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COLORFUL PARTITIONS OF CARDINAL NUMBERS

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1. Introduction. Use the two element subsets of κ, denoted by $[\kappa]^2$, as the edge set for the complete graph on κ points. Write $\mathrm{CP}(\kappa, \mu, \nu)$ if there is an edge coloring R: $[\kappa]^2 \rightarrow \mu$ with μ colors so that for every proper ν element set $X \subsetneqq \kappa$, there is a point $x \in \kappa \smallsetminus X$ so that the edges between x and X receive at least the minimum of μ and ν colors. Write $\mathrm{CP}\#(\kappa, \mu, \nu)$ if the coloring is one-to-one on the edges between x and elements of X.

Peter W. Harley III [**5**] introduced CP and proved that for $\kappa \geqq \omega$, $\mathrm{CP}(\kappa^+, \kappa, \kappa)$ holds to solve a topological problem, since the fact that $\mathrm{CP}(\aleph_1, \aleph_0, \aleph_0)$ holds implies the existence of a symmetrizable space on $\aleph_1$ points in which no point is a G_δ.

G. McNulty showed that $\mathrm{CP}(\kappa, \mu, \nu)$ holds for $\kappa^\nu = \kappa$ and $\nu \geqq \mu \geqq \omega$. We heard about the problem from him and from Trotter. The paper owes its title to McNulty. We would like to thank the referee for several useful suggestions.

Many people have worked on the problem of determining for which finite m and k with $m \geqq k + 1 \geqq 3$ these relations hold. The following list summarizes the known results and is based on notes from W. T. Trotter, Jr.

1. $\mathrm{CP}(k + 1, k, k)$ if and only if k is odd (many people)
2. not $\mathrm{CP}(3, 2, 2)$ (from 1) CP $(m, 2, 2)$ for $m \geqq 4$ (Gauter, McNulty, Sumner, Trotter)
3. $\mathrm{CP}(4, 3, 3)$ (from 1) not $\mathrm{CP}(5, 3, 3)$ (many people) not $\mathrm{CP}(6, 3, 3)$ $\mathrm{CP}(7, 3, 3)$ (Sumner and Trotter) $\mathrm{CP}(10, 3, 3)$ and $\mathrm{CP}(11, 3, 3)$ (Weese) $\mathrm{CP}(19, 3, 3)$ (Gauter and Rosa)
4. CP (m, k, k) if $k \geqq 3$ and $m \geqq k^{3/2}e^k$ (Erdös)
5. For every $\epsilon > 0$ there is a k_0 so that if $k + 2 \leqq m \leqq k^{-1/2-\epsilon}e^k$ and $k \geqq k_0$ then not $\mathrm{CP}(m, k, k)$ (Erdös and Spencer).

For the last two results, Erdös and Spencer use the "probabilistic" method. It would be desirable to obtain an asymptotic formula for $\mathrm{CP}(m, k, k)$ but this does not seem to be easy. Sumner and Trotter, and Gauter and Rosa construct the colorings for $\mathrm{CP}(7, 3, 3)$ and $\mathrm{CP}(19, 3, 3)$ respectively. Not much else has been done to construct colorings in the other cases for which the relation is known to hold.

In this paper, we consider only infinite parameters. We shall prove in Lemma 5.1 that if μ is regular and $\kappa \geqq \mu$, then $\mathrm{CP}(\kappa, \mu, \mu)$, and if $\mu < \kappa$, then $\mathrm{CP}(\kappa, \mu,$

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μ^+). In Theorem 5.2 we characterize CP under the assumption of GCH, by proving that for κ, μ, ν with $\kappa \geqq \mu$, $\kappa \geqq \nu$, the relation $\mathrm{CP}(\kappa, \mu, \nu)$ fails only if $\kappa > \mu \geqq \nu > \mathrm{cf}\nu = \mathrm{cf}\kappa$. In Theorem 5.3, we characterize CP# under the assumption of GCH, by proving that for κ, μ, ν with $\kappa \geqq \mu \geqq \nu$, the relation $\mathrm{CP}\#(\kappa, \mu, \nu)$ fails only if $\kappa > \mu \geqq \nu \geqq \mathrm{cf}\ \kappa$.

To prove the theorems about CP and CP#, we introduce two related relations BP and BP#. Write $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ if there is a coloring of the complete bipartite κ, λ graph, $R: \kappa \times \lambda \rightarrow \mu$, with μ colors, so that for every ν element subset $X \subseteq \kappa$, there is a point $x \in \lambda$, so that the edges from elements of X to x receive at least the minimum of μ and ν colors. That is, $|R''X \times \{x\}| \geqq \min(\mu, \nu)$. Write $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$ if R restricted to $X \times \{x\}$ is one-to-one.

In Section 2, we reduce problems about CP and CP# to problems about BP and BP#. In Section 3, we study BP#, giving a complete characterization under GCH. In Section 4, we study BP. Here we get a complete characterization only with the assumption of $V = L$. With GCH, there is still an open problem which is formulated in terms of the existence of a tree together with a family of its branches satisfying certain properties. In Section 5, we draw the conclusions for CP and CP# from the results of the previous sections.

The set theoretic terminology is standard. The letters $\kappa, \lambda, \mu, \nu, k, n$, are reserved for cardinal numbers, while $\alpha, \beta, \gamma, \delta, a, b$ are used for ordinals. Each ordinal number is identified with the set of its predecessors. Since the axiom of choice is assumed throughout, cardinals are identified with initial ordinals. Therefore, in particular, if α is an ordinal and λ is a cardinal, then $\alpha < \lambda$ if and only if $\alpha \in \lambda$. The set of natural numbers is denoted by ω.

If A is a set, then $|A|$ is the cardinality of A. The cardinal successor of κ is denoted by κ^+. The nth cardinal successor of κ is denoted by $\kappa^{+(n)}$. Let ν^- be the immediate predecessor if ν is a successor cardinal, and let $\nu^- = \nu$ otherwise.

If α is an ordinal, then $\mathrm{cf}\,\alpha$ is the least ordinal which can be mapped onto a cofinal subset of α.

A cardinal κ is *regular* if $\mathrm{cf}\ \kappa = \kappa$. It is well known that for any ordinal α, $\mathrm{cf}\alpha$ is regular, and that any successor cardinal is regular. Cardinals which are not successor cardinals are *limit cardinals.* Cardinals which are not regular are *singular.*

Cardinal arithmetic plays an important role here. At points the Generalized Continuum Hypothesis, or GCH, is used, which says that for every cardinal κ, $2^\kappa = \kappa^+$.

We denote by $[\kappa]^\nu$ the family of all ν element subsets of κ. We have already used this notation for $\nu = 2$. We write ${}^\alpha\sigma$ for the set of all functions of domain α and range a subset of σ. Write $R''A = \{R(a) : a \in A\}$ for the image of a set A under a function R.

In the following, various sets of appropriate cardinality will be used as the basis for the graphs and the sets of colors. The colorings themselves will be considered as functions from the set of edges into the set of colors, and thus may also be thought of as labelings of the edges or as partitions.

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2. Write BP $(\kappa, \lambda, \mu, \nu)$ if there is a coloring $R: \kappa \times \lambda \to \mu$, of $\kappa \times \lambda$ with μ colors, so that for every ν element subset $X \subseteq \kappa$, there is a point $x \in \lambda$ so that $X \times \{x\}$ has $\min(\mu, \nu)$ colors, that is, $|R''X \times \{x\}| = \min(\mu, \nu)$. Write BP#$(\kappa, \lambda, \mu, \nu)$ if in addition, every edge gets a different color, namely if R is one-to-one on $X \times \{x\}$.

LEMMA 2.1. *If $\kappa \geqq \nu$ and $\kappa \geqq \mu$, then* CP(κ, μ, ν) *if and only if* BP$(\kappa, \kappa, \mu, \nu)$.

Proof. If $R: \kappa \to \mu$ is a coloring which attests to CP(κ, μ, ν), then $S: \kappa \times \kappa \to \mu$ defined by $S(x, y) = R(\{x, y\})$ if $x \neq y$, $S(x, x) = 0$ attests to BP$(\kappa, \kappa, \mu, \nu)$.

Suppose $S: \kappa \times \kappa \to \mu$ attests to BP$(\kappa, \kappa, \mu, \nu)$. Without loss of generality we may assume $|S''\kappa \times \{x\}| = \mu$ for all $x \in \kappa$. We may also assume S is symmetric (otherwise replace S by S', where

$$S'(x, y) = \{S(x, y), S(y, x)\}).$$

Now define $R: [\kappa \times \kappa]^2 \to \mu \times \mu$ by

$$R\{(x_1, y_1), (x_2, y_2)\} = (S(x_1, x_2), S(y_1, y_2)).$$

Suppose $X \subseteq \kappa \times \kappa$, $|X| = \nu$, X is proper. Let $X_1 = \{x: \exists y(x, y) \in X\}$ and $X_2 = \{y: \exists x(x, y) \in X\}$. If $X_1 = \kappa$ or $X_2 = \kappa$ we are done, so suppose not. Either $|X_1| = \nu$ or $|X_2| = \nu$. Say $|X_1| = \nu$. Choose $x \in \kappa$ so that $|S''X_1 \times \{x\}| = \mu$ and choose $y \in \kappa - X_2$. Then $|R''X \times \{(x, y)\}| = \mu$. The case $|X_2| = \nu$ is symmetric.

LEMMA 2.2. *For all κ, μ, ν with $\kappa \geqq \mu \geqq \nu$, if* CP#$([\kappa]^2, u, \nu)$, *then* BP#$(\kappa, \kappa, \mu, \nu)$.

Proof. We prove the contrapositive. So assume not BP#$(\kappa, \kappa, \mu, \nu)$, and suppose $R: [\kappa]^2 \to \mu$ is a coloring. Define $S: \kappa \times \kappa \to \mu$ by $S(x, y) = R(\{x, y\})$ if $x \neq y$ and $S(x, y) = 0$ if $x = y$. Choose $X \subseteq \kappa$ with $|X| = \nu$ so that for all $x \in \kappa$, S restricted to $X \times \{x\}$ is not one-to-one. Then X has the corresponding property for R, so the lemma follows.

LEMMA 2.3. *For all κ, μ, ν with $\kappa \geqq \mu \geqq \nu$, if* BP#$(\kappa, \kappa, \mu, \nu)$, *then* CP#$(\kappa, \mu, \nu)$.

