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NEAR-RINGS OF CONTINUOUS FUNCTIONS ON DISCONNECTED GROUPS

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Abstract

$\mathcal{N}(G)$ denotes the near-ring of all continuous selfmaps of the topological group G (under composition and the pointwise induced operation) and $\mathcal{N}_0(G)$ is the subnear-ring of $\mathcal{N}(G)$ consisting of all functions having the identity element of G fixed. It is known that if G is discrete then (a) $\mathcal{N}_0(G)$ is simple and (b) $\mathcal{N}(G)$ is simple if and only if G is not of order 2. We begin a study of the ideal structure of these near-rings when G is a disconnected group.

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1. Introduction

Let G be a topological group. $\mathcal{N}(G)$ denotes the collection of all continuous selfmaps of G, and $\mathcal{N}_0(G)$ denotes that subset of $\mathcal{N}(G)$ which consists of all maps which have the identity of G as a fixed point. Under the pointwise operation on $\mathcal{N}(G)$ which is induced by the binary operation on G, and under the usual composition of functions, $\mathcal{N}(G)$ is a near-ring and $\mathcal{N}_0(G)$ is a subnear-ring of $\mathcal{N}(G)$. In Berman and Silverman (1959) and Nöbauer and Philipp (1962) it was shown that if G has the discrete topology then $\mathcal{N}_0(G)$ is simple (that is, contains no non-trivial near-ring ideals, see Pilz (1975), p. 15) and $\mathcal{N}(G)$ is simple provided G is not of order 2. It is then natural to ask whether $\mathcal{N}(G)$ or $\mathcal{N}_0(G)$ is simple when G is endowed with some topology other than the discrete topology. In this paper we answer the question when G is disconnected, and obtain the results of Berman and Silverman (1959) as special cases. We show that if G is disconnected then $\mathcal{N}_0(G)$ is simple if and only if G is discrete. Moreover, if G is disconnected and $\mathcal{N}(G)$ is simple then G is totally disconnected.

We also begin the study of the ideal structure of these near-rings. We find a class of disconnected groups with the property that if G is a totally disconnected group in this class then $\mathcal{N}_0(G)$ contains a unique minimal ideal. And if G is in this class but is not totally disconnected then $\mathcal{N}(G)$ contains a unique maximal ideal. This class of groups properly contains those infinite groups that have proper, open, disconnected subgroups as well as those groups that are 0-dimensional subgroups of locally compact, Hausdorff groups. We call this class the class of f-locally precompact groups. We do not know whether every infinite totally disconnected group is f-locally precompact.

In Section 2 we consider near-rings on groups with quite general properties. In Section 3 we introduce the notion of a compatible decomposition as a tool to obtain our most important theorems. In Section 4 we examine some properties of f-locally precompact groups and prove that every infinite, totally disconnected, f-locally precompact group has a compatible decomposition. We finish the paper with a number of examples. We will assume throughout that all groups are Hausdorff and that our near-rings are right near-rings.

2. General properties

The following lemma will be used to shorten upcoming proofs.

LEMMA (2.1). *Let $(N, +, \cdot)$ be a near-ring with identity* 1. *A subset I of N is an ideal of N if and only if*

a) *$(I, +)$ is a subgroup of $(N, +)$,*

(b) *$I \cdot N \subseteq I$, and*

(c) *for each $i \in I$ and each $m, n \in N$, $m(n+i) - mn \in I$.*

PROOF. Any ideal in a near-ring satisfies conditions (a) through (c). Conversely suppose I satisfies these three conditions. We need only prove that $(I, +)$ is a normal subgroup of $(N, +)$. Since $n+i-n = 1\cdot(n+i)-1\cdot n$, it follows that I is an ideal.

If X is any set and $x \in X$ we will let $\langle x \rangle$ denote the constant function which carries all of X onto x and we will let id denote the identity function. We will let $+$ denote the (not necessarily commutative) binary operation of each group.

LEMMA (2.2). *Let G be a topological group, let $\mathcal{M}(G)$ be a near-ring of selfmaps of G under pointwise addition and composition, and let $\mathcal{U}$ be a filterbase of sets on G with the property that $h^{-1}(U) \in \mathcal{U}$ for each $U \in \mathcal{U}$ and each $h \in \mathcal{M}(G)$.*

(a) *The set $I = \{f \in \mathcal{M}(G)\colon U \subseteq f^{-1}(0)$ for some $U \in \mathcal{U}\}$ is an ideal in $\mathcal{M}(G)$.*

(b) *If $0 \in \operatorname{int} U$ for each $U \in \mathcal{U}$ and $\mathrm{id} \in I$ then G is discrete.*

PROOF. We first prove (a). Suppose $f, g \in I$. Then there exist $U_1, U_2 \in \mathscr{U}$ such that $U_1 \subseteq f^{-1}(0)$ and $U_2 \subseteq g^{-1}(0)$. Since $\mathscr{U}$ is a filterbase there exists $U_3 \in \mathscr{U}$ such that $U_3 \subseteq U_1 \cap U_2$. Then

$$U_3 \subseteq U_1 \cap U_2 \subseteq f^{-1}(0) \cap g^{-1}(0) \subseteq (f-g)^{-1}(0).$$

Therefore $f-g \in I$. Next suppose $f \in I$ and $\mathscr{D}|\ h \in \mathscr{M}(G)$. Then $U \subseteq f^{-1}(0)$ for some $U \in \mathscr{U}$. Since $h^{-1}(U) \in \mathscr{U}$ and $h^{-1}(U) \subseteq h^{-1}(f^{-1}(0)) = (f \circ h)^{-1}(0)$, $f \circ h \in I$. Finally, suppose $f \in I$ and $g_1, g_2 \in \mathscr{M}(G)$. Since

$$f^{-1}(0) \subseteq (g_1 \circ (g_2 + f) - g_1 \circ g_2)^{-1}(0),$$

it follows from Lemma (2.1) that I is an ideal in $\mathscr{M}(G)$.

Suppose now that $\mathrm{id} \in I$ and $0 \in \operatorname{int} U$ for each $U \in \mathscr{U}$. Since $\mathrm{id} \in I$, $\{0\} \in \mathscr{U}$ and $0 \in \operatorname{int} \{0\}$. Therefore $\{0\}$ is an open set in G, and since any topological group is homogeneous, G is discrete.

In Magill (1967) a topological space X is defined to be an S^*-space if X is T_1 and if for each closed subset F of X and each $p \notin F$ there exists $y \in X$ and a continuous selfmap f of X such that $f(x) = y$ if $x \in F$ and $f(p) \neq y$. We will say that a topological group G is an S^*-group if the topology on G is that of an S^*-space. Since any T_1 group is Hausdorff, each S^*-group is Hausdorff.

THEOREM (2.3). *If G is an S^*-group and $\mathscr{N}_0(G)$ is simple then G is discrete.*

PROOF. Let $\mathscr{U}$ be the filterbase of all open sets about 0 and let

$$I = \{f \in \mathscr{N}_0(G) \colon U \subseteq f^{-1}(0) \text{ for some } U \in \mathscr{U}\}.$$

Since $h^{-1}(U) \in \mathscr{U}$ for every $U \in \mathscr{U}$ and every $h \in \mathscr{N}_0(G)$, it follows from Lemma (2.2) that I is an ideal in $\mathscr{N}_0(G)$. Let $x \in G$ and $x \neq 0$. Since G is Hausdorff there exist disjoint open sets U, V such that $0 \in U$ and $x \in V$. Since G is an S^*-group there exists $y \in G$ and a continuous selfmap f of G such that $f[G-V] = \{y\}$ and $f(x) \neq y$. Since $U \subseteq (f - \langle y \rangle)^{-1}(0)$, $f - \langle y \rangle \in I$, and I is a nonzero ideal. From this it follows that since $\mathscr{N}_0(G)$ is simple, $\mathrm{id} \in I$ and by Lemma (2.2)(b), G is discrete.

