

On the existence and uniqueness of solutions of parabolic equations

K.L. Teo

Recently, Eklund (*Proc. Amer. Math. Soc.* 47 (1975), 137-142) has shown that to each continuous function F on $\partial_p Q \cong \{\partial\Omega \times [0, T]\} \cup \{\Omega \times (0)\}$ there is a unique solution to the boundary value problem

$$\begin{cases} (L\phi)(x, t) = 0 & , (x, t) \in \Omega \times (0, T] , \\ \phi(x, t) = F(x, t) & , (x, t) \in \partial_p Q , \end{cases}$$

where L is a linear second order parabolic operator in divergence form, $\Omega \subset \mathbb{R}^n$ is a bounded domain with compact closure and $\partial\Omega$ denotes its boundary. In this note, it is shown that the existence theorem of Eklund remains valid for the following boundary problem

$$\begin{cases} (L\phi)(x, t) = f(x, t) + \{g_i(x, t)\}_{x_i} & , (x, t) \in \Omega \times (0, T] , \\ \phi(x, t) = F(x, t) & , (x, t) \in \partial_p Q . \end{cases}$$

1. Introduction and preliminary results

Let $\Omega \subset \mathbb{R}^n$ be open with compact closure $\bar{\Omega} = \Omega \cup \partial\Omega$. Let $T > 0$ and $Q \cong \Omega \times (0, T]$. Further, let $\partial_p Q \cong \{\partial\Omega \times [0, T]\} \cup \{\Omega \times (0)\}$ denote the parabolic boundary of Q .

Consider the system described by the following linear second order partial differential equation of parabolic type with non-homogeneous

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boundary conditions:

$$S : \begin{cases} (L\phi)(x, t) = f(x, t) + (g_i(x, t))_{x_i}, & (x, t) \in Q, \\ \phi(x, t) = F(x, t) & , (x, t) \in \partial_p Q, \end{cases}$$

where the (parabolic) differential operator L is given by

$$Lu \triangleq u_t - \left\{ a_{ij}(x, t)u_{x_i} + a_j(x, t)u \right\}_{x_j} - b_j(x, t)u_{x_j} - c(x, t)u$$

with

$$\psi_t \triangleq \frac{\partial \psi}{\partial t}, \quad \psi_{x_j} \triangleq \frac{\partial \psi}{\partial x_j}, \quad \text{and} \quad \psi_{x_i x_j} \triangleq \frac{\partial^2 \psi}{\partial x_i \partial x_j}.$$

Note that the convention adopted above and throughout the rest of the note is to take summation up to n over repeated indices.

The given functions and solutions will lie in multidimensional L^p spaces and Sobolev space $L^2[0, T, H^{1,2}(\Omega)]$. These spaces are defined in detail by Aronson and Serrin [2].

Throughout the note, the coefficients of L are assumed to satisfy the following assumptions which will be referred to collectively as (A).

(i) The $a_{ij}(x, t)$ are measurable functions in (x, t) with $|a_{ij}(x, t)| \leq M < \infty$ almost everywhere in Q , and for some $\lambda > 0$,

$$a_{ij}(x, t)z_i z_j \geq \lambda |z|^2 = \lambda \sum_{i=1}^n z_i^2 \quad \text{for all } z \in \mathbb{R}^n \quad \text{and almost all}$$

$(x, t) \in Q$.

(ii) $c(x, t) \in L^q[0, T; L^p(\Omega)]$ for some pair p, q satisfying

$$1 < p, \quad q \leq \infty,$$

(*) $n/2p + 1/q < 1$.

(iii) $b_j(x, t), a_j(x, t) \in L^q[0, T, L^p(\Omega)]$ for some pair p, q satisfying

$$2 < p, \quad q \leq 0,$$

$$(**) \quad n/2p + 1/q < \frac{1}{2} .$$

For convenience of reference, one basic definition and two basic theorems of Aronson [1], and the main theorem of Eklund [3] are quoted without proof.

DEFINITION 1. Let L be as described above,

$f(x, t) \in L^q[0, T; L^p(\Omega)]$ where p, q satisfy (*) and $g_i(x, t) \in L^q[0, T; L^p(\Omega)]$ where p, q satisfy (**). Then, $u(x, t)$ is said to be a weak solution of the boundary value problem

$$(1) \quad \begin{cases} (Lu)(x, t) = f(x, t) + \{g_i(x, t)\}_{x_i} & , (x, t) \in Q , \\ u(x, t) = 0 & , (x, t) \in S \triangleq \partial\Omega \times [0, T] , \\ u(x, 0) = u_0(x) & , x \in \Omega , \end{cases}$$

if

$$(i) \quad u \in L^2[\delta, T; H^1_{loc}(\Omega)] \cap L^\infty[\delta, T; L^2_{loc}(\Omega)] \quad \text{for each } \delta > 0 ,$$

