SIMPLEX ALGEBRAS AND THEIR REPRESENTATION

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0. Summary and preliminaries. This paper establishes a relationship (Theorem 4.1) between the approaches of A. C. Thompson **[8, 9]** and E. G. Effros **[2]** to the representation of simplex algebras, that is, real unital Banach algebras that are simplex spaces with the unit for order identity. It proves that the (nonempty) interior of the associated cone is contained in the principal component of the set of all regular elements of the algebra. It also conjectures that each maximal ideal (in the order sense—see below) of a simplex algebra contains a maximal left ideal of the algebra. This conjecture and other aspects of the relationship are illustrated by considering algebras of $n \times n$ real matrices.

A general reference for this paper is **[4]**, by Graham Jameson. The following is a summary of results taken from **[2]**; these are used in the sequel.

Let $(A, \leqq)$ be a partially ordered real linear space, and let P be the associated cone. We call A an *L-space* (respectively, an *M-space*) if it is a Banach lattice such that the norm is absolutely monotonic on A and additive on P (respectively, preserves the sup operation on P). It is called a *simplex space* if it is a Banach space, P is closed, and the dual space A' is an L-space with respect to the dual norm and dual cone P'. We call it a *complete function system* if it has an Archimedean order unit e inducing a complete order unit norm. In this case, the set $P(A)$ of all positive linear functionals on A is a cone in the space dual to A with respect to the order-unit norm. Further, the set $S(A) = \{f \in P(A) : f(e) = 1\}$ is a base for $P(A)$.

Proposition 2.8 of **[2]** proves that A is a complete function system with $S(A)$ for a *simplex* (that is, with $P(A)$ for a lattice cone) if and only if A is a simplex space with e for *order identity* (i.e., $f(e) = 1$ whenever f is in P' and $\|f\| = 1$).

We call a subspace J of A (i) an *order ideal* if $J_P = J \cap P$ is a face of P, (ii) an *ideal* (a *proper ideal*) if, in addition, J_P generates J (and $J \neq A$), and (iii) a *maximal ideal* if it is a proper ideal that coincides with any proper ideal containing it.

Let A be a simplex space with order identity and B the set of all positive functionals of norm one. Section 4 of **[2]** shows that maximal ideals are all closed and in one-to-one correspondence with $\operatorname{ext} B$, the set of extreme points of B (Indeed, a subspace J is a maximal ideal if and only if it is the null space of a member of $\operatorname{ext} B$). Further, Theorem 4.8 of **[2]** shows that there is a *linear, isometric, order isomorphic representation of A onto* $C(\operatorname{ext} B)$ if and only if $\operatorname{ext} B$ is closed in the $\sigma(A', A)$-topology (where A' denotes the dual space of A), which is so if and only if A is an M-space.

We abbreviate maximal left ideal to m.l.i. and maximal ideal (always considered in the order sense) to m.i.

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1. A representation theorem for real unital Banach algebras. This theorem is similar to that for complex unital Banach algebras, proved in [9]. However, some important modifications, characteristic of the real case, occur in its proof, an outline of which is presented below.

Let E be a real Banach algebra with unit e of norm one. Let J be an m.l.i. of E. Then J is closed, $e \notin J$, and, by the Hahn–Banach theorem, $f(e) = \|f\| = 1$ and $f(J) = (0)$ for some f in E', the dual space of E. The following sets are therefore nonempty:

$$B = \{f \in E' : f(e) = \|f\| = 1\},$$

$$M = \{f \in B : f(J) = (0) \text{ for some m.l.i. } J \text{ of } E\},$$

$$\Omega = w^*\text{-cl } M \text{ (weak*-closure of } M),$$

$$S = w^*\text{-cl co } M \text{ (} w^*\text{-cl of convex span of } M),$$

$$K' = \bigcup\{\lambda S : \lambda \geqq 0\},$$

$$K = \{x \in E : f(x) \geqq 0 \text{ for all } f \text{ in } K'\}.$$

The construction of these sets is intrinsic to the norm and algebraic structures of E. B, S, and Ω are compact Hausdorff spaces in the w^*-topology. B and S are also convex. Hence, by Theorems 15.1 and 15.2 of [6], we have $\text{ext}\, S = \text{ext}\, \Omega \neq \emptyset$. The set $X = w^*\text{-cl ext}\, \Omega$ is w^*-compact and Hausdorff and $X \subseteq \Omega \subseteq S \subseteq B$.