Proof. If $\nu = \kappa$, then we have $\kappa = \mu = \nu$. So every coloring which is one-to-one on $[\kappa]^2$ attests to CP#(κ, μ, ν), thus for $\nu = \kappa$ the lemma holds.

So assume $\nu < \kappa$. Let $R: \kappa \times \kappa \to \mu$ attest to BP#$(\kappa, \kappa, \mu, \nu)$. Since $\nu < \kappa$, we have $\nu^+ \leqq \kappa$. Write κ as the disjoint union of ν^+ subsets each of power κ, $\kappa = \bigcup\{A_\alpha: \alpha < \nu^+\}$. For each $\alpha < \nu^+$, let $\{a_\alpha(\beta): \beta < \kappa\}$ enumerate A_α in order type κ. Then

$$\kappa = \{a_\alpha(\beta): \alpha < \nu^+ \text{ and } \beta < \kappa\}.$$

Define $S: [\kappa]^2 \to \mu$ by

$$S(\{a_\alpha(\beta), a_\gamma(\delta)\}) = R(a_\alpha(\beta), \delta) \text{ if } \alpha < \gamma \text{ and}$$
$$S(\{a_\alpha(\beta), a_\gamma(\delta)\}) = 0 \text{ if } \alpha = \gamma.$$

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Suppose $X \subseteq \kappa$ and $|X| = \nu$. Find $\gamma < \nu^+$ so that $X \subseteq \bigcup\{A_\alpha : \alpha < \gamma\}$. Find $\delta \in \kappa$ so that R restricted to $X \times \{\delta\}$ is one-to-one. Then $a_\gamma(\delta) \notin X$ and S restricted to

$$\{\{a_\alpha(\beta), a_\gamma(\delta)\} : a_\alpha(\beta) \in X\}$$

is one-to-one. So the lemma holds for $\nu < \kappa$.

3. BP#. In this section we discuss BP#(κ, λ, μ, ν). Since the relation makes no sense if $\nu > \kappa$ and cannot hold if $\nu > \mu$, in discussing BP#(κ, λ, μ, ν) we always assume $\kappa \geqq \nu$ and $\mu \geqq \nu$. First we give arguments showing no coloring exists. Then we construct colorings under various assumptions. We show how to use the assumption of the relation in some cases to prove it in others. Finally, we discuss the relation under the assumption of GCH.

LEMMA 3.1. *If $\kappa > \mu$ and $\nu \geqq \lambda$, then the relation* BP#(κ, λ, μ, ν) *fails to hold.*

Proof. Let $R: \kappa \times \lambda \to \mu$ be a coloring. For each $y \in \lambda$, $R^y: \kappa \to \mu$ is defined by $R^y(x) = R(x, y)$. Using the fact that $\kappa > \mu$, for each $\alpha < \lambda$ choose two points u_α, v_α so that $R_\alpha(u_\alpha) = R_\alpha(v_\alpha)$. Let Y be any set of power ν having all the u_α's and v_α's as elements. Then Y works for R.

LEMMA 3.2. *If $\nu \geqq \mathrm{cf}\lambda$ and for all $\rho < \lambda$, $\mu^\rho < \kappa$, then the relation* BP#(κ, λ, μ, ν) *fails to hold.*

Proof. Let $R: \kappa \times \lambda \to \mu$ be a coloring. Divide

$$\lambda = \bigcup\{A_\alpha : \alpha < \mathrm{cf}\lambda\}$$

into cfλ disjoint sets each of power less than λ. For each $\alpha < \mathrm{cf}\,\lambda$, since $|A_\alpha| = \rho < \lambda$, also $\mu^\rho < \kappa$. So there are fewer than κ functions from A_α into μ. For each $\alpha < \mathrm{cf}\,\lambda$ choose two points $u_\alpha, v_\alpha \in \kappa$ so that R restricted to $\{u_\alpha\} \times A_\alpha$ induces the same function on A_α as R restricted to $\{v_\alpha\} \times A_\alpha$. Let Y be a set of power ν having all the u_α's and v_α's as elements. Then Y works for R.

LEMMA 3.3. *If $\mu^\lambda < \kappa$, then the relation* BP#(κ, λ, μ, ν) *fails to hold.*

Proof. Let $R: \kappa \times \lambda \to \mu$ be a coloring. For each $x \in \kappa$, $R_x: \lambda \to \mu$ is defined by $R_x(y) = R(x, y)$. There are at most $\mu^\lambda < \kappa$ functions from λ into μ. So for some $S: \lambda \to \mu$ and some $X \subseteq \kappa$ of power ν, we have $R_x = S$ for all $x \in X$. Then X works for R.

LEMMA 3.4. *If μ is singular, $(\mathrm{cf}\,\mu)^\lambda < \mathrm{cf}\,\kappa$, and $\mathrm{cf}\,\mu \leqq \nu < \mu$, then* BP#($\kappa$, λ, μ, ν) *holds if and only if for some ρ with $\nu \leqq \rho < \mu$, the relation* BP#(κ, λ, ρ, ν) *holds.*

Proof. One direction follows from the definition. We prove the contrapositive of the other direction. Assume for all ρ with $\nu \leqq \rho < \mu$, the relation BP#(κ, λ, ρ, ν) fails to hold. Let $R: \kappa \times \lambda \to \mu$ be a coloring. Divide $\mu = \bigcup\{A_\alpha : \alpha < \mathrm{cf}\,\mu\}$ into the disjoint union of cf μ sets each of power between cf μ

and μ. Define $\hat{R}$: $\kappa \times \lambda \to \mathrm{cf}\mu$ by $\hat{R}(x, y) = \alpha$ where $R(x, y) \in A_\alpha$. Since $\mathrm{cf}\kappa > (\mathrm{cf}\mu)^\lambda$, there is a set $X \subseteq \kappa$ of power κ and a function S: $\lambda \to \mathrm{cf}\mu$ so that for all $(x, y) \in X \times \lambda$, $\hat{R}(x, y) = S(y)$. Now S induces a partition of $\lambda = \bigcup\{B_\alpha: \alpha < \mathrm{cf}\, \mu\}$, where $B_\alpha = \{y: S(y) = \alpha\}$. For each $\alpha < \mathrm{cf}\, \mu$, R restricted to $X \times B_\alpha$ maps into A_α. So for each $\alpha < \mathrm{cf}\, \mu$, let $Y_\alpha \subseteq X$ be a set attesting to not $\mathrm{BP}\#(\kappa, \lambda, |A_\alpha|, \nu)$ for R restricted to $X \times B_\alpha$. Then $Y = \bigcup\{Y_\alpha: \alpha < \mathrm{cf}\, \mu\}$ works for R.

LEMMA 3.5. *If* λ *is singular and* $\mathrm{cf}\, \lambda \leqq \nu$, *then* $\mathrm{BP}\#(\lambda^+, \lambda, \mu, \nu)$ *if and only if for some* $\tau < \lambda$, $\mathrm{BP}\#(\lambda^+, \tau, \mu, \nu)$.

Proof. One direction follows from the definition. We prove the contrapositive of the other direction. Suppose for all $\tau < \lambda$, the relation $\mathrm{BP}\#((\lambda^+, \tau, \mu, \nu)$ fails to hold. Let R: $\lambda^+ \times \lambda \to \mu$ be a coloring. Divide $\lambda = \bigcup\{A_\alpha: \alpha < \mathrm{cf}\, \lambda\}$ into $\mathrm{cf}\, \lambda$ disjoint sets each of power less than λ. For each $\alpha < \mathrm{cf}\, \lambda$, let $X_\alpha \subseteq \lambda^+$ be a set of power ν attesting to not $\mathrm{BP}\#(\lambda^+, |A_\alpha|, \mu, \nu)$, for R restricted to $\lambda^+ \times A_\alpha$. Then $X = \bigcup\{A_\alpha: \alpha < \mathrm{cf}\, \lambda\}$ works for R.

LEMMA 3.6. *If* $\mu \geqq \kappa$, *then* $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$.

Proof. Define R: $\kappa \times \lambda \to \mu$ by $R(x, y) = x$.

LEMMA 3.7. *If there is a family* $F \subseteq [\kappa]^\nu$ *covering all subsets of* κ *of power* ν, (*that is, if* $A \in [\kappa]^\nu$, *then there is* $B \in F$ *with* $A \subseteq B$), *then the relation* $\mathrm{BP}\#(\kappa, |F|, u, \nu)$ *holds*.

Proof. Let R: $\kappa \times F \to \mu$ be any coloring with the property that for each $B \in F$, R restricted to $B \times \{B\}$ is one-to-one.

Given a disjoint family of sets $\{A_\alpha: \alpha < \rho\}$, a transversal of the family is a set B such that for every $\alpha < \rho$, $|B \cap A_\alpha| = 1$. Two transversals are almost disjoint if their intersection has cardinality $< \rho$.

LEMMA 3.8. (TRANSVERSAL LEMMA) *Let* ρ *be a cardinal, let* $\{A_\alpha: \alpha < \rho\}$ *be a disjoint family of sets, and let* D *be a family of almost disjoint transversals. Let* $\mu < \mathrm{cf}\, \rho$, *and let* $F \subseteq \bigcup_{\alpha<\rho}[A_\alpha]^\mu$ *be a collection of sets so that every member of* $\bigcup_{\alpha<\rho}[A_\alpha]^\nu$ *is a subset of some member of* F. *Then* $\mathrm{BP}\#(|D|, |F|, \mu, \nu)$.

Proof. For each $x \in F$, let f_x: $x \to \mu$ be a bijection. Define R: $D \times F \to \mu$ so that $R(B, x) = f_x(x \cap B)$ if $x \cap B \neq \emptyset$.

Suppose $X \in [D]^\nu$. Choose α so that if $B, C \in X$ and $B \neq C$, then $B \cap A_\alpha \neq C \cap A_\alpha$. Let

$$x = \bigcup\{B \cap A_\alpha: B \in X\},$$

and choose $y \in F$ so that $x \subseteq y$. Then for all $B, C \in X$, if $B \neq C$, then $R(B, y) \neq R(C, y)$.

COROLLARY 3.9. *For all* n *with* $0 < n < \omega$, *the relation* $\mathrm{BP}\#(\mu^{+(n+1)}, \mu^{+(n)}, \mu, \mu)$ *holds*.

This corollary is derived using two lemmas which are proved by induction.