THEOREM (2.4). *If G is the additive group of a topological division ring then $\mathscr{N}(G)$ is simple.*

PROOF. Suppose I is a nonzero ideal in $\mathscr{N}(G)$. Then there exists $f \in I$ and $x, y \in G$ such that $f(x) = y$ and $y \neq 0$. Then $\langle y \rangle = f \circ \langle x \rangle \in I$. Since y has an inverse and each multiplicative left translation is continuous, $\langle 1 \rangle \in I$. We seek a function g such that $g \circ (\mathrm{id} + \langle 1 \rangle) - g \circ \mathrm{id} = \mathrm{id}$. That is, we seek a function g such that $g(x+1) - g(x) = x$ for each $x \in G$. In the language of finite differences (see Wylie (1975), for instance) we seek a solution for the difference equation $\Delta g_0(x) = x$.

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The antidifference of the right-hand side is the desired solution and is given by $g(x)=\frac{1}{2}x(x-1)$. Since $\mathrm{id}=g\circ(\mathrm{id}+\langle 1\rangle)-g\circ\mathrm{id}\in I$ it follows immediately that $\mathcal{N}(G)$ is simple.

The following Lemma has a direct proof which will not be given.

LEMMA (2.5). *If N is any near-ring, M is a subnear-ring of N, and I is an ideal in N then $M\cap I$ is an ideal in M.*

Let G be a disconnected group. *Henceforth we will let C denote the connected component of G which contains* 0. Let $P(C)$ denote the set $\{f\in\mathcal{N}(G)\colon R(f)\subseteq C\}$ and $P_0(C)=P(C)\cap\mathcal{N}_0(G)$, where $R(f)$ denotes the range of f.

THEOREM (2.6). t*If G is a disconnected group then $P(C)$ and $P_0(C)$ are ideals in $\mathcal{N}(G)$ and $\mathcal{N}_0(G)$ respectively.*

PROOF. It is well known that C is a normal subgroup of G (see Hewitt and Ross (1963), p. 60), and that the connected components of G are precisely the cosets of C in G. One may use a direct proof to show that $P(C)$ satisfies (a) and (b) of Lemma (2.1). To prove that (c) is satisfied let $h\in P(C)$ and $f,g\in\mathcal{N}(G)$. Then for each $x\in G$, $g(x)+h(x)-g(x)\in C$ and $g(x)+h(x)\in g(x)+C$. Since f is continuous it maps connected sets to connected sets. Therefore $f(g(x)+h(x))$ and $f(g(x))$ are in the same connected component of G. Hence $f(g(x)+h(x))-f(g(x))\in C$ and $P(C)$ is an ideal in $\mathcal{N}(G)$. It follows directly from Lemma (2.5) that $P_0(C)$ is an ideal in $\mathcal{N}_0(G)$.

COROLLARY (2.7). *If G is a disconnected group and $\mathcal{N}(G)$ is simple then G is totally disconnected.*

PROOF. Since $P(C)$ is an ideal in $\mathcal{N}(G)$ and $\mathrm{id}\notin P(C)$ it follows that if $\mathcal{N}(G)$ is simple then $P(C)=\{\langle 0\rangle\}$. This implies that $C=\{0\}$ and hence that G is totally disconnected.

We will let M_0 denote the set $f\in\mathcal{N}_0(G)\colon f^{-1}(0)$ contains a clopen set about $0\}$, where a clopen set is one which is both open and closed.

THEOREM (2.8). *If G is a disconnected group then M_0 is an ideal in $\mathcal{N}_0(G)$ such that $M_0-P_0(C)\neq\varnothing$.*

PROOF. Let $\mathcal{U}$ denote the collection of all clopen sets about 0. Then $\mathcal{U}$ is a filterbase and $h^{-1}(U)\in\mathcal{U}$ for every $U\in\mathcal{U}$ and every $h\in\mathcal{N}_0(G)$. Since

$$M_0=\{f\in\mathcal{N}_0(G)\colon U\subseteq f^{-1}(0)\quad\text{for some } U\in\mathcal{U}\},$$

it follows from Lemma (2.2) that M_0 is an ideal in $\mathcal{N}_0(G)$.

To prove that $M_0-P_0(C)\neq\varnothing$ let $a\in G-C$ and let U and V be disjoint clopen sets such that $0\in U$, $a\in V$ and $U\cup V=G$. Let g be given by $g(x)=0$ if $x\in U$ and $g(x)=a$ if $x\in V$. It then follows that $g\in M_0-P_0(C)$.

By making use of Theorem (2.3) we obtain the following.

COROLLARY (2.9). *If G is an S^*-group or is disconnected and if $\mathcal{N}_0(G)$ is simple then G is discrete.*

PROOF. The case when G is an S^*-group has been dealt with in Theorem (2.3). Suppose that G is disconnected. Then by Theorem (2.8) M_0 is a nonzero ideal in $\mathcal{N}_0(G)$. Therefore if $\mathcal{N}_0(G)$ is simple then $M_0=\mathcal{N}_0(G)$ and $\mathrm{id}\in M_0$. It then follows from Lemma (2.2) that G is discrete.

LEMMA (2.10). *Let G be a disconnected group. If U is a clopen set about 0 and $a\in U-C$, there exists a clopen set V about 0 such that $V\subseteq U$, $V+a\subseteq U$ and $V\cap V+a=\varnothing$.*

PROOF. Since $a\notin C$ there exist disjoint clopen sets U_1 and U_2 such that $0\in U_1$, $a\in U_2$ and $U_1\cup U_2=G$. Let

$$V=U\cap(U-a)\cap U_1\cap(U_2-a).$$

It then follows directly that $V\subseteq U$, $V+a\subseteq U$ and $V\cap V+a=\varnothing$.

LEMMA (2.11). *If G is a disconnected group and I is an ideal in $\mathcal{N}(G)$ such that $I-P(C)\neq\varnothing$, then*

(a) *I contains all constant functions, and*

(b) *if $|G/C|>2$ then $(I\cap\mathcal{N}_0(G))-P_0(C)\neq\varnothing$.*

PROOF. Since $I-P(C)\neq\varnothing$ there exists $f\in I$ and $a,b\in G$ such that $b\notin C$ and $f(a)=b$. Then $\langle b\rangle=f\circ\langle a\rangle\in I$. Let $0\neq z\in G$ be arbitrary, let U and U' be disjoint clopen sets such that $0\in U$, $b\in U'$ and $U\cup U'=G$. Let f_1 be given by $f_1(x)=0$ if $x\in U$ and $f_1(x)=z$ if $x\in U'$. Then $f_1\in\mathcal{N}_0(G)$ and

$$\langle z\rangle=f_1\circ(\langle 0\rangle+\langle b\rangle)-f_1\circ\langle 0\rangle\in I.$$

Therefore I contains all constant functions.

