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Three formulae for eigenfunctions of integrable Schrödinger operators

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Abstract. We give three formulae for meromorphic eigenfunctions (scattering states) of Sutherland's integrable N-body Schrödinger operators and their generalizations. The first is an explicit computation of the Etingof–Kirillov traces of intertwining operators, the second an integral representation of hypergeometric type, and the third is a formula of Bethe ansatz type. The last two formulas are degenerations of elliptic formulas obtained previously in connection with the Knizhnik–Zamolodchikov–Bernard equation. The Bethe ansatz formulas in the elliptic case are reviewed and discussed in more detail here: Eigenfunctions are parametrized by a 'Hermite–Bethe' variety, a generalization of the spectral variety of the Lamé operator. We also give the q-deformed version of our first formula. In the scalar sl_N case, this gives common eigenfunctions of the commuting Macdonald–Ruijsenaars difference operators.

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1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra, with a non-degenerate ad-invariant bilinear form $(\,,)$, and a fixed Cartan subalgebra $\mathfrak{h}$. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$. For each $\alpha \in \Delta$ let $e_\alpha \in \mathfrak{g}$ be a corresponding root vector, normalized so that $(e_\alpha, e_{-\alpha}) = 1$.

Suppose that U is a highest weight representation of $\mathfrak{g}$ with finite dimensional zero-weight space (the space of vectors in U annihilated by $\mathfrak{h}$) $U[0]$. We consider in this paper the differential operator

$$H = -\triangle + \sum_{\alpha\in\Delta} \frac{\pi^2}{\sin^2(\pi\alpha(\lambda))} e_\alpha e_{-\alpha}, \tag{1}$$

acting on functions on $\mathfrak{h}$ with values in $U[0]$. The Laplacian $\triangle$ is the operator $\sum_\nu \partial^2/\partial\lambda_\nu^2$ in terms of coordinates $\lambda_\nu = (b_\nu, \lambda)$ for any orthonormal basis $b_1, \dots, b_r$ of $\mathfrak{h}$. As remarked in [3], if $\mathfrak{g} = \mathrm{sl}_N$, and U is the symmetric power

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$S^{pN}\mathbb{C}^N$ of the defining representation $\mathbb{C}^N$, then $U[0]$ is one-dimensional and this differential operator reduces to Sutherland's integrable N-body Schrödinger operator [16] in one dimension with coupling constant $p(p+1)$. In suitable variables, H is in this case proportional to

$$H_S = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + p(p+1)\sum_{j\neq l} V(x_j - x_l), \tag{2}$$

with $V(x) = 1/\sin^2(x)$ (trigonometric model) or $V(x) = 1/\sinh^2(x)$ (hyperbolic model). We refer to these special cases as the scalar cases.

We consider in this paper eigenfunctions (functions ψ with $H\psi = \varepsilon\psi$ for some $\varepsilon \in \mathbb{C}$) of the form $\mathrm{e}^{2\pi i(\xi,\lambda)} f(\lambda)$, where f is meromorphic and regular as $\alpha(\lambda) \to i\infty$, $\forall\alpha \in \Delta$ (a more precise definition is given below). In the language of N-body Schrödinger operators these are *scattering states* for the hyperbolic models. It turns out that for generic ξ the space $E(\xi)$ of such eigenfunctions is finite dimensional and isomorphic to $U[0]$.

We give three formulae for these eigenfunctions. The first one (Theorem 3.1) is an explicit computation of an expression of Etingof and Kirillov in terms of traces of certain intertwining operators. The reason that these combinations of traces can be computed explicitly is that they turn out to be the same as for the corresponding Lie algebra without Serre relations (Proposition 9.2). This observation reduces the computation of traces to a tractable combinatorial problem. We also give the q-deformed version of this formula (Theorem 8.1). The second formula, an integral representation (Theorem 4.1), follows from the fact that eigenfunctions can be obtained as suitable limits of solutions of the Knizhnik–Zamolodchikov–Bernard equation of conformal field theory, for which we gave explicit solutions in [7]. The third formula (Theorem 5.1) is of the Bethe ansatz (or Hermite) type: one has an explicit expression of a function depending on parameters $T \in \mathbb{C}^n$. This function is an eigenfunction if the parameter lie on an algebraic variety which we call Hermite–Bethe variety. We have a regular map p from the Hermite–Bethe variety to $\mathfrak{h}$, sending T to the corresponding value of ξ. Conjecturally, p is a covering map and a basis of $E(\xi)$ for generic ξ is obtained by taking the eigenfunctions corresponding to the points in the fiber $p^{-1}(\xi)$. This conjecture is proved in some cases including the scalar case (Theorem 5.3).

The three formulae hold for generic values of the spectral parameter ξ. In Section 6 we study trigonometric polynomial solutions of the igenvalue problem $H\psi = \varepsilon\psi$. These (Weyl antiinvariant) eigenfunctions are related to multivariable Jacobi polynomials (or Jack polynomials). Formulae for these polynomials are obtained as the spectral parameters tends to an antidominant integral weight. The construction involves the construction of the *scattering matrices* for the problem, which give a solution of the Yang–Baxter equation. It would be interesting to generalize this construction to the q-deformed case, which would give formulae for Macdonald polynomials. We give such formulae in Section 8 as a conjecture.

This conjecture can be proved by the same method as in the classical case for sl_2 and sl_3, but the regularity property of Weyl antiinvariant eigenfunction appears to be more difficult to prove in the general case.