Let $C(X)$ be the space of all real-valued continuous functions on X with sup norm. The usual evaluation mapping $x \to \hat{x}$ is a linear, homomorphic representation of E into $C(X)$. Our aim is to study this representation with and without additional conditions on the sets constructed above and on the norm.

As in the complex case [9, § 1], the set K' is a w^*-closed cone. It is contained in the cone generated by B, that is, the set of all functionals that attain their norm at e. It is also normal (indeed, the dual norm is monotonic on K': if f, g, and $g-f$ are in K', then $0 \leqq \|g-f\| = g(e)-f(e) = \|g\|-\|f\|$).

The set K remains unchanged if, in its definition, we replace K' by X (or M or Ω or S). It is a wedge with e for an order unit (indeed, it is an interior point of K: K contains the e-translate of the unit ball). These facts imply that, for each x in E, there is an $\alpha > 0$ such that $\alpha e \pm x$ are in K; equivalently, $-\alpha \leqq f(x) \leqq \alpha$ for all f in X. Therefore the order unit and function seminorms (say $\| \ \|_e$ and $\| \hat{} \ \|$) induced by e and X, respectively, are identical:

$$\begin{aligned} \|x\|_e &= \inf\{\alpha > 0 : |f(x)| \leqq \alpha \quad \text{for all} \quad f \in X\} \\ &= \sup\{|f(x)| : f \in X\} \\ &= \|\hat{x}\|, \end{aligned}$$

where, by evaluation mapping, $\hat{x}$ is an element of $C(X)$. As in the complex case [9, §2, §3], K is weakly (hence norm) closed; it induces an Archimedean ordering (say $\leqq$); its (nonempty) interior is contained in the principal component of the set of all left regular elements of E and Proposition 7.1 shows that, in general, K is not closed under multiplication.

The evaluation mapping is, as in the complex case [**9**, §4], a linear, continuous, order homomorphic representation (in fact, it maps K into the cone of nonnegative functions in $C(X)$). It is further an isomorphism (respectively, a homeomorphism) if and only if K is a cone (respectively, a normal cone).

2. The unit e and one–one representation. Consider the real plane $\mathbb{R}^2$ with complex multiplication. Let $E = \mathbb{R}^2 \oplus \mathbb{R}^2$; then E is, with respect to the maximum norm induced by the usual inner product norms on its components and the usual pointwise operations, a unital Banach algebra with $\mathbb{R}^2 \times \{(0, 0)\}$ and $\{(0, 0)\} \times \mathbb{R}^2$ for its maximal ideals and $e = (1, 0, 1, 0)$ for its unit (identify E with $\mathbb{R}^4$). One has: $B = \{(\lambda, 0, 1-\lambda, 0) \in E' : 0 \leqq \lambda \leqq 1\}$ and $\Omega = M = X = \{(1, 0, 0, 0), (0, 0, 1, 0)\}$. The unit e is clearly not a vertex of the unit ball, that is, $\{e\}$ is properly contained in the intersection of all the hyperplanes supporting the unit ball at e. The representation on X is evidently not one–one.

(Since $K = \{(x_1, x_2, x_3, x_4) \in E : x_1 \geqq 0, x_3 \geqq 0\}$, this example also shows that, in general, K is not a cone, but a wedge.)

This vertex property of e is therefore necessary for one–one representation; however, it may not be sufficient, for, even if e is a vertex, S, Ω, and X may be too small.