To prove (b) we first consider the case when C is open. Let $b\in G-C$ and $d\in G-(C\cup C+b)$, and let g be given by

$$g(x)=-d \text{ if } x\in C+d \quad\text{and}\quad g(x)=0 \text{ if } x\notin C+d.$$

We proved above that $\langle b\rangle\in I$. Since C is open, $g\in\mathcal{N}_0(G)$ and the function

$$h=g\circ(\mathrm{id}+\langle b\rangle)-g\circ\mathrm{id}\in I.$$

For each $x \in C$, $h(x) = g(b) - g(0) = 0$. Therefore $h \in I \cap \mathcal{N}_0(G)$. Since $b \notin C$,

$$d + b \notin C + d \quad \text{and } h(d) = 0 - g(d) = d \notin C,$$

which implies that $h \notin P_0(C)$. Therefore $h \in (I \cap \mathcal{N}_0(G)) - P_0(C)$.

Now suppose that C is not open and that $b \notin C$. By Lemma (2.10) (with $U = G$) there exists a clopen set V about 0 such that $V \cap V + b = \varnothing$. Since a connected set cannot contain a proper clopen subset, $V \nsubseteq C$ and there is an element $e \in V - C$. Let f_2 be given by $f_2(x) = 0$ if $x \in V$ and $f_2(x) = e$ if $x \notin V$. Moreover, let f_3 be given by $f_3(x) = x - b$ if $x \in V + b$ and $f_3(x) = 0$ if $x \notin V + b$. Then $f_2, f_3 \in \mathcal{N}_0(G)$,

$$f_2 = (f_3 \circ (\mathrm{id} + \langle b \rangle) - f_3 \circ \mathrm{id}) \circ f_2 \in (I \cap \mathcal{N}_0(G)) - P_0(C),$$

and the proof is complete.

In the proof of the following lemma we make use of the fact that an ideal I in $\mathcal{N}_0(G)$ is closed under composition on the left by members of $\mathcal{N}_0(G)$. If $f \in I$ and $g \in \mathcal{N}_0(G)$ then $g \circ f = g \circ (\langle 0 \rangle + f) - g \circ \langle 0 \rangle \in I$.

LEMMA (2.12). *Let G be a disconnected group.*

(a) *If I is an ideal in $\mathcal{N}_0(G)$ such that $I - P_0(C) \neq \varnothing$, then I contains all functions whose range is finite.*

(b) *If $|G/C| > 2$ and I is an ideal in $\mathcal{N}(G)$ for which $I - P(C) \neq \varnothing$, then I contains all functions whose range is finite.*

PROOF. The proof of (a) is by induction. Since $I - P_0(C) \neq \varnothing$ there exists $f \in I$ and $a, b \in G$ such that $b \notin C$ and $f(a) = b$. Let U and U' be disjoint open sets such that $0 \in U$, $a \in U'$ and $U \cup U' = G$. Then the function f_1 given by $f_1(x) = 0$ if $x \in U$ and $f_1(x) = a$ if $x \in U'$ is in $\mathcal{N}_0(G)$. Let $f_2 = f \circ f_1$. Then $f_2 \in I$ and $f_2(x) = 0$ if $x \in U$ and $f_2(x) = b$ if $x \in U'$. Therefore I contains at least one function whose range consists of exactly two elements. Let $g \in \mathcal{N}_0(G)$ be such that $|R(g)| = 2$ and let $0 \neq c \in R(g)$. Then $g^{-1}(0)$ and $g^{-1}(c)$ are nonempty, disjoint clopen sets whose union is G. Let g_1 be given by $g_1(x) = 0$ if $x \in g^{-1}(0)$ and $g_1(x) = a$ if $x \in g^{-1}(c)$. Moreover, let V and V' be disjoint clopen sets such that $0 \in V$, $b \in V'$ and $V \cup V' = G$, and let g_2 be given by $g_2(x) = 0$ if $x \in V$ and $g_2(x) = c$ if $x \in V'$. Then $g_1, g_2 \in \mathcal{N}_0(G)$ and $g = g_2 \circ f_2 \circ g_1$. Since ideals in $\mathcal{N}_0(G)$ are closed under composition on the left by members of $\mathcal{N}_0(G)$, $g \in I$ and I contains all functions g such that $|R(g)| \leqslant 2$.

Suppose I contains all functions g such that $|R(g)| \leqslant n$ where $n \geqslant 2$, and let $h \in \mathcal{N}_0(G)$ be such that $R(h) = \{0 = a_0, a_1, \ldots, a_n\}$ where $a_i \neq a_j$ if $i \neq j$. Let h_1 be given by $h_1(x) = 0$ if $x \in h^{-1}(0)$ and $h_1(x) = a_1$ if $x \notin h^{-1}(0)$. Then

$$|R(h_1)| = 2, \quad |R(h - h_1)| = n$$

and by the induction hypothesis, h_1 and $h-h_1$ are in I. It follows that $h \in I$. Therefore all functions h such that $|R(h)| \leqslant n+1$ are in I, and by induction I contains all functions whose range is finite.

For the proof of (b) suppose that I is an ideal in $\mathcal{N}(G)$ such that $I - P(C) \neq \varnothing$. It follows from Lemmas (2.5) and (2.11) that $I \cap \mathcal{N}_0(G)$ is an ideal in $\mathcal{N}_0(G)$ which is not a subset of $P_0(C)$. If $g \in \mathcal{N}(G)$ and $R(g) = \{a_0, a_1, \ldots, a_n\}$, where $g(0) = a_0$, then $g - \langle a_0 \rangle$ is in $\mathcal{N}_0(G)$ and by (a) above, $g - \langle a_0 \rangle \in I \cap \mathcal{N}_0(G)$. By Lemma (2.11) $\langle a_0 \rangle \in I$ and it follows that $g = (g - \langle a_0 \rangle) + \langle a_0 \rangle \in I$. Thus I contains all functions whose range is finite.

3. Compatible decompositions

The collection of all left cosets of an open, disconnected subgroup of a topological group G forms a partition of G whose members are clopen sets. Such a partition enables us to obtain more information about the ideal structure of $\mathcal{N}(G)$ and $\mathcal{N}_0(G)$. The following Definition sets forth the essential properties which we will require of these partitions. We leave it to the reader to verify that the collection of left cosets of an open, disconnected subgroup satisfies this Definition.

Definition (3.1). A partition $\mathscr{F}$ of a topological group G which consists of clopen sets is called a compatible decomposition of G if there exists a subset X of G which indexes the partition and $a \in G - C$ such that

(a) $0 \in X$,

(b) $x, x+a \in F_x$ for each $x \in X$,

(c) there exists a symmetric clopen subset U_0 of F_0 such that $-z+x \in U_0$ for each $z \in F_x$ and each $0 \neq x \in X$, and

(d) there exists $b \in G$ such that $F_0 \cap b + F_0 = \varnothing$.

We will say that a subset I of a near-ring $(N, +, \cdot)$ with identity is a left ideal if I satisfies (a) and (c) of Lemma (2.1). Note that a left ideal in $\mathcal{N}_0(G)$ is closed under composition on the left by members of $\mathcal{N}_0(G)$.