$$(ii) \quad u_0(x) \in L^2(\Omega) ,$$

and if, for each $z(x, t) \in C^1(\bar{Q})$ with compact support in Ω ,

$$(iii) \quad \int_Q \left[a_{ij} u_{x_i} z_{x_j} + a_j z_{x_j} u - b_j u_{x_j} z - c u z - u z_t \right] dx dt \\ = \int_Q \left[f z - g_i z_{x_i} \right] dx dt + \int_\Omega u_0(x) z(x, 0) dx ,$$

and

$$(iv) \quad \lim_{t \downarrow 0} \int_\Omega u(x, t) z(x, t) dx = \int_\Omega u_0(x) z(x, 0) dx .$$

Aronson and Serrin [2] have shown that every weak solution of (1) in Q has a representative that is continuous in Q . Henceforth, u will denote the continuous representative of a given weak solution.

THEOREM A. *Suppose that the assumption (A) is satisfied and that $u_0(x)$, $f(x, t)$, $g_i(x, t)$ are as described in Definition 1. Then there is a unique weak solution u of the boundary value problem (1).*

A proof of this theorem is given by Aronson in [1].

THEOREM B. *Let u be a solution of the problem (1) with $f \equiv 0$, $g_i \equiv 0$ on Q , and $u_0 \in L^2(\Omega)$. Further, let $\zeta = \zeta(x)$ be a non-negative smooth function such that $\zeta u \in L^2[0, T, H_0^1, 2(\Omega)]$. Then there is a positive constant C such that*

$$\|\zeta u\|_{2, \infty}^2 + \|\zeta u_x\|_{2, 2}^2 \leq C \left\{ \int_{\Omega} \zeta^2 u_0^2 dx + \|\zeta_x u\|_{2, 2}^2 \right\}.$$

A proof of this theorem can be found in [1].

THEOREM C. *Suppose that the assumption (A) is satisfied and that $F(x, t)$ is continuous on S so that $F(x, 0) \in L^2(\Omega)$. Then there is a unique weak solution u of the boundary value problem*

$$(2) \quad \begin{cases} (Lu)(x, t) = 0 & , (x, t) \in Q, \\ u(x, t) = F(x, t) & , (x, t) \in \partial_p Q. \end{cases}$$

A proof of this theorem is given by Eklund in [3].

2. Main result

We are now to present our main result in the following theorem. Its proof is based on Theorems (A), (B), and (C) given in the previous section.

THEOREM 1. *Consider the boundary value problem (S). Suppose that the assumption (A) is satisfied and that $F(\cdot, \cdot)$ is continuous on S and $F(\cdot, 0) \in L^2(\Omega)$. Then there is a unique weak solution u of the problem (S).*

Proof. In view of Theorems (A) and (C), we note that the boundary value problems (1) and (2) possess unique solutions u and v respectively. Define

$$\psi \triangleq u + v.$$

We shall show that ψ is the unique weak solution of the problem (S). First of all, we see that for each $z \in C^1(\bar{Q})$ with compact support in Ω ,

$$\begin{aligned} \int_Q (L\psi)z dxdt &= \int_Q (Lu)z dxdt + \int_Q (Lv)z dxdt \\ &= \int_Q \left[fz - g_i z x_i \right] dxdt + \int_\Omega F(x, 0)z(x, 0) dx . \end{aligned}$$

Further,

$$\begin{aligned} \lim_{t \downarrow 0} \int_\Omega \psi(x, t)z(x, t) dx &= \lim_{t \downarrow 0} \left\{ \int_\Omega u(x, t)z(x, t) dx + \int_\Omega v(x, t)z(x, t) dx \right\} \\ &= \lim_{t \downarrow 0} \int_\Omega u(x, t)z(x, t) dx + \lim_{t \downarrow 0} \int_\Omega v(x, t)z(x, t) dx \\ &= \int_\Omega F(x, 0)z(x, 0) dx . \end{aligned}$$

This implies that ψ is a weak solution of the problem (S).

It remains to show the uniqueness. For this, we assume that u_1 and u_2 are two distinct solutions of the problem (S). Then, define

$$\phi \equiv u_1 - u_2$$

and consider the following boundary value problem

$$(3) \quad \begin{cases} (L\phi)(x, t) = 0 , & (x, t) \in Q , \\ \phi(x, t) = 0 , & (x, t) \in \partial_p Q . \end{cases}$$

Therefore, it follows from Theorems (A) and (B) that ϕ is also the unique solution of the problem (3) and is equal to zero identically on \bar{Q} . This simply implies that $u_1 \equiv u_2$ on \bar{Q} . Thus, we can conclude that the problem (S) admits only a unique weak solution ψ . This completes the proof.

References

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School of Mathematics,
University of New South Wales,
Kensington,
New South Wales.