to the quasi-classic limit (with no spin–orbit interaction and no quantum Bohm potential) of the hydrodynamic equations (3.1), (3.8), (3.13) with the following definitions of the momentum flux

$$\Pi_a^{\alpha\beta}(\boldsymbol{r},t) = \int \frac{p^\alpha p^\beta}{m} f_a(\boldsymbol{r},\boldsymbol{p},t)\,\mathrm{d}\boldsymbol{p}, \tag{D 5}$$

the pressure

$$p_a^{\alpha\beta}(\boldsymbol{r},t) = \Pi_a^{\alpha\beta}(\boldsymbol{r},t) - j_a^\alpha(\boldsymbol{r},t) j_a^\beta(\boldsymbol{r},t)/n_a(\boldsymbol{r},t), \tag{D 6}$$

and the spin–current

$$J_a^{\alpha\beta}(\boldsymbol{r},t) = \int \frac{p^\beta}{m} S_e^\alpha(\boldsymbol{r},\boldsymbol{p},t)\,\mathrm{d}\boldsymbol{p}, \tag{D 7}$$

where

$$S_e^z(\boldsymbol{r},\boldsymbol{p},t) = f_{e\uparrow}(\boldsymbol{r},\boldsymbol{p},t) - f_{e\downarrow}(\boldsymbol{r},\boldsymbol{p},t). \tag{D 8}$$

Integrating (D 1) and (D 2) over all momentum space and assuming that the distribution functions f_s (in the third, fifth and sixth terms) and the product $p^\alpha f_s$ (in the fourth term after integration by parts) go to zero at an infinite value of the momentum, we find the continuity equations.

Similarly, the kinetic equations (D 1) and (D 2) can be multiplied by the momentum p^α. Next, the obtained equations can be integrated over all momentum space. The calculations become a bit more complicated. Hence, let us present this calculation, which also would help in understanding the calculations described above for the continuity equations. Therefore, we show the following:

$$\begin{aligned}
\partial_t j_s^\alpha &= \partial_t \int \mathrm{d}\boldsymbol{p}\, v^\alpha f_s = \int \mathrm{d}\boldsymbol{p}\, v^\alpha \partial_t f_s \\
&= -\nabla^\beta \int \mathrm{d}\boldsymbol{p}\, v^\alpha v^\beta f_s - q_e E^\beta \int \mathrm{d}\boldsymbol{p}\, v^\alpha \nabla_p^\beta f_s \\
&\quad - \frac{q_e}{c}\varepsilon^{\beta\gamma\delta} B^\gamma \int \mathrm{d}\boldsymbol{p}\, v^\alpha v^\beta \nabla_p^\delta f_s \mp \gamma_e \nabla^\beta B_z \int \mathrm{d}\boldsymbol{p}\, v^\alpha \nabla_p^\beta f_s \\
&\quad - \frac{1}{2}\gamma_e \nabla^\beta B_x \int \mathrm{d}\boldsymbol{p}\, v^\alpha \nabla_p^\beta S_x - \frac{1}{2}\gamma_e \nabla^\beta B_y \int \mathrm{d}\boldsymbol{p}\, v^\alpha \nabla_p^\beta S_y \\
&\quad \pm \frac{\gamma_e}{\hbar} B_y \int \mathrm{d}\boldsymbol{p}\, v^\alpha S_x \mp \frac{\gamma_e}{\hbar} B_x \int \mathrm{d}\boldsymbol{p}\, v^\alpha S_y.
\end{aligned} \tag{D 9}$$

In the second and fourth terms we have

$$\int \mathrm{d}\boldsymbol{p}\, v^\alpha \nabla_p^\beta f_s = \int \mathrm{d}\boldsymbol{p}\, \nabla_p^\beta (v^\alpha f_s) \tag{D 10}$$

$$-\delta^{\alpha\beta} \int \mathrm{d}\boldsymbol{p}\, f_s/m = -\delta^{\alpha\beta} n_s/m, \tag{D 11}$$