Lemma (3.2). *Let G be a disconnected group having a compatible decomposition and let I be a left ideal in $\mathcal{N}_0(G)$ such that $I - P_0(C) \neq \varnothing$. If $f \in I$ and U is a clopen set about* 0 *then there exists $g \in I$ such that $g(z) = 0$ if $z \in f^{-1}(U)$ and $g(z) = z$ if $z \notin f^{-1}(U)$.*

Proof. If $f^{-1}(U) = G$ there is nothing to prove. So suppose $f^{-1}(U) \subseteq G$ and $\mathscr{F}$ is a compatible decomposition of G. Then there exists $X \subseteq G$ and $a \in G - C$ such that X indexes $\mathscr{F}$ and conditions (a)–(d) of Definition (3.1) are satisfied. Let f_1 be given by $f_1(z) = 0$ if $z \in U$ and $f_1(z) = a$ if $z \notin U$. Since U is a clopen set about 0, $f_1 \in \mathcal{N}_0(G)$ and $f_1 \circ f \in I$. Then $f_1 \circ f(z) = 0$ if $z \in f^{-1}(U)$ and $f_1 \circ f(z) = a$ if

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$z \notin f^{-1}(U)$. Let f_2 be defined as follows. For each $z \in G$ there is a unique $x \in X$ for which $z \in F_x$, and we let $f_2(z) = x$. Then for each $x \in X$, $f_2[F_x] = \{x\}$, and since each F_x is a clopen set, $f_2 \in \mathcal{N}_0(G)$. Since $a \notin C$, x and $x+a$ are in different connected components of G for each $x \in X$. Therefore there exist disjoint open sets A_x and B_x of F_x such that $x \in A_x$, $x+a \in B_x$ and $A_x \cup B_x = F_x$. Let

$$A = F_0 \cup (\bigcup\{A_x : 0 \neq x \in X\}) \quad \text{and} \quad B = \bigcup\{B_x : 0 \neq x \in X\},$$

and let g_1 be given by $g_1(z) = 0$ if $z \in A$ and $g_1(z) = z - a$ if $z \in B$. Since A and B are nonempty, disjoint open sets whose union is G, g_1 is continuous. Since $f_1 \circ f \in I$, the function $g_2 = g_1 \circ (f_2 + f_1 \circ f) - g_1 \circ f_2$ is in I. If $z \in f^{-1}(U) \cup F_0$ then $g_2(z) = 0$. If $z \notin f^{-1}(U) \cup F_0$ then $z \in F_x$ for some $0 \neq x \in X$ and

$$g_2(z) = g_1(x+a) - g_1(x) = x.$$

Let $U_0 \subseteq F_0$ be a symmetric clopen set such that $-z+x \in U_0$ for each $z \in F_x$ and each $0 \neq x \in X$, and let $h_1(z) = 0$ if $z \in F_0 \cup f^{-1}(U)$ and $h_1(z) = z$ if $z \notin F_0 \cup f^{-1}(U)$. Moreover, let $f_3(z) = 0$ if $z \in U_0$ and $f_3(z) = z$ if $z \notin U_0$. Since $g_2 \in I$ the function

$$g_3 = f_3 \circ ((-\mathrm{id} + g_2) - g_2) - f_3 \circ (-\mathrm{id} + g_2)$$

is in I. If $z \in F_0 \cup f^{-1}(U)$ then $g_2(z) = 0$ and therefore $g_3(z) = 0$. But if

$$z \notin F_0 \cup f^{-1}(U)$$

then $z \in F_x - (F_0 \cup f^{-1}(U))$ for some $0 \neq x \in X$ and $(-\mathrm{id} + g_2)(z) = -z + x \in U_0$. As a result, $f_3 \circ (-\mathrm{id} + g_2)(z) = 0$ and $g_3(z) = f_3 \circ (-\mathrm{id})(z) = -z$ since $-z$ is not a member of the symmetric set U_0. Therefore $h_1 = -g_3 \in I$.

If $F_0 \subseteq f^{-1}(U)$ then h_1 is the desired function g. Suppose then that

$$F_0 - f^{-1}(U) \neq \varnothing.$$

According to (d) of Definition (3.1) there exists $b \in G$ such that $F_0 \cap b + F_0 = \varnothing$. Let the function f_4 be given by $f_4(z) = 0$ if $z \in U$ and $f_4(z) = b$ if $z \notin U$. Then $f_4 \circ f \in I$ and $f_4 \circ f(z) = 0$ if $z \in f^{-1}(U)$ and $f_4 \circ f(z) = b$ if $z \notin f^{-1}(U)$. Let f_5 be given by $f_5(z) = z$ if $z \in F_0 - f^{-1}(U)$ and $f_5(z) = 0$ otherwise, let $g_4(z) = 0$ if $z \in F_0$ and $g_4(z) = z - b$ if $z \notin F_0$, and let

$$h_2 = g_4 \circ (f_5 + f_4 \circ f) - g_4 \circ f_5.$$

If $z \in f^{-1}(U)$ then $f_4 \circ f(z) = 0$ and $h_2(z) = 0$. If $z \notin F_0 \cup f^{-1}(U)$ then

$$h_2(z) = g_4(0+b) - g_4(0) = 0$$

since $b \notin F_0$. Finally, if $z \in F_0 - f^{-1}(U)$ then $h_2(z) = g_4(z+b) - g_4(z) = z$. Therefore $h_2(z) = z$ if $z \in F_0 - f^{-1}(U)$ and $h_2(z) = 0$ otherwise. By letting $g = h_1 + h_2$ we obtain the desired function. Since, $h_1 h_2 \in I$, $g \in I$ and the proof is complete.

Recall that $M_0 = \{f \in \mathcal{N}_0(G) : f^{-1}(0) \text{ contains a clopen set about } 0\}$.

THEOREM (3.3). *If the disconnected group G has a compatible decomposition then $M_0 \subseteq I$ for every ideal I in $\mathcal{N}_0(G)$ for which $I - P_0(C) \neq \varnothing$.*

PROOF. Let $f \in M_0$ be arbitrary, let I be an ideal in $\mathcal{N}_0(G)$ such that $I - P_0(C) \neq \varnothing$, let U be a clopen set about 0 such that $U \subseteq f^{-1}(0)$, let $a \in G - C$ and let f_1 be given by $f_1(z) = 0$ if $z \in U$ and $f_1(z) = a$ if $z \notin U$. According to Lemma (2.12), $f_1 \in I$. Since G is disconnected and $a \in G - C$, $U = f_1^{-1}(V)$ for some clopen set V in G, and according to Lemma (3.2) the function g, which is given by $g(z) = 0$ if $z \in U$ and $g(z) = z$ if $z \notin U$, is a member of I. Therefore $f \circ g \in I$. One may readily verify that $f \circ g = f$. Thus $f \in I$ and $M_0 \subseteq I$.

If G is a totally disconnected group then $P_0(C) = \{\langle 0 \rangle\}$. Therefore we have the following.

COROLLARY (3.4). *If G is a totally disconnected group having a compatible decomposition then $M_0 \subseteq I$ for every nonzero ideal I in $\mathcal{N}_0(G)$.*

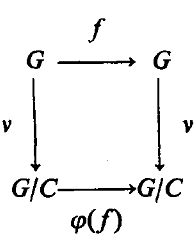
If G is a discrete group then $\{0\}$ is a clopen set about 0 and $\mathrm{id} \in M_0$. Therefore $M_0 = \mathcal{N}_0(G)$. Any infinite discrete group has a proper open subgroup and therefore, by the remarks preceding Definition (3.1), has a compatible decomposition. It then follows from Corollary (3.4) that if G is an infinite discrete group then $\mathcal{N}_0(G)$ is simple. If G is a finite discrete group it follows from Lemma (2.12) that $\mathcal{N}_0(G)$ is simple. Thus we obtain one of the results given in Berman and Silverman (1959).