LEMMA 3.10. *For all $n < \omega$, there is a covering family $F \subseteq [\mu^{+(n)}]^{\mu}$ of power $\mu^{+(n)}$ so that for all $x \in [\mu^{+(n)}]^{\mu}$, there is $y \in F$ with $x \subseteq y$.*

LEMMA 3.11. *If for all $\alpha < \lambda$, $|A_\alpha| = \lambda$, then there is a family D of almost disjoint transversals with $|D| = \lambda^+$.*

Lemma 3.11 is proved in [**3**, Lemma 4.1].

COROLLARY 3.12. *If $2^{\aleph_0} < \aleph_{\omega_1}$ and $2^{\aleph_0} < 2^{\aleph_1}$, then for all $\lambda \leqq 2^{\aleph_1}$,* $\mathrm{BP}\#(\lambda, \aleph_1, \aleph_0, \aleph_0)$.

Proof. Start with a disjoint family of $\aleph_1$ sets each of power $\aleph_1$. Lemma 3.10 guarantees the existence of the required covering family. To obtain the required set of transversals, employ the techniques of [**1**] which were used there to construct almost disjoint families of subsets of a given set.

COROLLARY 3.13. *For any cardinals σ, τ, ν, if $\nu <$ cf τ, then the relation* $\mathrm{BP}\#(\sigma^\tau, \tau, \sigma^{\underset{\smile}{\tau}}, \nu)$ *holds.*

Proof. Let T be a complete σ-branching tree of height τ. Then $|T| = \sigma^{\underset{\smile}{\tau}}$, and $|B| = \sigma^\tau$ where B is the set of branches of T of length τ. Define R: $B \times \tau \to T$ by $R(f, \alpha) = f|\alpha = f$ restricted to α. Suppose $X \subseteq B$ and $|X| = \nu$. Then for some α, if f and g are in X and $f \neq g$, then $f(\alpha) \neq g(\alpha)$. So R restricted to $X \times \{\alpha + 1\}$ is one-to-one.

COROLLARY 3.14. (GCH) *For all $\nu <$ cf μ, the relation* $\mathrm{BP}\#(\mu^+, \mathrm{cf}\,\mu, \mu, \nu)$ *holds.*

Proof. With GCH, $u^{\mathrm{cf}\mu} = \mu^+$ and $\mu^{\underset{\smile}{\mathrm{cf}\mu}} = \mu$.

COROLLARY 3.15. *Assume $\mu <$ cf τ. Then* $\mathrm{BP}\#(\sigma^\tau, \sigma^{\underset{\smile}{\tau}}, \mu, \nu)$.

Proof. For $\alpha < \tau$, let $A_\alpha = {}^\alpha\sigma$ be the collection of all functions from α into σ. Let D be the collection of branches of length τ through the tree $T = \bigcup_{\alpha<\tau} {}^\alpha\sigma$. Let $F = \bigcup_{\alpha<\tau} [{}^\alpha\sigma]^\mu$.

COROLLARY 3.16. (GCH) *For all $\mu <$ cf λ, the relation* $\mathrm{BP}\#(\lambda^+, \lambda, \mu, \nu)$ *holds.*

Proof. With GCH, $\lambda^+ = \lambda^{\mathrm{cf}\lambda}$ and $\lambda = \lambda^{\underset{\smile}{\mathrm{cf}\lambda}}$.

COROLLARY 3.17. *If λ is strongly inaccessible, then for all $\mu < \lambda$, the relation* $\mathrm{BP}\#(\lambda^\lambda, \lambda, \mu, \nu)$ *holds.*

Proof. Here $\lambda^{\underset{\smile}{\lambda}} = \lambda$.

LEMMA 3.18. *For all n with $0 < n < \omega$, if* $\mathrm{BP}\#(\kappa, \lambda, \mu^{+(n)}, \mu)$ *holds, then* $\mathrm{BP}\#(\kappa, \lambda \cdot \mu^{+(n)}, \mu, \mu)$ *holds.*

Proof. The proof is by induction on n. Suppose $\mathrm{BP}\#(\kappa, \lambda, \mu^{+(k+1)}, \mu)$ holds, and S: $\kappa \times \lambda \to \mu^{+(k+1)}$ attests to the fact. For each $\alpha < \mu^{+(k+1)}$, let f_α: $\alpha \to$

$\mu^{+(k)}$ be a one-to-one function. Define

$$R: \kappa \times (\lambda \times \mu^{+(k+1)}) \to \mu^{+(k)}$$

so that

$$R(\alpha, (\beta, \gamma)) = f_\gamma(S(\alpha, \beta)) \text{ if } S(\alpha, \beta) < \gamma.$$

Suppose $X \subseteq \kappa$ and $|X| = \mu$. Then for some $\beta < \lambda$, S is one-to-one on $X \times \{\beta\}$. Choose γ so large that $S(X \times \{\beta\}) \subseteq \gamma$. Then (β, γ) works for X and R.

LEMMA 3.19. *If* $\mathrm{BP}\#(\sigma, \tau, \mu, \nu)$ *and* $\mathrm{BP}\#(\kappa, \lambda, \sigma, \nu)$ *holds, then* $\mathrm{BP}\#(\kappa, \tau \cdot \lambda, \mu, \nu)$ *holds.*

Proof. Let $S: \sigma \times \tau \to \mu$ and $T: \kappa \times \lambda \to \sigma$ attest to $\mathrm{BP}\#(\sigma, \tau, \mu, \nu)$ and $\mathrm{BP}\#(\kappa, \lambda, \sigma, \nu)$ respectively. Define $R: \kappa \times (\lambda \times \tau) \to \mu$ by $R(\alpha, (\beta, \gamma)) = S(T(\alpha, \beta), \gamma)$.

Suppose $X \in [\kappa]^\nu$. Then there is $\beta < \lambda$ so that T is one-to-one on $X \times \{\beta\}$. So $T''X \times \{\beta\} \in [\sigma]^\nu$. Thus there is $\gamma < \tau$, so that S is one-to-one on $(T''X \times \{\beta\}) \times \{\gamma\}$. Therefore R is one-to-one on $X \times \{(\beta, \gamma)\}$.

The following corollary gives some insight into the uses of this lemma.

COROLLARY 3.20. *If* $\mathrm{BP}\#(2^{\aleph_0}, \aleph_1, \aleph_0, \aleph_0)$ *holds, then* $\mathrm{BP}\#(2^{\aleph_1}, \aleph_1, \aleph_0, \aleph_0)$ *also holds.*

Proof. From Corollary 3.13 to the Transversal Lemma, it follows that $\mathrm{BP}\#(2^{\aleph_1}, \aleph_1, 2^{\aleph_0}, \aleph_0)$ holds. Set

$$\sigma = 2^{\aleph_0}, \tau = \lambda = \aleph_1, \mu = \nu = \aleph_0, \text{ and } \kappa = 2^{\aleph_1},$$

to derive the above statement from the previous lemma.

LEMMA 3.21. *If ν is singular and for all $\nu' < \nu$, the relation* $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu^+)$ *holds, then the relation* $\mathrm{BP}\#(\kappa, \lambda^{\mathrm{cf}\,\nu}, \mu^{\mathrm{cf}\,\nu}, \nu)$ *holds.*

Proof. Let $\{\nu_\alpha: \alpha < \mathrm{cf}\,\nu\}$ be an increasing sequence cofinal in ν. For each $\alpha < \mathrm{cf}\,\nu$, let $S_\alpha: \kappa \times \lambda \to \mu$ be a coloring attesting to $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu_\alpha)$. Define $R: \kappa \times {}^{\mathrm{cf}\,\nu}\lambda \to {}^{\mathrm{cf}\,\nu}\mu$ by $R(\beta, f) = g$ where for all $\alpha < \mathrm{cf}\,\nu$, $g(\alpha) = S_\alpha(\beta, f(\alpha))$.

Suppose $X \in [\kappa]^\nu$. Express $X = \bigcup\{X_\alpha: \alpha < \mathrm{cf}\,\nu\}$ as the union of a chain of increasing sets where $|X_\alpha| = \nu_\alpha$. Let $f: \mathrm{cf}\,\nu \to \mu$ be a function so that for each $\alpha < \mathrm{cf}\,\nu$, the value $f(\alpha)$ attests to $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu_\alpha)$ for X_α and S_α. If β and γ are in X and $\beta \neq \gamma$, then choose α so large that β and γ are both in X_α. Since S_α restricted to $X_\alpha \times \{f(\alpha)\}$ is one-to-one, it follows that $R(\beta, f) \neq R(\gamma, f)$. Thus R is one-to-one on $X \times \{f\}$.

Now we use the lemmas already proved to give a characterization of the relation BP# under the assumption of GCH.

THEOREM 3.22. (GCH) *If κ is a limit cardinal, then* $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$ *holds if and only if* $\mu \geqq \kappa$ *or* $\lambda > \kappa$ *or* ($\lambda = \kappa$ *and* $\nu < \mathrm{cf}\ \kappa$).

Proof. If $\mu \geqq \kappa$, then Lemma 3.6 gives the desired coloring. So assume $\kappa > \mu$. If $\lambda > \kappa$, or $\lambda = \kappa$ and $\nu < \mathrm{cf}\ \kappa$, then GCH implies that $\lambda \geqq |[\kappa]^\nu|$, so Lemma 3.7 gives the desired coloring.

If $u < \kappa$ and $\lambda < \kappa$, then $\mu^\lambda < \kappa$, so by Lemma 3.3, the relation $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$ fails. So assume not only that $\mu < \kappa$, but also that $\lambda = \kappa$, and $\nu \geqq \mathrm{cf}\ \kappa$. If κ is regular, then our assumptions would give the contradiction $\nu \geqq \mathrm{cf}\ \kappa = \kappa > \mu \geqq \nu$. So we may assume κ is singular. In this case, by Lemma 3.2, the relation $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$ fails to hold. So the theorem follows.

THEOREM 3.23. (GCH) *If κ is a successor cardinal, $\kappa \geqq \mu$, $\mu \geqq \nu$, then* $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$ *holds if and only if one of the following conditions holds:*

(*a*) $\mu \geqq \kappa$,

(*b*) $\lambda \geqq \kappa$,

(*c*) $\kappa = \mu^+$ *and* $\lambda \geqq \mathrm{cf}\ \mu$ *and* $\nu < \mathrm{cf}\ \mu$,

(*d*) $\kappa = \lambda^+$ *and* $\nu < \mathrm{cf}\ \lambda$.