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where we used $\nabla_p^\beta v^\alpha = \delta^{\alpha\beta}/m$. Similarly we make for the fifth and sixth terms

$$\int \mathrm{d}\boldsymbol{p}\, v^\alpha \nabla_p^\beta S_i = \int \mathrm{d}\boldsymbol{p}\, \nabla_p^\beta (v^\alpha S_i) \tag{D 12}$$

$$-\delta^{\alpha\beta} \int \mathrm{d}\boldsymbol{p}\, S_i/m = -\delta^{\alpha\beta} S_i(r, t)/m, \tag{D 13}$$

where $i = x, y$. In the third term we make the following transformation

$$\begin{aligned}\int \mathrm{d}\boldsymbol{p}\, v^\alpha v^\beta \nabla_p^\delta f_s &= \int \mathrm{d}\boldsymbol{p}\, \nabla_p^\delta (v^\alpha v^\beta f_s) - \int \mathrm{d}\boldsymbol{p}\, \nabla_p^\delta (v^\alpha v^\beta) f_s \\ &= -\int \mathrm{d}\boldsymbol{p}\, (\delta^{\alpha\delta} v^\beta + \delta^{\beta\delta} v^\alpha) f_s.\end{aligned} \tag{D 14}$$

Next, we multiply the last equation on $\varepsilon^{\beta\gamma\delta}$ in accordance with the third term in (D 9) and find

$$\int \mathrm{d}\boldsymbol{p}\, (\varepsilon^{\beta\gamma\alpha} v^\beta + \varepsilon^{\beta\gamma\beta} v^\alpha) f_s = \varepsilon^{\alpha\beta\gamma} \int \mathrm{d}\boldsymbol{p}\, v^\beta f_s = \varepsilon^{\alpha\beta\gamma} j^\beta. \tag{D 15}$$

Finally, we obtain the Euler equation for the spin-s electrons

$$\begin{aligned} m\partial_t j_s^\alpha &= -\nabla^\beta \Pi_s^{\alpha\beta} + q_e n_s E^\alpha + \frac{q_e}{c} \varepsilon^{\alpha\beta\gamma} j^\beta B^\gamma \\ &\quad \pm \gamma_e n_s B_z + \frac{1}{2}\gamma_e (S_x \nabla^\alpha B_x + S_y \nabla^\alpha B_y) \pm \frac{m\gamma_e}{\hbar}(J_{x\alpha} B_y - J_{y\alpha} B_x). \end{aligned} \tag{D 16}$$

In the same way we integrate (D 3) and (D 4) to find the quasi-classic part of the x- and y-projections of the spin evolution equation (5.6). To find the z-projection we need to consider and integrate the difference between the kinetic equations (D 1) and (D 2).

Let us mention the area of applicability of the equations derived in this paper. The SSE-QHD is obtained in the self-consistent (the mean field) approximation for fully degenerate electrons (the temperature is equal to zero). The SSE-QHD includes an extra degree of freedom following from the existence of spin. The single-fluid hydrodynamics of electrons includes the spin via the spin density evolution which is proportional to the magnetization and can be introduced is classical physics as well. The SSE-QHD explicitly incorporates the discrete nature of the spin projections which are presented in the two line structure of the Pauli equation and revealed in the Stern–Gerlach experiment. Neglecting the SSE we drop the SEAWs which spectrum is the low frequency part of the electron dynamics at $\boldsymbol{k} \parallel \boldsymbol{B}_{ext}$. At $\boldsymbol{k} \perp \boldsymbol{B}_{ext}$ the SEAW spectra are located in the relatively small area above the electron cyclotron frequency. Hence, the single-fluid model of electrons, in a way, is the high frequency regime of the plasma dynamics. Recovering the ion motion, we see that the SSE dynamics corresponds to the intermediate frequency range, but it can overlap with the low frequency area of the ion sound in the limit cases (see for instance Andreev 2015*b*). One relativistic effect (the SO interaction) is considered while others, such as the relativistic part of the kinetic energy, the spin–current interaction, the current–current interaction the Darwin interaction, are outside of the focus of this paper.

The quantum Bohm potential is not included in the equations demonstrated in Andreev (2016*b*) and repeated here. Andreev (2016*b*) focused on the quasi-classical effects of the separate spin evolution. However, the method of derivation used there is shown in Andreev (2012). Andreev (2012) does not give details of the derivation,

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but shows an intermediate form of the kinetic equations, where all electrons are considered as a single fluid. The quantum terms are shown there. Particularly, an equation for the traditional distribution function f_e contains a term proportional to $\hbar^2$ along with $(\boldsymbol{E}\cdot\nabla_{\boldsymbol{p}})f_e$. This term is also traditional for Wigner kinetics. It provides the quantum Bohm potential contribution to the spectrum of collective excitations. A similar term appears in the quantum kinetics with separate spin evolutions, but it was neglected in the published papers.

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