COROLLARY (3.5). *If G is a discrete group then $\mathcal{N}_0(G)$ is simple.*

Combining this result with Corollary (2.9) we have the following

COROLLARY (3.6). *If G is an S^*-group or is disconnected then $\mathcal{N}_0(G)$ is simple if and only if G is discrete.*

LEMMA (3.7). *Let G be a disconnected group, let $\nu: G \to G/C$ be the natural map and let $\varphi: \mathcal{N}(G) \to \mathcal{N}(G/C)$ be defined so that for each $f \in \mathcal{N}(G)$, the following diagram commutes:*



Then

(a) *φ is a near-ring homomorphism with kernel $P(C)$, and*
(b) *if C is open then φ is onto.*

PROOF. For each $x \in G$ and each $f \in \mathcal{N}(G)$, $\varphi(f)(x+C) = f(x)+C$. If

$$x+C = y+C$$

then x and y are in the same connected component of G. Since f is continuous, it carries connected sets to connected sets and it follows that $f(x)$ and $f(y)$ are in the same component of G. Therefore $f(x)+C = f(y)+C$ and it follows that φ is a well-defined map. Since

$$\begin{aligned}\varphi(f+g)(x+C) &= (f+g)(x)+C = f(x)+g(x)+C = (f(x)+C)+(g(x)+C)\\ &= \varphi(f)(x+C)+\varphi(g)(x+C) = (\varphi(f)+\varphi(g))(x+C)\end{aligned}$$

and

$$\varphi(f \circ g)(x+C) = f(g(x))+C = \varphi(f)(g(x)+C) = \varphi(f) \circ \varphi(g)(x+C),$$

φ is a near-ring homomorphism. It is easy to see that $\ker \varphi = P(C)$.

Now suppose that C is open. Let k: $G/C \to G$ be such that $\nu \circ k$ is the identity on G/C and let $g \in \mathcal{N}(G/C)$ be arbitrary. For each $x \in G$ let $f(x) = k(g(x+C))$. Since $R(k) \cap (x+C)$ is a singleton for each $x \in G$, the restriction of f to each coset of C is a constant function. Since each coset of C is the (homeomorphic) image of C by a translation of G, each coset of C is open in G. Therefore $f^{-1}(x)$ is open in G for each $x \in G$ and $f \in \mathcal{N}(G)$. But

$$\varphi(f)(x+C) = f(x)+C = k(g(x+C))+C = g(x+C)$$

since $k(g(x+C)) \in g(x+C)$. Thus $\varphi(f) = g$ and φ is onto.

THEOREM (3.8). *Let G be a disconnected group.*

(a) *If G/C is an infinite set and G has a compatible decomposition then $P(C)$ is the unique maximal ideal in $\mathcal{N}(G)$.*
(b) *If $|G/C| = n > 2$ for some natural number n then $P(C)$ is the unique maximal ideal in $\mathcal{N}(G)$.*
(c) *If $|G/C| = 2$ then $K = \{f \in \mathcal{N}(G)$: $R(f)$ is a subset of some coset of $C\}$ is the unique maximal ideal in $\mathcal{N}(G)$.*

PROOF. Suppose first that G/C is an infinite set and that G has a compatible decomposition. Let $a \in G - C$. According to Lemma (2.10), there exists a clopen set V about 0 such that $V \cap V+a = \varnothing$. Define $f_1(x) = x$ if $x \in V$ and $f_1(x) = 0$ if $x \notin V$, and define $f_2(x) = x-a$ if $x \in V+a$ and $f_2(x) = 0$ if $x \notin V+a$. Since V is a clopen set about 0, f_1 and f_2 are in $\mathcal{N}(G)$. Suppose I is a proper ideal in $\mathcal{N}(G)$ and that $I - P(C) \neq \varnothing$. Then according to Lemma (2.11), $\langle a \rangle \in I$ and the function

$$g = (f_2 \circ (\mathrm{id}+\langle a \rangle) - f_2 \circ \mathrm{id}) \circ f_1$$

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is a member of I. If $x \in V$ then $g(x) = x$ and if $x \notin V$ then $g(x) = 0$. Therefore $g = f_1 \in I$. By Lemmas (2.5) and (2.11) $I \cap \mathcal{N}_0(G)$ is an ideal in $\mathcal{N}_0(G)$ such that $(I \cap \mathcal{N}_0(G) - P_0(C)) \neq \varnothing$. Since G has a compatible decomposition it follows from Theorem (3.3) that $\mathrm{id} - f_1 \in M_0 \subseteq I \cap \mathcal{N}_0(G) \subseteq I$. Therefore $\mathrm{id} - f_1$ and f_1 are in I, from which it follows that $I = \mathcal{N}(G)$. Since we assumed I was a proper ideal this is a contradiction. Thus $I \subseteq P(C)$ and $P(C)$ is the unique maximal ideal in $\mathcal{N}(G)$.

Now suppose that G/C is finite. Then C is the intersection of a finite number of clopen sets and therefore is open. Since the natural map $v: G \to G/C$ is continuous and open, G/C is discrete and the map $\varphi: \mathcal{N}(G) \to \mathcal{N}(G/C)$ is a surjective near-ring homomorphism. If G/C is not of order 2 it follows from Lemma (2.12) that any nonzero ideal in $\mathcal{N}(G/C)$ contains all functions having finite range. Since G/C is finite, this implies that $\mathcal{N}(G/C)$ is simple. Therefore $P(C) = \ker \varphi$ is the unique maximal ideal in $\mathcal{N}(G)$, and (b) is proved. To prove (c) suppose G/C is of order 2. One can show by checking all possible cases (see also Nöbauer and Philipp (1962)) that the set J of both constant functions is the only ideal in $\mathcal{N}(G/C)$. Then $K = \varphi^{-1}(J)$ and is the unique maximal ideal in $\mathcal{N}(G)$.

If G is a totally disconnected group then $P(C) = \{\langle 0 \rangle\}$ and we obtain the following partial converse of Corollary (2.7).

COROLLARY (3.9). *If G is a totally disconnected group which is either finite and not of order 2, or is infinite and has a compatible decomposition, then $\mathcal{N}(G)$ is simple.*

If G is an infinite discrete group then G has a proper open subgroup, and by the remarks preceding Definition (3.1), G has a compatible decomposition. We thus have another of the results presented in Berman and Silverman (1959).

COROLLARY (3.10). *If G is a discrete group having more than two elements then $\mathcal{N}(G)$ is simple.*

4. f-locally precompact groups

If $\mathcal{U}$ is a uniformity on a set X (see Kelley (1975), Chapter 6) and $T \in \mathcal{U}$, $T[x]$ is defined to be the set $\{y: (x,y) \in T\}$, and a topology on X is obtained from $\mathcal{U}$ by defining a set V to be open if for every $x \in V$ there exists $T \in \mathcal{U}$ such that $T[x] \subseteq V$. Moreover, a subset A of X is said to be precompact (or totally bounded) with respect to the uniformity $\mathcal{U}$ if for each $T \in \mathcal{U}$ there exists a finite subset F of A such that $A \subseteq \bigcup\{T[x]: x \in F\}$.