Proof. If $\mu \geqq \kappa$, then Lemma 3.6 gives the desired coloring. So assume $\mu < \kappa$. If $\lambda \geqq \kappa$, then $\lambda \geqq \kappa = \kappa^\nu = |[\kappa]^\nu|$, so Lemma 3.7 gives the desired coloring. So assume $\lambda < \kappa$.

If $\kappa > \lambda^+$ and $\kappa > \mu^+$, then $\kappa > \mu^\lambda$, so by Lemma 3.3, $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$ fails to hold. So assume either $\kappa = \mu^+$ or $\kappa = \lambda^+$.

First assume $\kappa = \mu^+$. If $\lambda < \mathrm{cf}\ \mu$, then $\kappa > \mu^\lambda$, and Lemma 3.3 gives the desired result. So assume $\lambda \geqq \mathrm{cf}\ \mu$. If $\nu < \mathrm{cf}\ \mu$, then Corollary 3.14 yields $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$. If $\nu \geqq \mathrm{cf}\ \mu$ and μ is regular, then $\mu^+ = \kappa > \lambda \geqq \mathrm{cf}\ \mu = \mu$, so $\nu \geqq \mathrm{cf}\ \mu = \mu = \lambda$ and Lemma 3.1 yields not $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$. If $\nu \geqq \mathrm{cf}\ \mu$ and μ is singular, then we shall show that not $\mathrm{BP}\#(\kappa, \lambda, \mu, \nu)$. Looking at the definition, we see that it is enough to show that $\mathrm{BP}\#(\mu^+, \mu, \mu, \mathrm{cf}\ \mu)$ fails. Since μ is singular, by Lemma 3.5, it suffices to show that for all $\tau < \mu$, $\mathrm{BP}\#(\mu^+, \tau, \mu, \mathrm{cf}\ \mu)$ fails to hold. If $\tau < \mu$, then $(\mathrm{cf}\ \mu)^\tau < \mu^+$, so by Lemma 3.4, to show that $\mathrm{BP}\#(\mu^+, \tau, \mu, \mathrm{cf}\ \mu)$ fails, it suffices to show for all $\rho < \mu$ with $\mathrm{cf}\ \mu \leqq \rho < \mu$, that $\mathrm{BP}\#(\mu^+, \tau, \rho, \mathrm{cf}\ \mu)$ fails. But if $\rho < \mu$ and $\tau < \mu$, then $\mu^+ > \rho^\tau$, so Lemma 3.3 yields the desired result.

Now assume $\kappa = \lambda^+$ and $\mu < \lambda$. If $\nu < \mathrm{cf}\ \lambda$, then by Corollary 3.16, $\mathrm{BP}\#(\lambda^+, \lambda, \nu, \nu)$ holds and $\mathrm{BP}\#(\lambda^+, \lambda, \mu, \nu)$ holds. If $\nu \geqq \mathrm{cf}\ \lambda$, then by Lemma 3.2, $\mathrm{BP}\#(\lambda^+, \lambda, \mu, \nu)$ fails

This completes the proof of the theorem.

4. BP. Recall that we write $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ if there is a coloring $R: \kappa \times \lambda \rightarrow \mu$ so that for every ν element set $X \in [\kappa]^\nu$, there is a point $x \in \lambda$ so that $|R''X \times \{x\}| \geqq \min(\mu, \nu)$. If $\nu > \kappa$, the relation makes no sense, so we assume that $\kappa \geqq \nu$. We discuss the relation first in general, and then under the assumption of GCH. Even under GCH, we do not have a complete characterization, but we do have a complete characterization if $V = L$.

LEMMA 4.1. (MONOTONICITY) (*a*) *Assume that* BP$(\kappa, \lambda, \mu, \nu)$ *holds, and that* $\kappa' \leqq \kappa$, $\lambda' \geqq \lambda$, $\mu' \geqq \mu$. *Assume also that if* $\nu > \mu$, *then* $\mu = \mu'$. *Then* BP$(\kappa', \lambda', \mu', \nu)$ *holds.*

(*b*) *If* BP$(\kappa, \lambda, \mu, \nu)$ *holds and* $\nu' \geqq \nu \geqq \mu$, *then* BP$(\kappa, \lambda, \mu, \nu')$ *holds.*

The above lemma and the following one follow straightforwardly from the definitions.

LEMMA 4.2. *If* BP#$(\kappa, \lambda, \mu, \nu)$ *holds, then for all* $\nu' \leqq \nu$, *the relation* BP $(\kappa, \lambda, \mu, \nu')$ *holds.*

Thus for BP, attention may be restricted to those cardinals for which the sharp relation is not settled positively.

LEMMA 4.3. (*a*) *If* $\mu^\lambda < \kappa$ *and either* κ *is regular or* $\nu < \kappa$, *then* BP$(\kappa, \lambda, \mu, \nu)$ *fails to hold.*

(*b*) *If* $\kappa > \mu^\lambda \geqq \mu > \operatorname{cf} \kappa$, *then* BP$(\kappa, \lambda, \mu, \nu)$ *fails to hold.*

(*c*) *If* $\kappa > \mu^\lambda$ *and* $\kappa > \operatorname{cf} \kappa \geqq \mu$, *then* BP$(\kappa, \lambda, \mu, \kappa)$ *holds if and only if* BP$(\operatorname{cf} \kappa, \lambda, \mu, \operatorname{cf} \kappa)$.

Proof. Let $R: \kappa \times \lambda \rightarrow \mu$ be a coloring. For each $x \in \kappa$, let $R_x: \lambda \rightarrow \mu$ be defined by $R_x(y) = R(x, y)$. There are only $\mu^\lambda < \kappa$ functions from λ into μ.

For part (*a*), select a set $X \in [\kappa]^\nu$ so that for all $x, y \in X$, $R_x = R_y$. Then X works for R.

For parts (*b*) and (*c*), express $\kappa = \bigcup_{\alpha < \operatorname{cf} \kappa} A_\alpha$ as the disjoint union of $\operatorname{cf} \kappa$ sets each of power a regular cardinal less than κ but greater than μ^λ. For each $\alpha < \operatorname{cf} \kappa$, select $X_\alpha \subseteq A_\alpha$ with $|X_\alpha| = |A_\alpha|$ so that for all $x, y \in X_\alpha$, $R_x = R_y$. Let $X = \bigcup_{\alpha < \operatorname{cf} \kappa} X_\alpha$. Then $|X| = \kappa$, and for all $x \in \lambda$, $|R''X \times \{x\}| \leqq \operatorname{cf} \kappa$. So (*b*) is proved.

We continue this argument to prove part (*c*). Using R restricted to $X \times \lambda$, define $S: \operatorname{cf} \kappa \times \lambda \rightarrow \mu$ by $S(\alpha, y) = R(x, y)$ for any $y \in X_\alpha$. A set $Y \subseteq \operatorname{cf} \kappa$ attesting to not BP$(\operatorname{cf} \kappa, \lambda, \mu, \operatorname{cf} \kappa)$ gives rise to a set $Z = \bigcup\{X_\alpha: \alpha \in Y\}$ attesting to not BP$(\kappa, \lambda, \mu, \kappa)$. So if BP$(\operatorname{cf} \kappa, \lambda, \mu, \operatorname{cf} \kappa)$ fails to hold, then also BP$(\kappa, \lambda, \mu, \nu)$ fails. Using similar arguments, one can show that a coloring $S: \operatorname{cf} \kappa \times \lambda \rightarrow \mu$ which attests to BP$(\operatorname{cf} \kappa, \lambda, \mu, \operatorname{cf} \kappa)$ gives rise to a coloring of $R: \kappa \times \lambda \rightarrow \mu$ by setting $R(x, y) = S(\alpha, y)$ for $x \in A_\alpha$, and this coloring attests to BP$(\kappa, \lambda, \mu, \kappa)$. So part (*c*) is proved.

LEMMA 4.4. *If* $\kappa > \operatorname{cf} \kappa$, *then* BP$(\kappa, \lambda, \operatorname{cf} \kappa, \kappa)$ *holds.*

Proof. Write $\kappa = \bigcup_{\alpha < \operatorname{cf} \kappa} A_\alpha$ as the disjoint union of $\operatorname{cf} \kappa$ sets each of power less than κ. Define $R: \kappa \times \lambda \rightarrow \operatorname{cf} \kappa$ by $R(x, y) = \alpha$ where $x \in A_\alpha$.

LEMMA 4.5. (*a*) *If* BP(σ, τ, μ, ν) *and* BP$(\kappa, \lambda, \sigma, \nu)$, *then* BP$(\kappa, \tau \cdot \lambda, \mu, \nu)$.

(*b*) *If* $\nu > \sigma$ *and* BP$(\sigma, \tau, \mu, \sigma)$ *and* BP$(\kappa, \lambda, \sigma, \nu)$, *then* BP$(\kappa, \tau \cdot \lambda, \mu, \nu)$.

The proof of this lemma is essentially the same as the proof of the analogous lemma for BP#, Lemma 3.19.

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LEMMA 4.6. *For all* μ *and* $\nu \leqq \mu^+$, *the relation* $\mathrm{BP}(\mu^+, \mu^+, \mu, \nu)$ *holds.*

Proof. Use Lemma 4.2, Corollary 3.9, and if $\nu > \mu$, also Lemma 4.1 (*b*).

LEMMA 4.7. *If* $\mathrm{BP}(\kappa, \lambda, \mu^+, \nu)$, *then* $\mathrm{BP}(\kappa, \lambda \cdot \mu^+, \mu, \nu)$.

Proof. By Lemma 4.6, $\mathrm{BP}(\mu^+, \mu^+, \mu, \rho)$ holds, where $\rho = \min(\nu, \mu^+)$. If $\mathrm{BP}(\kappa, \lambda, \mu^+, \nu)$ holds, then by Lemma 4.5, $\mathrm{BP}(\kappa, \lambda \cdot \mu^+, \mu, \nu)$ holds.

LEMMA 4.8. *For all* n *with* $0 < n < \omega$, *if* $\mathrm{BP}(\kappa, \lambda, \mu^{+(n)}, \mu)$ *holds, then also* $\mathrm{BP}(\kappa, \lambda \cdot \mu^{+(n)}, \mu, \mu)$ *holds.*

Proof. Use Lemma 4.7 and induction.