As we shall see, for simplex algebras, e is a vertex of the unit ball. This is due to a result of A. C. Thompson [**8**], presented below in a strengthened form; it is proved for the complex case only, the proof being similar for the real case with obvious modifications. See also Bohnenblust and Karlin [**1**] and L. Ingelstam [**3**].

Let A be a complex normed linear space, e an element of A of norm one, and A' the dual of A. The following sets are nonempty:

$$B = \{f \in A' : f(e) = \|f\| = 1\},$$
$$C'' = \text{cone generated by } B,$$
$$P' = C'' - iC'',$$
$$P = \{x \in A : \operatorname{Re} f(x) \geqq 0 \text{ for all } f \text{ in } P'\}.$$

(In the real case, $P' = C''$ and $\operatorname{Re} f(x) = f(x)$.) Clearly, P is a wedge.

2.1. THEOREM. *The following statements are equivalent*:

(a) *$P' - P'$ is a w^*-dense subspace of A'.*
(b) *P is a cone.*
(c) *e is a vertex of the unit ball.*

Proof. It is easily seen that (a) implies (b) and (c).

If $P' - P'$ is not w^*-dense in A', then some element of A' can be strictly separated from the w^*-closure of $P' - P'$, by a w^*-closed hyperplane. Thus, for some *nonzero* v in A, $f(v) = 0$ for all f in $P' - P'$, or, equivalently, for all f in P'.

Hence v and $-v$ are in P and $v \neq 0$. Thus (b) implies (a).

Again, for any complex number μ, $f(e + \mu v) = f(e) = \|f\|$ for all f in C''. Therefore the e-translate of the complex plane determined by v is contained in the intersection of all hyperplanes supporting the unit ball at e. Thus e is not a vertex and (c) implies (a).

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3. In § 1, we saw that $\|x\|_e = \|\hat{x}\|$ for all x in E, where e is the unit in E. Using this fact, we shall now prove the following lemma.

3.1. LEMMA. *$S = \{f \in E' : f(e) = \|f\|_e = 1\}$ and K' is the cone dual to K.*

Proof. Let K'' be the wedge dual to K:

$$K'' = \{f \in E' : f(x) \geqq 0 \quad \text{for all} \quad x \text{ in } K\}.$$

Since K generates E, K'' is in fact a cone. Clearly, K'' contains K'.

Since the ordering induced by K is Archimedean, the $\|\ \|_e$-unit ball is the order interval $[-e, e]$, $f(e) = \|f\|_e$ for all f in K'', and $\|f\|_e$ is a norm on K'' (These follow from Theorem 3.7.2 of **[4]**). Hence the convex set $S' = \{f \in K'' : f(e) = 1\}$ is a base for K''. We observe that

$$S \subseteq S' \subseteq S'' = \{f \in E' : f(e) = 1 = \|f\|_e\}.$$

The assertion follows by proving that $S = S''$.

If $S \neq S''$, then, since S is w^*-closed, some f_0 in S'' can be strictly separated from S by a w^*-closed hyperplane. Hence, for some x_0 in E, we have

$$\sup\{g(x_0) \colon g \in S\} < f_0(x_0). \tag{1}$$

We may now assume that x_0 is in K, without loss of generality. (To see this, observe that the number $\lambda = \inf\{g(x_0) : g \in S\}$ is finite, as S is w^*-compact. Clearly, $x_0 - \lambda e = y_0$ is in K and (1) holds with y_0 in place of x_0.) Thus $g(x_0) \geqq 0$ for all g in S. In (1), we can replace S by X; we therefore have

$$\|x_0\|_e = \|\hat{x}_0\| < f_0(x_0) \leqq \|f_0\|_e \|x_0\|_e = \|x_0\|_e,$$

a contradiction. Thus $S = S''$.