Let G be a topological group. We will say that S is a neighbourhood of 0 if

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$0 \in \operatorname{int} S$. For each neighbourhood S of 0 let $L(S) = \{(x, y)\colon -x+y \in S\}$. The collection $\mathscr{L}$ of all such $L(S)$ is a uniformity on G and is called the left uniformity for G. If $L(S) \in \mathscr{L}$ and $x \in G$ then

$$L(S)[x] = \{y\colon (x, y) \in L(S)\} = \{y\colon -x+y \in S\} = x+S,$$

the translation of S by x. The topology obtained from $\mathscr{L}$ is the original topology on G. Therefore a subset A of a topological group G is precompact (with respect to $\mathscr{L}$) if for every neighbourhood S of 0 there exists a finite subset F of A such that $A \subseteq \bigcup\{x+S\colon x \in F\}$. In other words, A is precompact if for every neighbourhood S of 0, the cover $\{x+S\colon x \in A\}$ has a finite subcover.

DEFINITION (4.1). A subset A of a topological group G is precompact with respect to the neighbourhood S of 0 if the cover $\{x+S\colon x \in A\}$ has a finite subcover.

Therefore a subset A of G is precompact if it is precompact with respect to each neighbourhood of 0. If U is an open set about 0, $2U$ denotes $U+U$, $3U$ denotes $U+U+U$, etc.

DEFINITION (4.2). A disconnected group G is f-locally precompact if there exist symmetric open sets W_1, W_2, W_3, W_4 about 0 such that
(a) $2W_{i+1} \subseteq W_i$ for each $i = 1, 2, 3$,
(b) W_2 and W_4 are clopen sets,
(c) W_1 is precompact with respect to W_3, and
(d) $G - 2W_1 \neq \varnothing$.

Note that if G has a proper open subgroup H, then by taking $W_i = H$ for each $i = 1, 2, 3, 4$ it follows that G is f-locally precompact.

THEOREM (4.3). *If G/C is infinite and G is f-locally precompact then G has a compatible decomposition.*

PROOF. If G has a proper, disconnected, open subgroup then by the remarks preceding Definition (3.1) G has a compatible decomposition. On the other hand, if C is open then G/C is discrete and infinite. It then follows that G has a proper, disconnected, open subgroup. We therefore assume that G has no proper open subgroups. Let W_1, W_2, W_3 and W_4 be open sets about 0 which satisfy conditions (a)–(d) of Definition (4.2).

Since G has no proper open subgroups, $2W_i - W_i \neq \varnothing$ for each $i = 1, 2, 3, 4$. Let $\mathfrak{A}$ be the collection of all subsets A of $G - W_1$ such that if $x, y \in A$ and $x \neq y$ then $(x+W_3) \cap (y+W_3) = \varnothing$. The collection $\mathfrak{A}$ is nonempty. For if $x \in W_1 - W_2$ and $W_3 \cap x+W_3 \neq \varnothing$ then $w = x+w'$ for some $w, w' \in W_3$ and since W_3 is symmetric, $x = w-w' \in 2W_3 \subseteq W_2$. Since x was assumed to lie in $W_1 - W_2$ this cannot happen.

Since G is f-locally precompact there exists $b \in G - 2W_1$. By an argument similar to the one just used to show $W_3 \cap x + W_3 = \emptyset$ for $x \in W_1 - W_2$, one can show that $W_1 \cap b + W_1 = \emptyset$. Also, $b \notin C$. (If it is, then $W_2 \cap C$ is a nonempty clopen subset of C and $b \notin W_2 \cap C$. Therefore C is disconnected, which is a contradiction.) Now since $W_3 \cap x + W_3 = \emptyset$ for any $x \in W_1 - W_2$, $(b + W_3) \cap (b + x + W_3) = \emptyset$. Therefore, $\{b, b+x\} \in \mathfrak{A}$ for any $x \in W_1 - W_2$. Moreover, $\mathfrak{A}$ is partially ordered by set inclusion. Using Zorn's lemma one can prove that $\mathfrak{A}$ contains a maximal element Y. From the maximality of Y it follows that $\{x + W_2 : x \in Y\}$ is an open cover of $G - W_1$. (For if $z \in G - W_1$ then there exists $x \in Y$ such that $(z + W_3) \cap (x + W_3) \neq \emptyset$. Then there exist $w, w' \in W_3$ such that

$$z + w = x + w' \quad \text{and} \quad z = x + w' - w \in x + 2W_3 \subseteq x + W_2$$

since W_3 is symmetric.) Therefore $\{W_1\} \cup \{x + W_2 : x \in Y\}$ is an open cover of G.

We now assert that this cover is neighbourhood finite (see Dugundji (1966), p. 81). In order to prove this it is enough to show that for each $z \in G$, the set $A_z = \{x \in Y : (z + W_2) \cap (x + W_2) \neq \emptyset\}$ is finite. If $A_z = \emptyset$ we are done. So suppose $x \in A_z$. Then there exist $w, w' \in W_2$ such that

$$z + w = x + w' \quad \text{and} \quad x = z + w - w' \in z + 2W_2 \subseteq z + W_1.$$

Therefore $A_z \subseteq z + W_1$ and $-z + A_z \subseteq W_1$. If $x, y \in -z + A_z$ and $x \neq y$ then there exist $x', y' \in A_z$ such that $x = -z + x'$ and $y = -z + y'$. Now $A_z \subseteq Y$ and therefore $(x' + W_3) \cap (y' + W_3) = \emptyset$. As a result,

$$\begin{aligned}(x + W_3) \cap (y + W_3) &= (-z + x' + W_3) \cap (-z + y' + W_3)\\ &= (x' + W_3) \cap (y' + W_3) = \emptyset.\end{aligned}$$

Using Zorn's lemma we find a maximal subset Y' of W_1 such that $-z + A_z \subseteq Y'$ and $(x + W_3) \cap (y + W_3) = \emptyset$ whenever x and y are distinct members of Y'. Now W_1 is precompact with respect to W_3. Therefore there is a finite subset K of W_1 such that $\{x + W_3 : x \in K\}$ is a cover of W_1. We claim that for each $x \in K$, $x + W_3$ can contain at most one member of Y'. For suppose that $y', y'' \in Y' \cap (x + W_3)$ for some $x \in K$ and $y' \neq y''$. Then there exist $w', w'' \in W_3$ such that $y' = x + w'$ and $y'' = x + w''$. Since W_3 is symmetric $-w', -w'' \in W_3$ and

$$x = y' - w' = y'' - w'' \in (y' + W_3) \cap (y'' + W_3).$$

This is a contradiction. It then follows that Y' is finite. Therefore $-z + A_z$ is finite, A_z is finite and the cover is neighbourhood finite.