LEMMA 4.9. *If* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ *holds, then also* $\mathrm{BP}(\kappa^+, \lambda \cdot \kappa^+, \mu, \nu)$ *holds.*

Proof. By Lemma 4.6, we have $\mathrm{BP}(\kappa^+, \kappa^+, \kappa, \nu)$. If $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ holds, then by Lemma 4.5 (*a*), $\mathrm{BP}(\kappa^+, \lambda \cdot \kappa^+, \mu, \nu)$ holds.

LEMMA 4.10. *If* κ *is a limit cardinal,* $\mathrm{cf}\, \nu \neq \mathrm{cf}\, \kappa$ *and* $\{\rho: \mathrm{BP}(\rho, \lambda, \mu, \nu)\}$ *is cofinal in* κ, *then* $\mathrm{BP}(\kappa, \lambda \cdot \mathrm{cf}\, \kappa, \mu, \nu)$.

Proof. If $\mu \geqq \kappa$, then by Lemmas 4.2 and 3.6, $\mathrm{BP}(\kappa, \lambda \cdot \mathrm{cf}\, \kappa, \mu, \nu)$ holds. So assume $\kappa > \mu$. Since $\kappa \geqq \nu$ and $\mathrm{cf}\, \kappa \neq \mathrm{cf}\, \nu$, we have $\kappa > \nu$. Express $\kappa = \bigcup_{\alpha < \mathrm{cf}\, \kappa} A_\alpha$ as the union of a chain of nested sets where $|A_0| > \mu, \nu$, and for each $\alpha < \mathrm{cf}\, \kappa$, $\mathrm{BP}(|A_\alpha|, \lambda, \mu, \nu)$ holds. For each α, let $R_\alpha: A_\alpha \times (\lambda \times \{\alpha\}) \to \mu$ be a function attesting to $\mathrm{BP}(|A_\alpha|, \lambda, \mu, \nu)$. Then any extension of $\bigcup_{\alpha < \mathrm{cf}\, \kappa} R_\alpha$ to a function from $\kappa \times (\lambda \times \mathrm{cf}\, \kappa)$ into μ attests to $\mathrm{BP}(\kappa, \lambda \cdot \mathrm{cf}\, \kappa, \mu, \nu)$.

LEMMA 4.11. *If* μ *is regular and* $\mu \leqq \kappa$, *then* $\mathrm{BP}(\kappa, \kappa, \mu, \mu)$ *holds.*

Proof. The proof proceeds by induction on κ. $\mathrm{BP}(\mu, \mu, \mu, \mu)$ holds by Lemmas 4.2 and 3.6. If $\mathrm{BP}(\lambda, \lambda, \mu, \mu)$ and $\kappa = \lambda^+$, then $\mathrm{BP}(\kappa, \kappa, \mu, \mu)$ holds by Lemma 4.9. If κ is a limit and $\mathrm{cf}\, \kappa \neq \mathrm{cf}\, \mu$, then $\mathrm{BP}(\kappa, \kappa, \mu, \mu)$ holds by Lemma 4.10. So suppose κ is a limit cardinal, $\mathrm{cf}\, \kappa = \mathrm{cf}\, \mu$ and for all λ with $\kappa > \lambda \geqq \mu$, $\mathrm{BP}(\lambda, \lambda, \mu, \mu)$ holds. Let $\{\lambda_\alpha: \alpha < \mathrm{cf}\, \kappa\}$ be an increasing sequence of cardinals cofinal in κ with $\lambda_0 \geqq \mu$. For each α with $0 < \alpha < \mathrm{cf}\, \kappa$, let $R_\alpha: \lambda_\alpha \times (\lambda_\alpha \times \{\alpha\}) \to \mu$ attest to $\mathrm{BP}(\lambda_\alpha, \lambda_\alpha, \mu, \mu)$. Let $A_0 = \lambda_0$, and for $\alpha > 0$, $A_\alpha = \lambda_\alpha - \bigcup_{\beta < \alpha} \lambda_\beta$. Then $\kappa = \bigcup_{\alpha < \mathrm{cf}\, \kappa} A_\alpha$. Define $R_0: \kappa \times (\kappa \times \{0\}) \to \mu$ by $R_0(x, (y, 0)) = \alpha$ where $x \in A_\alpha$. Let $R: \kappa \times (\kappa \times \mathrm{cf}\, \kappa) \to \mu$ be any function which extends $\bigcup_{\alpha < \mathrm{cf}\, \kappa} R_\alpha$. Now suppose $X \in [\kappa]^\mu$. If $X \subseteq \lambda_\alpha$ for some $\alpha < \mathrm{cf}\, \kappa$, then using the induction hypothesis, we can find $(x, \alpha) \in \lambda_\alpha \times \{\alpha\}$ so that $|R_\alpha{}''X \times \{(x, \alpha)\}| \geqq \mu$. Then $|R''X \times \{(x, \alpha)\}| = \mu$. If X is not a subset of λ_α for any $\alpha < \mathrm{cf}\, \kappa$, then X is cofinal in κ, and $|R_0{}''X \times \{(0, 0)\}| = \mu$, so $|R''X \times \{(0, 0)\}| = \mu$. In either case, the lemma follows.

LEMMA 4.12. *If* $\mu < \kappa$, *then* $\mathrm{BP}\ (\kappa, \kappa, \mu, \mu^+)$ *holds.*

Proof. Use Lemmas 4.11 and 4.7.

LEMMA 4.13. *For any* κ *and* μ *with* $\mu \leqq \kappa$, $\mathrm{BP}(\kappa, \kappa^{\mathrm{cf}\,\mu}, \mu, \mu)$ *holds. Hence if* $\kappa \geqq \mu \geqq \nu$, $\mathrm{BP}(\kappa, \kappa^{\mathrm{cf}\,\nu}, \mu, \nu)$ *holds.*

Proof. The proof is by induction on κ using Lemmas 4.9 and 4.10. The only difficult case is $\mathrm{cf}\,\kappa = \mathrm{cf}\,\mu$. If $\kappa = \mu$ then $\mathrm{BP}(\kappa, \kappa^{\mathrm{cf}\,\mu}, \mu, \mu)$ holds by Lemma 3.6. It also holds if $\mu = \mathrm{cf}\,\mu$ by Lemmas 4.11 and 4.1. So suppose $\kappa > \mu > \mathrm{cf}\,\mu = \mathrm{cf}\,\kappa$ and that $\mathrm{BP}(\lambda, \lambda^{\mathrm{cf}\,\mu}, \mu, \mu)$ holds for $\mu \leqq \lambda < \kappa$. Let $\{\mu_\alpha : \alpha < \mathrm{cf}\,\mu\}$ be an increasing sequence of regular cardinals with limit μ.

Write $\kappa = \bigcup\{A_\alpha : \alpha < \mathrm{cf}\,\mu\}$ as the disjoint union of $\mathrm{cf}\,\mu$ sets with $\mu \leqq |A_\alpha| < \kappa$, and for $\alpha < \mathrm{cf}\,\mu$ put $B_\alpha = \bigcup_{\beta \leqq \alpha} A_\beta$. Also, write $\kappa^{\mathrm{cf}\,\mu} = \bigcup\{C_\alpha : \alpha < \mathrm{cf}\,\mu\}$ as the disjoint union of $\mathrm{cf}\,\mu$ sets each of power $\kappa^{\mathrm{cf}\,\mu}$. Let I denote the set of all ordered pairs, (f, g), of functions f, g with $f \in {}^{\mathrm{cf}\,\mu}\,\mathrm{cf}\,\kappa$, $g \in {}^{\mathrm{cf}\,\mu}\,\kappa$ and such that $g(\alpha) \in A_\alpha$ for all $\alpha < \mathrm{cf}\,\kappa$. Clearly $|I| \leqq \kappa^{\mathrm{cf}\,\mu}$ and we may assume without loss of generality that $I \subseteq C_0$.

Now for $0 < \alpha < \mathrm{cf}\,\mu$ it follows from the induction hypothesis that there is a coloring $S_\alpha : B_\alpha \times C_\alpha \to \mu$ which attests to $\mathrm{BP}(|B_\alpha|, |C_\alpha|, \mu, \mu)$. Also, by Lemma 4.11, $\mathrm{BP}(|A_\alpha|, |A_\alpha|, \mu_\beta, \mu_\beta)$ holds for $\alpha, \beta < \mathrm{cf}\,\mu$. Let $T_{\alpha\beta} : A_\alpha \times A_\alpha \to \mu_\beta$ be a coloring which attests to this fact. Now define a coloring $S_0 : \kappa \times I \to \mu$ by setting $S_0(x, (f, g)) = T_{\alpha f(\alpha)}(x, g(\alpha))$ for $x \in A_\alpha$ and $\alpha < \mathrm{cf}\,\mu$. The required coloring $R : \kappa \times \kappa^{\mathrm{cf}\,\kappa} \to \mu$ is any extension of $\bigcup_{\alpha < \mathrm{cf}\,\mu} S_\alpha$.

To see that P works, let $X \in [x]^\mu$. If for some α, $0 < \alpha < \mathrm{cf}\,\mu$, we have $|X \cap B_\alpha| = \mu$, then there is $y \in C_\alpha$ such that $|S_\alpha''(X \cap B_\alpha) \times \{y\}| = \mu$. Suppose $|X \cap B_\alpha| < \mu$ for all $\alpha < \mathrm{cf}\,\mu$. Then there is an increasing sequence $\{\alpha(\beta) : \beta < \mathrm{cf}\,\mu\}$ of ordinals less than $\mathrm{cf}\,\mu$ such that $|A_{\alpha(\beta)} \cap X| \geqq \mu_\beta$. Let f: $\mathrm{cf}\,\mu \to \mathrm{cf}\,\mu$ be any function which satisfies $f(\alpha(\beta)) = \beta$ $(\beta < \mathrm{cf}\,\mu)$. Choose g: $\mathrm{cf}\,\mu \to \kappa$ so that, for each $\beta < \mathrm{cf}\,\mu$, $g(\alpha(\beta)) \in A_{\alpha(\beta)}$ and

$$|T_{\alpha(\beta)\beta}''(A_{\alpha(\beta)} \cap X) \times \{g(\alpha(\beta))\}| \geqq \mu_\beta.$$

Then

$$|R'' X \times \{(f, g)\}| \geqq \lim \mu_\beta = \mu.$$

LEMMA 4.14. *Assume* $\lambda > \mathrm{cf}\,\lambda = \mathrm{cf}\,\nu$, $\mu \geqq \nu > \mathrm{cf}\,\nu$, *and for all* $\rho < \lambda$, $\mu^\rho < \kappa$. *Then* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ *fails to hold.*

Proof. Let $R : \kappa \times \lambda \to \mu$ be a coloring. Express $\lambda = \bigcup_{\alpha < \mathrm{cf}\,\lambda} A_\alpha$ as the union of a chain of nested sets each of power less than λ. Pick $\{\nu_\alpha : \alpha < \mathrm{cf}\,\nu\}$ a sequence of cardinals cofinal in ν. For each $\alpha < \mathrm{cf}\,\lambda$, since if $\rho = |A_\alpha|$, then $\rho < \lambda$, so $\mu^\rho < \kappa$. So there are fewer than κ functions from A_α into μ. Choose a set $X_\alpha \subset \kappa$ of power ν_α so that for every $x, y \in X_\alpha$, R restricted to $\{x\} \times A_\alpha$ and R restricted to $\{y\} \times A_\alpha$ are the same function. Then $X = \bigcup_{\alpha < \mathrm{cf}\,\lambda} X_\alpha$ works for R.