3.2. REMARK. Since $S = S''$, the dual unit balls for the given norm and the order unit seminorm have some parts of their surfaces in common; in particular, the associated dual norms are identical on K'.

3.3. REMARK. The second part of the above lemma, which is an immediate consequence of the first, also follows from the results due to V. L. Klee [**6**, Theorems 3.1 and 2.5]. I am grateful to the referee for bringing this to my attention.

3.4. REMARK. Since the given dual norm is additive on K', we find that, if K' is also a generating lattice cone, E in general lacks just one property for being an L-space, namely, the absolute monotonicity of the norm (Example 4.2 illustrates this). However, this property of the norm is satisfied under the conditions stated in the next theorem, which is our main result.

4. In what follows, we need the results from **[2]**, a summary of which is given in § 0.

4.1. THEOREM. *Let E be a real Banach algebra with unit e of norm one. Then the following statements are equivalent* (notations are as before):

(a) *E is a simplex space with e for order identity.*
(b) $\|x\| = \|x\|_e$ *for all x in E, and S is a simplex.*

When these conditions are satisfied we have $S = B$.

Proof. By Lemma 3.1, K' is the cone dual to K. Hence (a) implies (b), by Proposition 2.8 of [2]. If (b) holds, then the order unit seminorm is a norm, identical with the given norm. Since K is also closed, E is a complete function system such that S is a simplex. By Proposition 2.8 of [2], (a) holds. Thus (a) and (b) are equivalent. Since $S = S''$ and $\|x\| = \|x\|_e$ for all x in E, it follows that $S = B$.

REMARK. Since $S = B$, S and B both induce the cone K and so, by Theorem 2.1, e is a vertex of the unit ball, which is the order interval $[-e, e]$. The dual unit ball is the convex span of B and $-B$. The cones K and K' are normal, closed and generating.

4.2. EXAMPLE. This shows that, in (b) above, the second condition does not imply the first. This is true even if $S = B$.

Let $E = \mathbb{R}^2$, the real plane, with pointwise multiplication and let the unit ball have vertices at the points $e = (1, 1)$ (the unit), $-e$, $\pm(1, 0)$ and $\pm(0, 1)$. The dual norm is given by

$$\|(f_1, f_2)\| = \max\{|f_1|, |f_2|, |f_1+f_2|\}.$$

We have: $B = S = \text{co}\{(1, 0), (0, 1)\}$, which is clearly a simplex; but, since the $\|\ \|_e$-unit ball is the l_∞-unit ball, $\|x\| \neq \|x\|_e$ for all x. Further, $K = K' =$ the positive quadrant is a generating lattice cone; however, the points $(-1, 1)$ and $(3/4, 1/2)$ show that the dual norm is not absolutely monotonic (See Remark 3.4).

We later give Example 7.2 to show that the first condition in (b) does not imply the second.

In view of the above examples, Theorem 4.1, and the intrinsic nature of the sets in § 1, we define a *simplex algebra* as a real Banach algebra with unit of norm one such that it satisfies statement (b) of Theorem 4.1; the associated norm is called a *simplex norm.*

4.3. REMARK. We observe that a finite direct sum of simplex algebras E_i $(i = 1, 2, \ldots, n)$ is a simplex algebra with respect to pointwise multiplication and the l_∞-norm induced by the norms on E_i.

5. The set M and maximal right ideals. We have derived the above results by using m.l.i.'s in the definition of the set M. We clearly obtain results analogous to the above by using m.r.i.'s (maximal right ideals) instead. Let the sets in § 1 be now denoted by M_r, Ω_r, etc. and let us study some particular cases. Note that the set B is independent of m.l.i.'s and m.r.i.'s.

Suppose that S contains S_r. Then K_r contains K. The interior of K is therefore contained in that of K_r, which is itself contained in the principal component of the set of all right regular elements. Since every element of int K (interior of K) is left regular, it follows that int K is contained in the principal component of all regular elements. The same conclusion holds for K_r if S_r contains S.