Let $X = Y \cup \{0\}$ and let $V_0 = W_1 - \bigcup\{x + W_2 : x \in Y\}$. It is well known that if $\{A_\alpha : \alpha \in \Gamma\}$ is a neighbourhood finite family of sets in a topological space then $\text{cl}\,(\bigcup\{A_\alpha : \alpha \in \Gamma\}) = \bigcup\{\text{cl}\, A_\alpha : \alpha \in \Gamma\}$. Therefore $\bigcup\{x + W_2 : x \in Y\}$ is a clopen set,

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from which it follows that V_0 is a clopen set. For each $x \in Y$ let

$$B_x = \{y \in Y: (y+W_4) \cap (x+W_2) \neq \varnothing\}.$$

Since $W_4 \subseteq W_2$,

$$B_x \subseteq A_x = \{y \in Y: (x+W_2) \cap (y+W_2) \neq \varnothing\}.$$

Since we proved above that A_x is finite for each $x \in G$, B_x is finite and

$$V_x = (x+W_2) - \bigcup\{y+W_4: y \in B_x - \{x\}\}$$

is a clopen set for each $x \in Y$. We now assert that $\mathscr{V} = \{V_x: x \in X\}$ is a neighbourhood finite cover of G. For let $z \in G$ be arbitrary. If $z \in V_0$, then since V_0 is disjoint from all other sets in $\mathscr{V}$, there is a neighbourhood of z, namely V_0, which intersects only finitely many sets in $\mathscr{V}$. On the other hand, suppose $z \notin V_0$. If $z \in x+W_4$ for some $x \in Y$ then $x+W_4$ is an open set about z which intersects only one set in $\mathscr{V}$, namely V_x. If $z \notin x+W_4$ for every $x \in Y$ then since $z \notin V_0$ and $\{x+W_2: x \in Y\}$ is a cover of $G - V_0$, there is some $x \in Y$ such that $z \in x+W_2$, and therefore

$$z \in x+W_2 - \bigcup\{y+W_4: y \in B_x - \{x\}\} = V_x.$$

Since

$$\{x \in Y: (z+W_2) \cap V_x \neq \varnothing\} \subseteq \{x \in Y: (z+W_2) \cap (x+W_2) \neq \varnothing\} = A_z,$$

and A_z is finite, $z+W_2$ is an open set about z which intersects only finitely many sets in $\mathscr{V}$. Therefore $\mathscr{V}$ is an open, neighbourhood finite cover of G.

Let $\leqslant$ be a well-order for X having 0 as least element and let $F_0 = V_0$. For each $x \in Y$ let $F_x = V_x - \bigcup\{V_y: y < x\}$. For each $x \in Y$, $\bigcup\{V_y: y < x\}$ is a clopen set. Therefore F_x is a clopen set for each $x \in X$. We claim that $\{F_x: x \in X\}$ is a partition of G. For suppose $x, z \in X$ and $x \neq z$. Without loss of generality we suppose that $z < x$. Then

$$\begin{aligned} F_x \cap F_z &\subseteq (V_x - \bigcup\{V_y: y < x\}) \cap V_z \\ &\subseteq (V_x - \bigcup\{V_y: y < x\}) \cap (\bigcup\{V_y: y < x\}) = \varnothing. \end{aligned}$$

And if $z \in G$ is arbitrary let x be the least element of X such that $z \in V_x$. Then $z \in V_x - \bigcup\{V_y: y < x\} = F_x$. Thus $\{F_x: x \in X\}$ is a partition of G by clopen sets, it is indexed by X and $0 \in X$.

We now assert that $y+W_4 \subseteq F_y$ for every $y \in Y$. Suppose to the contrary that there is some $y \in Y$ such that $y+W_4 \nsubseteq F_y$. Since $\{F_x: x \in X\}$ is a partition of G there exists $z \in X$ such that $z \neq y$ and $(y+W_4) \cap F_z \neq \varnothing$. If $z = 0$ then

$$F_z \cap (y+W_4) \subseteq (W_1 - \bigcup\{x+W_2: x \in Y\}) \cap (y+W_2) = \varnothing$$

which is impossible. Therefore $z \in Y$ and $y \in B_z - \{z\}$. Consequently

$$y+W_4 \subseteq \bigcup\{x+W_4: x \in B_z - \{z\}\}$$

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and

$$\begin{aligned}\emptyset \neq (y+W_4)\cap F_z &\subseteq (y+W_4)\cap V_z \\ &\subseteq \left(\bigcup\{x+W_4\colon x\in B_z-\{z\}\}\right)\cap\left(z+W_2-\bigcup\{x+W_4\colon x\in B_z-\{z\}\}\right)=\emptyset.\end{aligned}$$

This is a contradiction. Thus $y+W_4 \subseteq F_y$ for every $y\in Y$. Now if $y\in Y$ and $(y+W_2)\cap W_2 \neq \emptyset$ then it follows that $y\in 2W_2 \subseteq W_1$. But $Y\subseteq G-W_1$, and it follows that $(y+W_2)\cap W_2=\emptyset$ for each $y\in Y$ and

$$W_2 \subseteq F_0 = W_1 - \bigcup\{y+W_2\colon y\in Y\}.$$

Since $W_4\subseteq W_2\subseteq F_0$, it follows that $x+W_4\subseteq F_x$ for every $x\in X$.

Since G has no proper open subgroups, G is not discrete. Since W_4 is clopen and C is not open, $W_4 \not\subseteq C$, and hence there exists an element $a\in W_4-C$. Since $x+W_4\subseteq F_x$ for each $x\in X$, and since x, $x+a\in x+W_4$ for every $x\in X$, it follows that x, $x+a\in F_x$ for every $x\in X$. Therefore, (b) of Definition (3.1) holds. To show that (c) holds let $U_0=W_2$, let $x\in Y$ and let $z\in F_x$. Since $F_x\subseteq x+W_2$, $z\in x+W_2$ and $-x+z\in W_2=U_0$. Recall from earlier in the proof that since $G-2W_1\neq\emptyset$ there exists $b\in G-C$ such that $W_1\cap b+W_1=\emptyset$ and since $F_0\subseteq W_1$,

$$F_0\cap b+F_0=\emptyset.$$

Thus $\{F_x\colon x\in X\}$ is a compatible decomposition of G.

LEMMA (4.4). *If G is a subgroup of some locally compact group, then there is a neighbourhood U of* 0 *such that if V is a neighbourhood of* 0 *and $V\subseteq U$ then V is precompact.*

PROOF. Suppose G' is a locally compact group which has G as a subgroup and K is a compact neighbourhood of 0 in G'. Let $U=K\cap G$ and let V be any neighbourhood of 0 such that $V\subseteq U$. Let W be any symmetric neighbourhood of 0 in G. Then there exists a symmetric neighbourhood W' of 0 in G' such that $W'\cap G=W$. If $z\in \mathrm{cl}_{G'}V$ then $(z+\mathrm{int}\,W')\cap V\neq\emptyset$ and there exists $x\in(z+\mathrm{int}\,W')\cap V$. Therefore $-z+x\in\mathrm{int}\,W'$. Since W' is symmetric, $\mathrm{int}\,W'$ is symmetric, and $-x+z\in\mathrm{int}\,W'$. Therefore $z\in x+\mathrm{int}\,W'$ and $\{x+\mathrm{int}\,W'\colon x\in V\}$ is an open cover of $\mathrm{cl}_{G'}V$. Recall that we are assuming all our groups to be Hausdorff. Since K is a compact subset of a Hausdorff space it is closed in G' and $V\subseteq K$. Therefore $\mathrm{cl}_{G'}V\subseteq K$. Since any closed subset of a compact space is compact, $\mathrm{cl}_{G'}V$ is compact and there is a finite subcover $\{x_1+\mathrm{int}\,W',\dots,x_n+\mathrm{int}\,W'\}$ of $\mathrm{cl}_{G'}V$. If $z\in V$ then

$$z\in x_k+\mathrm{int}\,W' \quad\text{and}\quad -x_k+z\in\mathrm{int}\,W'\subseteq W'$$

for some $k=1,2,\dots,n$. Then $-x_k+z\in W'\cap G=W$ and $z\in x_k+W$. Therefore $\{x_1+W,\dots,x_n+W\}$ is a finite cover of V.

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A topological space X is said to be 0-dimensional if there is a basis for the topology of X which consists of clopen sets. A group G is 0-dimensional if the topology on G is 0-dimensional.