THEOREM 4.15. (GCH). *Assume* $\lambda \geqq \kappa > \mu$. *Then* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ *holds if and only if it is not true that* $\lambda = \kappa$ *and* $\mu \geqq \nu > \mathrm{cf}\,\kappa = \mathrm{cf}\,\nu$.

Proof. If $\lambda > \kappa$, then $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ holds by Lemmas 4.1 and 4.13. So suppose $\lambda = \kappa$. If $\nu > \mu$, then by Lemma 4.12, $\mathrm{BP}(\kappa, \kappa, \mu, \mu^+)$ holds, so $\mathrm{BP}(\kappa, \kappa, \mu, \nu)$ holds. So suppose $\nu \leqq \mu$. If $\mathrm{cf}\ \nu \neq \mathrm{cf}\ \kappa$, then an easy induction on κ using Lemmas 4.6, 4.9, 4.10, and 4.13 shows $\mathrm{BP}(\kappa, \kappa, \nu, \nu)$, so $\mathrm{BP}(\kappa, \kappa, \mu, \nu)$ holds. So suppose $\mathrm{cf}\ \nu = \mathrm{cf}\ \kappa$. If $\nu = \mathrm{cf}\ \kappa$, then by Lemma 4.11, $\mathrm{BP}(\kappa, \kappa, \mathrm{cf}\ \kappa, \mathrm{cf}\ \kappa)$ holds, so $\mathrm{BP}(\kappa, \kappa, \mu, \nu)$ holds. The only remaining case is $\mu \geqq \nu > \mathrm{cf}\ \kappa = \mathrm{cf}\ \nu$, and in this case $\mathrm{BP}(\kappa, \kappa, \mu, \nu)$ fails by Lemma 4.14.

LEMMA 4.16. (*a*) *If* $\kappa > \mu^{\underset{\smile}{\lambda}}$ *and* $\mu \geqq \nu \geqq \mathrm{cf}\ \nu = \mathrm{cf}\ \lambda$, *then not* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$.

(*b*) *If* μ *is singular*, $\mu \geqq \nu$, $\mathrm{cf}\ \lambda = \mathrm{cf}\ \mu = \mathrm{cf}\ \nu$, κ *is regular, and* $\sigma^\tau < \kappa$ *whenever* $\sigma < \mu$, $\tau < \lambda$, *then not* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$.

Proof. First we prove part (*a*). Let $\{\tau_\alpha: \alpha < \mathrm{cf}\ \lambda\}$ be cofinal in λ. Let $R: \kappa \times \lambda \rightarrow \mu$ be a coloring. For $\alpha \in \kappa$ and $\beta \in \mathrm{cf}\ \lambda$, define $g_\alpha(\beta) \in {}^{\tau_\beta}\mu$ by $g_\alpha(\beta)(\gamma) = R(\alpha, \gamma)$. There are fewer than $(\mu^{\underset{\smile}{\lambda}})^+$ functions of the form $g_\alpha|\beta$. Since $\kappa \geqq (\mu^{\underset{\smile}{\lambda}})^+$, there is $\alpha \in \kappa$ such that for all $\beta < \mathrm{cf}\ \lambda$,

$$|\{\alpha' \in \kappa: g_{\alpha'}|\beta = g_\alpha|\beta\}| \geqq (\mu^{\underset{\smile}{\lambda}})^+.$$

Let $\{\nu_\beta: \beta < \mathrm{cf}\ \lambda\}$ be cofinal in ν if ν is singular; otherwise let $\nu_\beta = 1$ for all β. Choose $A_\beta \subseteq \kappa$ for each $\beta < \mathrm{cf}\ \lambda$ so that $|A_\beta| = \nu_\beta$ and for all $\alpha' \in A_\beta$ $g_{\alpha'}|\beta = g_\alpha|\beta$. Then $X = \bigcup_{\beta < \mathrm{cf}\lambda} A_\beta$ works.

Next we prove part (*b*). The proof is analogous to the proof of part (*a*). Let $\{\sigma_\alpha: \alpha < \mathrm{cf}\ \lambda\}$ and $\{\tau_\alpha: \alpha < \mathrm{cf}\ \lambda\}$ be cofinal in μ and λ respectively. Let $R: \kappa \times \lambda \rightarrow \mu$ be a coloring. For each $\alpha \in \kappa$ and $\beta \in \mathrm{cf}\ \lambda$, define $g_\alpha(\beta) \in {}^{(\tau_\beta)}(\sigma_\beta)$ by setting $g\alpha(\beta)(\gamma) = R(\alpha, \gamma)$ if $R(\alpha, \gamma) < \sigma_\beta$, and setting $g_\alpha(\beta)(\gamma) = 0$ otherwise. Since κ is regular and for all $\sigma < \mu$, $\tau < \lambda$, we have $\kappa > \sigma^\tau$, there are fewer than κ functions of the form $g_\alpha|\beta$. Hence there is $\alpha \in \kappa$ such that for all $\beta < \mathrm{cf}\ \lambda$,

$$|\{\alpha' \in \kappa: g_{\alpha'}|\beta = g_\alpha|\beta\}| = \kappa.$$

Letting $\{\nu_\beta: \beta < \mathrm{cf}\ \lambda\}$ be as in part (*a*), choose $A_\beta \subseteq \kappa$ for each $\beta < \mathrm{cf}\ \lambda$ so that $|A_\beta| = \nu_\beta$ and for all $\alpha' \in A_\beta$, $g_\alpha'|\beta = g_\alpha|\beta$. Then $\bigcup_{\beta < \mathrm{cf}\lambda} A_\beta$ works.

LEMMA 4.17. *If* $2^\lambda = \lambda^+$ *and* $\mu \leqq \lambda^+$, *then* $\mathrm{BP}(\lambda^+, \lambda, \mu, \lambda^+)$.

Proof. The proof follows from Lemma 14.1 (p. 222) of [**4**] which says the following:

There is a function $f: [\lambda^+]^2 \rightarrow \lambda^+$ so that whenever $X, Y \subseteq \lambda^+$ with $|X| = \lambda$ and $|Y| = \lambda^+$, then there is $x \in X$ so that the edges between x and members of Y receive all λ^+ colors.

LEMMA 4.18. *Suppose* μ *is singular*.

(*a*) *If* κ *is regular and for all* $\rho < \mu$, $\rho^\lambda < \kappa$, *then for all* $\nu > \mathrm{cf}\ \mu$ *with* $\mathrm{cf}\ \nu = \mathrm{cf}\ \mu$, *not* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$.

(*b*) *If* $\kappa \geqq \nu > \mu$ *and* $\mathrm{cf}\ \nu > (\mathrm{cf}\ \mu)^\lambda$, *then not* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$.

(*c*) *Suppose* $\mu \geqq \lambda \geqq \mathrm{cf}\ \mu$, $\nu \geqq \mathrm{cf}\ \mu$ *and either*

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(*i*) *for all* $\rho < \mu$, $\rho^\lambda \leqq \mu$ *or*

(*ii*) $\lambda = \mu$, $\rho^\sigma \leqq \mu$ *for all* $\rho < \mu$ *and* $\sigma < \lambda$, $\nu < \mu$, *and* $\operatorname{cf} \nu \neq \operatorname{cf} \mu$.

Then $\mathrm{BP}(\mu^+, \lambda, \mu, \nu)$ *if and only if* $\mathrm{BP}(\mu^+, \operatorname{cf} \mu, \mu, \nu)$.

Proof. First we prove part (*a*). Let $R: \kappa \times \lambda \to \mu$ be a coloring. Divide $\mu = \bigcup_{\alpha < \operatorname{cf}\mu} A_\alpha$ into the disjoint union of cf μ sets each of power less than μ. Let $\{\nu_\alpha: \alpha < \operatorname{cf} \mu\}$ be a sequence of cardinals cofinal in ν. Define $\hat{R}: \kappa \times \lambda \to \operatorname{cf} \mu$ by $\hat{R}(x, y) = \alpha$ where $R(x, y) \in A_\alpha$. Since $(\operatorname{cf} \mu)^\lambda < \kappa$, we can find $U \subseteq \kappa$, a set of power κ, and $S: \lambda \to \operatorname{cf} \mu$, a function, so that for all $x \in U$ and $y \in \lambda$, $\hat{R}(x, y) = S(y)$. Now S induces a partition of λ, $\lambda = \bigcup_{\alpha<\operatorname{cf}\mu} B_\alpha$. For each $x \in U$, let $R_x: \lambda \to \mu$ be defined by $R_x(y) = R(x, y)$. For $u \in U$, the function R_u maps $\bigcup_{\beta \leqq \alpha} B_\beta$ into $\bigcup_{\beta\leqq\alpha} A_\beta$. There are at most $\rho^\lambda < \kappa$ functions from $\bigcup_{\beta\leqq\alpha} B_\beta$ into $\bigcup_{\beta\leqq\alpha} A_\beta$ where $\rho = |\bigcup_{\beta\leqq\alpha} A_\beta| < \mu$. So for each $\alpha < \operatorname{cf} \mu$, choose $X_\alpha \subseteq U$ so that for all $u, u' \in X_\alpha$,

$$R_u | \bigcup_{\beta\leqq\alpha} B_\beta = R_{u'} | \bigcup_{\beta\leqq\alpha} B_\beta.$$

Then $X = \bigcup_{\alpha<\operatorname{cf}\mu} X_\alpha$ works for R.