It is clear that for simplex algebras we have: $K = K_r$ and $S = S_r = B$. Hence the above conclusions hold for simplex algebras. Here one may have $M \neq M_r$; Example 7.8 is a noncommutative simplex algebra for which $M \neq M_r$ (This follows from its counterpart for the complex case [9, Proposition 5.6(a)]). But if M and M_r are identical, then so are S and S_r; Example 7.3 is a simplex algebra that is noncommutative and for which $M = M_r$ (This follows from its counterpart for the complex case [9, Proposition 5.6(b)]).

6. (a) Representation of simplex algebras. Let E be a simplex algebra. Then it is clear from the foregoing that the evaluation mapping is a linear, isometric, order isomorphic representation of E onto a closed linear subspace of $C(X)$. Since $S = B$ implies that $\operatorname{ext} B = \operatorname{ext} \Omega$, Theorem 4.8 of [2] shows that, if $\operatorname{ext} B$ is w^*-closed, then the evaluation mapping is onto $C(X)$.

(b) Relationship between maximal ideals and maximal left ideals. Let E be a simplex algebra. If the set M is w^*-closed, then $M = \Omega$, so that $\operatorname{ext} B = \operatorname{ext} \Omega = \operatorname{ext} M$. Equivalently, every element of $\operatorname{ext} B$ annihilates a maximal left ideal, that is, each maximal ideal, being the kernel of a unique element in $\operatorname{ext} B$, contains an m.l.i. This may be true even if M is not w^*-closed (Example 7.8). Hence we make the following

CONJECTURE. *Each maximal ideal of a simplex algebra contains a maximal left ideal of the algebra.*

Indeed, the examples in the next section establish the following facts. An m.i. may contain uncountably many m.l.i.'s whose union may either be identical with it or be properly contained in it. Conversely, an m.l.i. may be contained in *not more than* a fixed number of m.i.'s, even though this number may not be the same for all simplex algebras. In addition, some m.l.i.'s may be ideals.

7. Examples. These are drawn from the algebra $E = M_n(\mathbb{R})$ of all $n \times n$ $(n \geqq 2)$ real matrices. With respect to the operator norm induced by the maximum modulus norm (l_∞-norm) on $\mathbb{R}^n$, E is a noncommutative real Banach algebra with unit I (the identity matrix) of norm one. The maximal left ideals of E and the computations of various sets and seminorms (See § 1) are similar, with some obvious modifications, to their counterparts in the complex case [9, § 5]. Hence we merely list the results here (the indices i, j, etc., range from 1 to n, and the unit basis vectors e_i of $\mathbb{R}^n$ are regarded as column vectors in describing the elements f_i (within braces) of $\operatorname{ext} B$ below).

$$\begin{aligned}
B = S &= \{f = (p_{ij}) \in E' : |p_{ij}| \leqq p_{ii} \leqq 1 = \textstyle\sum_r p_{rr}\}, \\
\operatorname{ext} B &= \{f_i \in E' : f_i = \{\alpha_t e_i\}, \alpha_i = 1, \alpha_t = \pm 1 \ (t \neq i)\}, \\
K' &= \{(p_{ij}) \in E' : |p_{ij}| \leqq p_{ii}\}, \\
K &= \{x = (a_{ij}) \in E : \textstyle\sum_{j \neq i} |a_{ij}| \leqq a_{ii}\}, \\
\|x\|_e &= \|\hat{x}\| = \|x\| = \max_i \{\textstyle\sum_j |a_{ij}|\}.
\end{aligned} \tag{2}$$

7.1. PROPOSITION. *In general, K is not closed under multiplication.*

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REMARK. This shows that the representation theorem obtained above is different from the representation theorem obtained by Kung Fu Ng [7] for Banach algebras with given cones closed under multiplication.