THEOREM (4.5). *If G is an infinite, 0-dimensional group which is a subgroup of some locally compact group then G is f-locally precompact.*

PROOF. Let $0 \neq b \in G$. According to Lemma (2.10) there is an open set W' about 0 such that $W' \cap b + W' = \varnothing$. According to Lemma (4.4) there is an open set U' about 0 such that any subset of U' which is a neighbourhood of 0 is precompact. Let W_1 be any symmetric open subset of $U' \cap W'$ which is a neighbourhood of 0. Since G is 0-dimensional there exist symmetric clopen sets W_2, W_3, W_4 about 0 such that $2W_{i+1} \subseteq W_i$ for each $i = 1, 2, 3$. Since $W_1 \subseteq U'$, W_1 is precompact, and in particular W_1 is precompact with respect to W_3. And since $W_1 \subseteq W'$,

$$W_1 \cap b + W_1 = \varnothing.$$

Thus G is f-locally precompact.

THEOREM (4.6). *Suppose φ is a continuous homomorphism from a topological group G onto a topological group H. If H is f-locally precompact then G is f-locally precompact.*

PROOF. Suppose W_i, $i = 1, 2, 3, 4$, are symmetric open sets about 0 in H which satisfy conditions (a)–(d) of Definition (4.2). For each $i = 1, 2, 3, 4$ let $V_i = \varphi^{-1}(W_i)$. One may use a direct proof to show that the V_i satisfy conditions (a)–(d) of Definition (4.2). Thus G is f-locally precompact.

For information on the free topological group generated by a topological space see Hewitt and Ross (1963), p. 72.

COROLLARY (4.7). *If X is any disconnected, completely regular space, then the free topological group $F(X)$ is f-locally precompact.*

PROOF. Let U and V be nonempty, disjoint, open sets whose union is X, and let Z denote the additive group of integers with the discrete topology. Let $g: X \to Z$ be given by $g(x) = 0$ if $x \in U$ and $g(x) = 1$ if $x \in V$. Then g is continuous and its extension $G: F(X) \to Z$ is a continuous, surjective homomorphism. Since Z has a proper open subgroup, it is f-locally precompact and by Theorem (4.6) $F(X)$ is f-locally precompact.

The natural map $\nu: G \to G/C$ is a continuous homomorphism. Therefore Theorem (4.5) can be strengthened as follows.

COROLLARY (4.8). *If G/C is an infinite, 0-dimensional group which is a subgroup of some locally compact group, then G is f-locally precompact.*

If τ denotes the topology on a group G and τ' is a finer topology for which (G, τ') is a topological group, then (G, τ') is f-locally precompact if (G, τ) is, because the identity map is a continuous homomorphism. Any direct product of groups with the product topology is f-locally precompact if at least one of the factors is, because each projection map is a continuous, surjective homomorphism.

THEOREM (4.9). *Suppose H is a disconnected subgroup of G and that W_i, $i = 1, 2, 3, 4$ are symmetric open sets about 0 in G such that*

(a) $2W_{i+1} \subseteq W_i$ *for each* $i = 1, 2, 3$,

(b) W_2 *and* W_4 *are clopen sets,*

(c) W_1 *is precompact with respect to* W_4, *and*

(d) $H - 2W_1 \neq \varnothing$.

Then H is f-locally precompact.

PROOF. For each $i = 1, 2, 3, 4$ let $V_i = W_i \cap H$. It then follows directly that $2V_{i+1} \subseteq V_i$ for each $i = 1, 2, 3$ and that V_2 and V_4 are clopen sets. And since $2V_1 \subseteq 2W_1$, $H - 2V_1 \neq \varnothing$. Since W_1 is precompact with respect to W_4, there is a finite set $K \subseteq W_1$ such that $\{x + W_4 : x \in K\}$ is a cover of W_1. Let K' be the set of all members x of K such that $(x + W_4) \cap H \neq \varnothing$. For each $x \in K'$ choose one $b_x \in (x + W_4) \cap H$. We claim that $\{b_x + V_3 : x \in K'\}$ is a cover of V_1. For let $z \in V_1$. Then there exists $x \in K$ such that $z \in x + W_4$. Since $z \in H$, $(x + W_4) \cap H \neq \varnothing$ and $x \in K'$. Since $b_x \in x + W_4$ there exist $w, w' \in W_4$ such that $z = x + w$ and $b_x = x + w'$. Then $z = b_x - w' + w \in b_x + 2W_4 \subseteq b_x + W_3$. And since z, $b_x \in H$,

$$-b_x + z \in W_3 \cap H = V_3.$$

Therefore $z \in b_x + V_3$. Thus H is f-locally precompact.

5. Examples

There are infinite, 0-dimensional groups which are f-locally precompact but are not subgroups of any locally compact groups. Let Q denote the additive group of rational numbers with its usual topology, and let $Q^{\aleph_0}$ denote the additive group of all rational sequences endowed with the product topology. $Q^{\aleph_0}$ is not a subgroup of any locally compact group. Suppose it is. Then according to Lemma (4.4) there is some neighbourhood U of 0 with the property that any subset of U which is a neighbourhood of 0 is precompact. However if U is any neighbourhood of 0 in $Q^{\aleph_0}$ there exist $n_1, n_2, \ldots, n_k \in N$ and open intervals $U_1, U_2, \ldots, U_k$ about 0 in Q

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such that $V = p_{n_1}^{-1}(U_{n_1}) \cap \ldots \cap p_{n_k}^{-1}(U_{n_k})$ is a subset of U, where each p_{n_i} is a projection map. Choose $n \notin \{n_1, \ldots, n_k\}$ and let W be any bounded interval about 0 in Q. Then V is not precompact with respect to $p_n^{-1}(W)$ and therefore is not precompact. Thus $Q^{\aleph_0}$ is not a subgroup of any locally compact group. Since Q is a 0-dimensional subgroup of the additive group R of real numbers with its usual topology and R is locally compact, Q is f-locally precompact. Since each projection map is a continuous, surjective homomorphism, by Theorem (4.6) $Q^{\aleph_0}$ is f-locally precompact.

Not every totally disconnected group is an S^*-group. Let E denote the additive group consisting of all rational sequences in l_2, where the topology on E is the subspace topology inherited from the topology on l_2 induced by its usual norm. In Hewitt and Ross (1963), p. 65, it is proved that E is totally disconnected and that $U = E \cap \{x \in l_2 : \|x\| < 1\}$ contains no closed and open neighbourhood of 0. If E were an S^*-group there would exist a continuous selfmap f of E and $0 \neq y \in E$ such that $f(0) = 0$ and $f(x) = y$ for every $x \notin U$. Since E is totally disconnected there is some clopen set V which contains 0 but not y. Then $f^{-1}(V)$ is a clopen subset of U, which is a contradiction. Thus E is not an S^*-group and Corollary (2.9) does indeed generalize Theorem (2.3).

Not every infinite disconnected group is f-locally precompact. Suppose G is an infinite, disconnected group such that G/C is of order 2. Using the fact that C cannot contain a proper subset which is clopen in C, it can be shown that G has no proper open disconnected subgroups, and that if U is any open set about 0 such that $U \cap b + U = \varnothing$ for some $b \in G - 2U$, then U cannot properly contain any clopen sets about 0, and hence G cannot be f-locally precompact.

Unfortunately, we do not know whether every group G with the property that G/C is infinite is f-locally precompact.

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