Next we prove part (*b*). Let $R: \kappa \times \lambda \to \mu$ be a coloring. Let $\{\mu_\alpha: \alpha < \operatorname{cf} \mu\}$ be cofinal in μ, and for $\alpha \in \kappa$, define $g_\alpha: \lambda \to \operatorname{cf} \mu$ by $g_\alpha(\beta) =$ least γ such that $R(\alpha, \beta) < \mu_\gamma$. Choose $X \in [\kappa]^\nu$ so that for all $\alpha, \beta \in X$, the functions g_α and g_β are equal, $g_\alpha = g_\beta$. Then X works.

Next we prove part (*c*) (*i*). Lemma 4.1 guarantees that if $\mathrm{BP}(\mu^+, \operatorname{cf} \mu, \mu . \nu)$, then $\mathrm{BP}(\mu^+, \lambda, \mu, \nu)$. So assume $\mathrm{BP}(\mu^+, \lambda, \mu, \nu)$ holds, and let $R: \mu^+ \times \lambda \to \mu$ attest to the fact. Let $\{\mu_\alpha: \alpha < \operatorname{cf} \mu\}$ be cofinal in μ. Define $\hat{R}: \mu^+ \times \lambda \to \operatorname{cf} \mu$ by $\hat{R}(x, y) =$ the least γ with $R(x, y) < \mu_\gamma$. Since $(\operatorname{cf} \mu)^\lambda \leqq \mu < \mu^+$, there is a set $U \subseteq \mu^+$ of cardinality μ^+, and a function $f: \lambda \to \operatorname{cf} \mu$ so that for all $u \in U$, for all $y \in \lambda$, $\hat{R}(u, y) = f(y)$. Then f induces a partition of $\lambda = \bigcup_{\alpha<\operatorname{cf}\mu} L_\alpha$ where $f''L_\alpha = \{\alpha\}$. Let

$$I = \bigcup_{\alpha<\operatorname{cf}\mu} {}^{(L\alpha)}\mu_{f(\alpha)}.$$

Since for all $\alpha < \operatorname{cf} \mu$, the set ${}^{(L\alpha)}\mu_{f(\alpha)}$ has power $|\mu_{f(\alpha)}|^{|L\alpha|} \leqq \mu$, the set I has power $\leqq \mu$. Define $S: U \times \operatorname{cf} \mu \to I$ by

$$S(x, \alpha) = h \in {}^{(L\alpha)}\mu_{f(\alpha)},$$

where $h(y) = R(x, y)$. Now S works. For suppose $X \in [U]^\nu$. Let $y \in \lambda$ be such that $R''X \times \{y\}$ has cardinality at least $\min(\mu, \nu)$. Now $y \in L_\alpha$ for some α, so $S''X \times \{\alpha\}$ has cardinality at least $\min(\mu, \nu)$.

Finally we prove part (*c*) (*ii*). As in the previous part, only one direction presents any difficulty to prove. Here $\lambda = \mu$. So suppose $\mathrm{BP}(\mu^+, \mu, \mu, \nu)$ holds, and let $R: \mu^+ \times \mu \to \mu$ be a witness. Define S on $\mu^+ \times \operatorname{cf} \mu$ by

$$S(x, y) = h \in {}^{(\mu_y)}(\mu_y)$$

where $h(\gamma) = R(x, y)$ if $R(x, y) < \mu_y$ and $h(\gamma) = 0$ otherwise. Then S is a coloring showing $\mathrm{BP}(\mu^+, \operatorname{cf} \mu, \mu, \nu)$. For if $X \in [\mu^+]^\nu$, then there is $\gamma < \lambda$ such that $|\{R(\alpha, \gamma): \alpha \in X\}| = \nu$. Since cf $\nu \neq$ cf μ, there is β such that $|\{R(\alpha, \gamma) < \mu_\beta: \alpha \in X\}| = \nu$ and $\gamma < \mu_\beta$. But then $|\{S(\alpha, \beta): \alpha \in X\}| = \nu$.

LEMMA 4.19. (TRANSVERSAL LEMMA) *Let $\{A_i: i \in I\}$ be a family of disjoint sets. Let T be a set of transversals of $\{A_i: i \in I\}$ such that for every $X \in [T]^\nu$ there is $i \in I$ such that $|\{t \cap A_i: t \in X\}| \geqq \rho = \min(\mu, \nu)$, Let $C \subseteq \bigcup_{i \in I}[A_i]^{\leqq \mu}$ be such that for all $i \in I$, for all $x \in [A_i]^\rho$, there is $y \in C$ with $|x \cap y| = \rho$. Then* $\mathrm{BP}(|T|, |C|, \mu, \nu)$.

Proof. For each $y \in C$, let $f_y: y \to \mu$ be a one-to-one function. Let R: $T \times C \to \mu$ be any function with $R(t, y) = f_y(x)$ whenever $t \cap y = \{x\} \neq \emptyset$. Then R attests to BP $(|T|, |C|, \mu, \nu)$. For suppose $X \in [T]^\nu$. Find $i \in I$ so that

$$|\{t \cap A_i: t \in X\}| \geqq \rho.$$

Choose $y \in C$ so that

$$|\{x \in y: x \in t \cap A_i\}| \geqq \rho.$$

Then $|R''X \times \{y\}| \geqq \min(\mu, \nu)$.

LEMMA 4.20. (TREE EQUIVALENCE). *Assume $\mu^{\lambda} = \max(\mu, \lambda)$ and $\kappa = (\mu^{\lambda})^+$. Then* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ *if and only if there is a $\leqq \mu$-branching tree T of height λ and a set B of branches of length λ such that $|B| = \kappa$ and for all $B' \in [B]^\nu$, there is some $\alpha < \lambda$ such that $\{t \in T$: level $(t) = \alpha$ and t occurs in some element of $B'\}$ has cardinality* $\geqq \min^{\bullet}(\mu, \nu)$.

Proof. ($\Rightarrow$) Suppose $R: \kappa \times \lambda \to \mu$ witnesses $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$. For each $\alpha \in \kappa$, let $R_\alpha: \lambda \to \mu$ be defined by $R_\alpha(\beta) = R(\alpha, \beta)$. Let $T = \{R_\alpha|\beta: \alpha \in \kappa, \beta \in \lambda\}$, ordered by inclusion. Then T is $\leqq \mu$-branching and of height λ. Let $B = \{\{R_\alpha|\beta: \beta \in \lambda\}: \alpha \in \kappa\}$. B is certainly a family of κ branches of length λ. If $B' \in [B]^\nu$, then $X = \{\alpha: \{R_\alpha|\beta: \beta \in \lambda\} \in B'\}$ has power at least ν. Find $y \in \lambda$ so that $|R''X \times \{y\}| \geqq \min(\mu, \nu)$. Choose $\beta \geqq y$. Then

$$|\{R_\alpha|\beta: \alpha \in X\}| \geqq \min(\mu, \nu),$$

so

$$\{t \in T: \text{level}(t) = \beta \text{ and } t \text{ occurs in some element of } B'\}$$

has cardinality $\geqq \min(\mu, \nu)$. So B works.

($\Leftarrow$) We use Lemma 4.19. The αth level of the tree is A_α. Each branch is a transversal. All we need exhibit is C of cardinality λ. If $\mu \geqq \lambda$, then

$$|A_\alpha| \leqq |{}^\alpha\mu| \leqq \mu^{\lambda} = \mu,$$

so that

$$C = \{A_\alpha: \alpha \in \lambda\}.$$

Suppose $\mu < \lambda$. Let $\rho = \min(\mu, \nu)$. Then

$$|[A_\alpha]^\rho| \leqq |{}^\alpha\mu|^\rho = \mu^{|\alpha| \cdot \rho} \leqq \mu^{\lambda} = \lambda,$$

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so let

$$C = \bigcup_{\alpha\in\lambda}[A_\alpha]^\rho.$$

LEMMA 4.21. (GCH) *Assume* $\mu^{\underset{\smile}{\lambda}} < \mu^\lambda = \kappa$, $\nu < \kappa$, $\mathrm{cf}\,\nu \neq \mathrm{cf}\,\lambda$ *and* $\mathrm{cf}\,\nu^- \neq \mathrm{cf}\,\lambda$. *Then* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ *holds.*

Proof. Our first proof used a theorem of E. C. Milner [**6**] which characterizes the cardinals possible for families of almost disjoint transversals. We give a direct proof.

We wish to apply Lemma 4.20. Let the tree T be the set $\bigcup_{\alpha<\lambda}{}^\alpha\mu$, ordered by inclusion, and let the set of branches be

$$B = \{\{f|\beta\colon \beta \in \lambda\}\colon f \in {}^\lambda\mu\}.$$

Everything is clear except the assertion about $B' \in [B]^\nu$. Let

$$F = \{f \in {}^\lambda\mu\colon \{f|\beta\colon \beta \in \lambda\} \in B'\} = \{\bigcup b\colon b \in B'\}.$$

Then $|B'| = |F|$. For each $f, g \in F$ with $f \neq g$, let $0(f, g)$ be the least γ with $f(\gamma) \neq g(\gamma)$. Note that if $\delta \geqq 0(f, g)$, then $f(\delta) \neq g(\delta)$. If $\nu < \mathrm{cf}\,\lambda$, then

$$\alpha = \sup\{0(f, g)\colon f, g \in F \text{ and } f \neq g\}$$

shows B' satisfies the conditions of Lemma 4.20. Thus we may assume $\nu \geqq \mathrm{cf}\,\lambda$. Since $\mathrm{cf}\,\nu \neq \mathrm{cf}\,\lambda$, this inequality must be strict, namely we are assuming $\nu > \mathrm{cf}\,\lambda$. Let $A \subseteq \lambda$ be a cofinal set of power $\mathrm{cf}\,\lambda$. Let

$$B'' = \{\{f|\alpha\colon \alpha \in A\}\colon f \in F\}.$$

Since each branch $\{f|\alpha\colon \alpha \in \lambda\}$ of T is uniquely determined by $\{f|\alpha\colon \alpha \in A\}$, we know that $|B''| = |B'| = \nu$. For each $\alpha \in \lambda$ let

$$\lambda_\alpha = |\{f|\alpha\colon f \in F\}|.$$

Since T is a tree, $\alpha < \beta$ implies $\lambda_\alpha \leqq \lambda_\beta$. If some $\lambda_\alpha = \nu$, we are done, so suppose not, that is, suppose $\lambda_\alpha < \nu$ for all $\alpha \in A$. Since $\mathrm{cf}\,\nu \neq \mathrm{cf}\,\lambda$, $\sup\lambda_\alpha < \nu$. Now

$$\nu = |B'| = |B''| \leqq \prod_{\alpha\in A}\lambda_\alpha \leqq (\sup\lambda_\alpha)^{\mathrm{cf}\lambda} = \max((\mathrm{cf}\,\lambda)^+, (\sup\lambda_\alpha)^+).$$