Proof. In (2) above, K' contains K. The element $x=(a_{ij})$ whose only nonzero entries are $a_{11}=a_{12}=a_{22}=2$ and $a_{21}=1$, belongs to K and (therefore) to K'. But the element $x^2=(b_{ij})$, whose only nonzero entries are $b_{11}=b_{22}=6$, $b_{21}=4$ and $b_{12}=8$, does not belong to K' and (therefore) does not belong to K. The assertion follows.

7.2. PROPOSITION. *In general, even if $S=B$, the first condition in statement* (b) *of Theorem* 4.1 *does not imply the second.*

Proof. From (2) it follows that for E one has: $S=B$ and $\|x\|=\|x\|_e$ for all x. However, $\operatorname{ext} B=\operatorname{ext} S$ has $n2^{n-1}$ points. Therefore S is a simplex if and only if $n2^{n-1}=n^2$ (dimension of E), which is so if and only if $n=2$. Hence, for all $n>2$, E is not a simplex algebra.

We now consider examples which illustrate the conjecture and several aspects of the relationship mentioned in §6(b).

7.3. *The algebra $M_2(\mathbb{R})$.* From the proof of Proposition 7.2 and Theorem 4.1, it follows that the l_∞-operator norm is a simplex norm for $M_2(\mathbb{R})$. B is thus a solid tetrahedron with vertices at the points f_i $(1\leqq i\leqq 4)$ given respectively by

$$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix},\quad \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix},\quad \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix},\quad \begin{bmatrix}0 & 0\\ -1 & 1\end{bmatrix}.$$

M is the union of the convex spans of the pairs of points $\{f_1,f_2\}$, $\{f_3,f_4\}$, $\{f_1,f_3\}$ and $\{f_2,f_4\}$. It is clearly a part of the boundary of B. We have: $K'=K$.

The kernels (say) $N(f_i)$ of the f_i are m.i.'s. The maximal left ideal (m.l.i.) generated by f_2 is $J_{f_2}=N(f_1)\cap N(f_3)$; similarly, $J_{f_3}=N(f_2)\cap N(f_4)$. Therefore the conjecture of §6(b) is verified and, in addition, each m.i. contains exactly one m.l.i., which is also an ideal. Each m.l.i. is contained in at most two m.i.'s.

7.4. REMARK. Other simplex norms for $M_2(\mathbb{R})$ include the operator norms induced by vector norms p of $\mathbb{R}^2$ such that the p-unit ball is a parallelogram. We conjecture that there are no other simplex norms.

7.5. REMARK. Noncommutative simplex algebras of dimension exceeding 4 include, in the sense of Remark 4.3, finite direct sums of $M_2(\mathbb{R})$ with itself and (or) $M_1(\mathbb{R})$. These are in one-to-one correspondence with decompositions of $\mathbb{R}^n$ into direct sums of $\mathbb{R}^2$ and (or) $\mathbb{R}$.

7.6. REMARK. Since the l_∞-operator norm is a simplex norm for $M_n(\mathbb{R})$ if and only if $n=2$, one may try to see whether it continues to be a simplex norm for suitable subalgebras of $M_n(\mathbb{R})$ $(n>2)$. One such is clearly the finite direct sum discussed above. Another, which is not such a finite direct sum, is considered below.

7.7. *A noncommutative subalgebra of $M_3(\mathbb{R})$.* This consists of the elements of the form (a_{ij}) with $a_{13}=a_{23}=a_{21}=a_{31}=0$. Clearly, it does not arise from any decomposition of $\mathbb{R}^3$.