Since we have assumed $\mathrm{cf}\,\lambda < \nu$ and $(\sup\lambda_\alpha) < \nu$, we may conclude that

$$\nu = \max((\mathrm{cf}\,\lambda)^+, (\sup\lambda_\alpha)^+).$$

Since $\nu = (\mathrm{cf}\,\lambda)^+$ is ruled out by the hypothesis $\mathrm{cf}\,\nu^- \neq \mathrm{cf}\,\lambda$, we must have $\nu = (\sup\lambda_\alpha)^+$. Also $\mathrm{cf}(\sup\lambda_\alpha) \neq \mathrm{cf}\,\lambda$. Hence there are $\alpha < \lambda$ and β so that for all $\beta \geqq \alpha$, $\lambda_\beta = \rho$. So $\nu = \rho^+$ and $\rho^{\mathrm{cf}\lambda} = \nu$. So $\mathrm{cf}\,\rho < \mathrm{cf}\,\lambda < \rho$. Let T' be the tree

$$\{f|\beta\colon f \in F \text{ and } \beta \in A\}$$

ordered by inclusion. Then

$$|T'| = |\bigcup\{\{f|\beta\colon f \in F\}\colon \beta \in A\}| = \sum_{\alpha\in A}\lambda_\alpha = \rho.$$

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Enumerate the elements of T' in order type ρ, $T' = \{t_\xi: \xi < \rho\}$. Now B'' is a set of branches through T', so for each $b \in B''$, let ξ_b be the least $\xi < \rho$ such that $b \cap \{t_\eta: \eta < \xi\}$ is cofinal in b. Since $|B''| = \nu = \rho^+ > \rho$, we can find ξ so that

$$|\{b \in B'': \xi_b = \xi\}| = \nu.$$

But since $2^{|\xi|} \leqq \rho < \nu$ and $|B''| = \nu$, there must be b and b' in B'' such that $b \neq b'$; but b and b' have identical cofinal subsets, a contradiction.

THEOREM 4.22. (GCH). *Assume* $\kappa > \lambda, \mu, \kappa \leqq \mu^\lambda$ (*so* $\kappa = \max(\mu^+, \lambda^+)$).

(*a*) *Assume* $\mu^{\underset{\smile}{\lambda}} \geqq \kappa$.

(*i*) *If* $\lambda \geqq \mu$ *and* $\nu = \kappa$ *then* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$.

(*ii*) *if* $\lambda \geqq \mu$ *and* $\mathrm{cf}\ \nu = \mathrm{cf}\ \mu$ *then not* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$.

(*iii*) *in all other cases*, $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ *if and only if* $\mathrm{BP}(\kappa, \mathrm{cf}\ \mu, \mu, \nu)$.

(*b*) *Assume* $\mu^{\underset{\smile}{\lambda}} < \kappa$.

(*i*) *if* $\mathrm{cf}\ \nu \neq \mathrm{cf}\ \lambda$ *and either* ($\nu = \kappa$ *and* $\mu \leqq \lambda$) *or* ($\nu < \kappa$ *and* $\mathrm{cf}\ \nu^- \neq \mathrm{cf}\ \lambda$), *then* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$.

(*ii*) *if* $\mathrm{cf}\ \nu = \mathrm{cf}\ \lambda$ *or if* $\mathrm{cf}\ \nu \neq \mathrm{cf}\ \lambda, \nu = \kappa$ *and* $\lambda < \mu$, *then not* $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$.

(*iii*) *otherwise, i.e., if* $\mathrm{cf}\ \nu \neq \mathrm{cf}\ \lambda$, $\nu < \kappa$ *and* $\mathrm{cf}\ \nu^- = \mathrm{cf}\ \lambda$, *then assuming* $V = L$, $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ *holds.*

Proof. (*a*) Note that $\mu^{\underset{\smile}{\lambda}} \geqq \kappa$ implies $\kappa = \mu^+$ and $\lambda \geqq (\mathrm{cf}\ \mu)^+$ so μ is singular.

(*i*) holds by Lemma 4.17.

(*ii*) holds by Lemma 4.16 (*b*).

If $\nu \geqq \mathrm{cf}\ \mu$ then (*iii*) holds by Lemma 4.18(*c*). If $\nu < \mathrm{cf}\ \mu$ then $\mathrm{BP}(\kappa, \mathrm{cf}\ \mu, \mu, \nu)$ holds by Lemma 4.21. See case (*b*) (*i*) below. (Note that this case is reduced to part (*b*).)

(*b*) If $\mu^{\underset{\smile}{\lambda}} < \kappa$, then either $\kappa = \mu^+$ and $\lambda = \mathrm{cf}\ \mu$, or $\kappa = \lambda^+$.

(*i*) Suppose $\mathrm{cf}\ \nu \neq \mathrm{cf}\ \lambda$, $\nu = \kappa$, and $\mu \leqq \lambda$. Then $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ by Lemma 4.17. Now suppose $\mathrm{cf}\ \nu \neq \mathrm{cf}\ \lambda$, $\nu < \kappa$, $\mathrm{cf}\ \nu^- \neq \mathrm{cf}\ \lambda$. Then $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ by Lemma 4.21.

(*ii*) Assume $\mathrm{cf}\ \nu = \mathrm{cf}\ \lambda$. Then not $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ by Lemma 4.16(*a*). Assume $\mathrm{cf}\ \nu \neq \mathrm{cf}\ \lambda$, $\nu = \kappa$, $\lambda < \mu$, (so $\lambda = \mathrm{cf}\ \mu$). Then not $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ by Lemma 4.18(*b*).

(*iii*) The result if $V = L$ follows from unpublished work of Prikry [**7**].

It may also be derived from the gap-1 two-cardinal theorem in L using the methods of Litman [Theorem 3.4 of **2**].

Theorem 2 completes our discussion of the BP property under GCH. For if $\kappa = \mu$ then $\mathrm{BP}(\kappa, \lambda, \mu, \nu)$ by Lemma 4.2. So assume $\kappa > \mu$. If $\kappa > \mu^\lambda$, then Lemma 4.3 either settles the problem or reduces it to the remaining cases. So assume $\kappa \leqq \mu^\lambda$. If $\kappa \leqq \lambda$, then Theorem 4.15 applies, and if $\kappa > \lambda$, then Theorem 4.22.

Unfortunately, in Theorem 4.15(*b*) (*iii*), all we can say under GCH is that BP is equivalent to the proposition about trees given in Lemma 4.20. The situation of Theorem 4.15(*b*) (*iii*) can occur in two ways: Let μ be singular, $\nu < \mu$, cf $\nu \neq$ cf μ, cf $\nu^- =$ cf μ. Then the open questions are $\mathrm{BP}(\mu^+, \mathrm{cf}\,\mu, \mu, \nu)$ and $\mathrm{BP}(\mu^+, \mu, \rho, \nu)$, where $\rho < \mu$ is arbitrary.

Next we observe that BP either holds or fails for both situations together.

LEMMA 4.23. (GCH) *Suppose μ is singular, $\nu < \mu$, cf $\mu \neq$ cf ν and cf $\nu^- =$ cf μ. Then $\mathrm{BP}(\mu^+, \mathrm{cf}\,\mu, \mu, \nu)$ if and only if $\mathrm{BP}(\mu^+, \mu, \rho, \nu)$ where $\rho < \mu$ is arbitrary.*

Proof. Assume $\mathrm{BP}(\mu^+, \mathrm{cf}\,\mu, \mu, \nu)$. By Theorem 4.15, $\mathrm{BP}(\mu, \mu, \rho, \nu)$ holds, so by Lemma 4.5, $\mathrm{BP}(\mu^+, \mu, \rho, \nu)$. Now assume $\mathrm{BP}(\mu^+, \mu, \rho, \nu)$. Then $\mathrm{BP}(\mu^+, \mu, \mu, \nu)$, so by Lemma 4.18(*c*), $\mathrm{BP}(\mu^+, \mathrm{cf}\,\mu, \mu, \nu)$.

5. In this section, we use the equivalence of $\mathrm{CP}(\kappa, \mu, \nu)$ with $\mathrm{BP}(\kappa, \kappa, \mu, \nu)$ and $\mathrm{CP}\#(\kappa, \mu, \nu)$ with $\mathrm{BP}\#(\kappa, \kappa, \mu, \nu)$ together with the results of the previous sections to draw some conclusions about CP and CP#.

LEMMA 5.1. (*a*) *If $\kappa \geqq \mu$ and μ is regular, then $\mathrm{CP}(\kappa, \mu, \mu)$.*
(*b*) *If $\kappa > \mu$, then $\mathrm{CP}(\kappa, \mu, \mu^+)$.*

Proof. For (*a*), use Lemmas 2.1 and 4.11. For (*b*), use Lemmas 2.1 and 4.12.

THEOREM 5.2. (GCH) *For all κ, μ, ν with $\kappa \geqq \mu$, $\kappa \geqq \nu$, the relation $\mathrm{CP}(\kappa, \mu, \nu)$ fails to hold if and only if*

$$\kappa > \mu \geqq \nu > \mathrm{cf}\,\nu = \mathrm{cf}\,\kappa.$$

Proof. Use Lemma 2.1 and Theorem 4.15 if $\kappa > \mu$. If $\kappa = \mu$, then any one-to-one coloring works.

THEOREM 5.3. (GCH) *For all κ, μ, ν with $\kappa \geqq \mu \geqq \nu$, the relation CP# fails to hold if and only if*

$$\kappa > \mu \geqq \nu \geqq \mathrm{cf}\,\kappa.$$

Proof. If $\kappa = \mu$, then any one-to-one coloring works. If $\kappa > \mu$, then use Lemma 2.3 and Theorem 3.23.

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