It can be verified that the l_∞-operator norm is a simplex norm for this algebra. Its m.l.i.'s (say) J_1, J_2 and $J_{\alpha\beta}$ consist of the elements that are respectively of the forms

$$\begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & b & 0 \\ 0 & c & 0 \\ 0 & d & e \end{bmatrix}, \quad \begin{bmatrix} a & -\alpha a & 0 \\ 0 & 0 & 0 \\ 0 & -\beta e & e \end{bmatrix},$$

where $a, b, c, d, e, \alpha, \beta$ are real numbers. The extreme points (say) f_i ($1 \leqq i \leqq 5$) of B are respectively given by

$$\begin{bmatrix} 1 & \pm 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \pm 1 & 1 \end{bmatrix}.$$

The m.i.'s, that is, the kernels (say) $N(f_i)$ of these (extreme) points consist of the elements that are respectively of the forms

$$\begin{bmatrix} a & -a & 0 \\ 0 & c & 0 \\ 0 & d & e \end{bmatrix}, \quad \begin{bmatrix} a & a & 0 \\ 0 & c & 0 \\ 0 & d & e \end{bmatrix}, \quad \begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & d & e \end{bmatrix}, \quad \begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & -e & e \end{bmatrix}, \quad \begin{bmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & e & e \end{bmatrix}.$$

Similarly, the m.l.i.'s annihilated by f_i consist of the elements that are respectively of the forms

$$\begin{bmatrix} a & -a & 0 \\ 0 & 0 & 0 \\ 0 & -\beta e & e \end{bmatrix}, \quad \begin{bmatrix} a & a & 0 \\ 0 & 0 & 0 \\ 0 & -\beta e & e \end{bmatrix}, \quad \begin{bmatrix} a & -a\alpha & 0 \\ 0 & 0 & 0 \\ 0 & -\beta e & e \end{bmatrix}, \quad \begin{bmatrix} a & -\alpha a & 0 \\ 0 & 0 & 0 \\ 0 & -e & e \end{bmatrix}, \quad \begin{bmatrix} a & -\alpha a & 0 \\ 0 & 0 & 0 \\ 0 & e & e \end{bmatrix}.$$

Here, α and β are arbitrary. Hence each f_i annihilates uncountably many m.l.i.'s; if I_i denotes the union of these m.l.i.'s, we then have: $I_3 = N(f_3)$ and, for all $i \neq 3$, $N(f_i)$ contains I_i strictly. It is clear that the m.l.i. $J_{\alpha\beta}$ (with $\alpha = \beta = 1$) is contained in (indeed, equal to) the intersection of $N(f_i)$ ($i = 1, 3, 4$). Thus each m.l.i. is contained in at most three m.i.'s, as compared with two for $M_2(\mathbb{R})$.

Of course, the conjecture in § 6(b) is verified together with some of the other observations made following it.

We now consider a simplex algebra of dimension less than four.

7.8. *A noncommutative subalgebra of* $M_2(\mathbb{R})$. This is the algebra E of all 2×2 upper-triangular real matrices. The l_∞-operator norm is a simplex norm for E. This follows from its counterpart for the complex case [**9**, Proposition 5.5]. The extreme points of B are

$$f_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore B is the triangle with f_i as vertices and, as in the complex case, M is B without the interior of the set co $\{f_1, f_2\}$. Thus M is not (w^*-) closed and still $\operatorname{ext} B \subseteq M$, i.e., $\operatorname{ext} B = \operatorname{ext} M$. This verifies the conjecture of § 6. In addition, the following are also true.

The m.l.i.'s are J_1 and J_{2k} consisting of the elements that are respectively of the forms

$\begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix}$ and $\begin{bmatrix} a & -ka \\ 0 & 0 \end{bmatrix}$. $N(f_i)$, the null spaces of f_i or the m.i.'s of E, consist of the elements that are respectively of the forms

$$\begin{bmatrix} a & a \\ 0 & c \end{bmatrix}, \quad \begin{bmatrix} a & -a \\ 0 & c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

It is clear that

$$J_{2(-1)} = N(f_1) \cap N(f_3),$$
$$J_{21} = N(f_2) \cap N(f_3),$$
$$\bigcup_{k \in R} J_{2k} = N(f_3).$$

These show that only two m.l.i.'s are ideals and no m.l.i. is contained in more than two m.i.'s.

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