The fact that we have three different formulae for the same thing can be explained in informal terms as follows. Our second formula is an integral depending on a complex parameter $\kappa \neq 0$. Its form is

$$\psi(\lambda) = \mathrm{e}^{2\pi i \xi(\lambda)} \int_{\gamma(\kappa)} \Phi_0(T)^{1/\kappa} \omega(T, \lambda).$$

The function Φ_0 is a many-valued holomorphic function on $D_n = (\mathbb{C} - \{0,1\})^n - \cup_{i<j}\{T \mid T_i = T_j\}$ and $\omega(T, \lambda)$ is a rational $U[0]$-valued differential n-form on D_n. The cycle $\gamma(\kappa)$ has coefficients in the local system determined by $\Phi_0(T)^{1/\kappa}$. It turns out that, for all κ, $\gamma(\kappa)$, ψ is an eigenfunction of H with the same eigenvalue $4\pi^2(\xi, \xi)$, and that (if ξ is generic) for any fixed generic κ all eigenfunctions can be obtained by suitable $\gamma(\kappa)$. One can thus consider suitable families of cycles parametrized by κ in the limits $\kappa \to 0$, $\kappa \to \infty$. In the former case the homology reduces to ordinary homology, and the answer is given in terms of residues of ω (our first formula). In the latter limit, the integral can be evaluated by the saddle point method and the eigenfunctions are given by evaluating ω at the critical points of Φ_0 (our third formula).

This reasoning turns out to be useful to write down the formulas, (and to ‘understand’ them) but it is then easier to prove them by other methods.

In Section 7, we generalize our results on Bethe ansatz eigenfunctions (the third formula) to the elliptic case. The Schrödinger operator in this case has the Weierstrass $\wp$-function as a potential:

$$H_e = -\triangle + \sum_{\alpha \in \Delta} \wp(\alpha(\lambda)) e_\alpha e_{-\alpha} + \text{const.} \tag{3}$$

In the scalar case $\mathfrak{g} = \mathrm{sl}_N$, $U = S^{pN}\mathbb{C}^N$, we obtain (2) with $V = \wp$. Bethe ansatz eigenfunctions of H_e were given in [7]. After reviewing this construction, we give a result on completeness of Bethe states (Theorem 7.3) parallel to the trigonometric case, in the scalar case and the case of the adjoint representation of sl_N: for generic values of the spectral parameter ξ there exist $\dim(U[0])$ solutions of the Bethe ansatz equations corresponding to linearly independent quasi-periodic eigenfunctions with multiplier given by ξ.

The formulae discussed here in the trigonometric case were obtained by considering the trigonometric limit of solutions of the KZB equations for an elliptic curve with one marked point. The same construction could be done in the case of N marked points. It turns out that in the trigonometric limit one of the equations always reduces to an eigenvalue equation for H. The new feature is that U is the tensor product of N representations. The remaining commuting operators define differential equations in the space of eigenfunctions of H with a fixed eigenvalue.

In the q-deformed case, we could only generalize the first formula. It would be interesting to find the second and third formula in the q-deformed case.

The organization of the paper is as follows: in Section 2 we define and describe the space of eigenfunctions we consider in the trigonometric case. Three formulae for these eigenfunctions are given in Section 3 in terms of coordinates of singular vectors, in Section 4 as integrals, and in Section 5 by a Bethe ansatz. In Section 6, the action of the Weyl group is discussed along with the relation to multivariable Jacobi polynomials at special values of the spectral parameter. In Section 7 we discuss the Bethe ansatz in the elliptic case. Section 8 gives a generalization of the first formula to the q-deformed case. The proof of the first formula is contained in Section 9.

We conclude this introduction by fixing some notations and conventions. The Cartan subalgebra $\mathfrak{h}$ will be often identified with its dual via $(\,,\,)$. We fix a set of simple roots $\alpha_1, \dots, \alpha_r \in \Delta$, and write Q for the root lattice $\oplus_i \mathbb{Z}\alpha_i$. Its positive part $\oplus_i \mathbb{N}\alpha_i$, with $\mathbb{N} = \{0, 1, \dots,\}$, will be denoted by Q_+. We denote by ρ half the sum of the positive roots. We will use the partial ordering $\beta \geqslant \beta'$ iff $\beta - \beta' \in Q_+$ on $\mathfrak{h}$ and write $\beta > \beta'$ if $\beta \geqslant \beta'$ but $\beta \neq \beta'$. We set $\Delta_+ = \Delta \cap Q_+$. If $\beta \in Q$, we write $|\beta|^2 = (\beta, \beta)$. We will also use the notation $|A|$ to denote the cardinality of a set A. The group of permutations of n letters is denoted by S_n.

2. The ψ functions

We introduce a space of meromorphic functions on $\mathfrak{h}$ that is preserved by the differential operator H. For $\beta \in \mathfrak{h}^*$, let

$$X_\beta(\lambda) = \mathrm{e}^{-2\pi i\beta(\lambda)} \tag{4}$$

and write $X_\beta = X_i$ if $\beta = \alpha_i$. Let A be the algebra of functions on $\mathfrak{h}$ which can be represented as meromorphic functions of $(X_1, \dots, X_r) \in \mathbb{C}^r$ with poles belonging to set $\cup_{\alpha \in \Delta}\{X_\alpha = 1\}$. For instance, if α is a positive root, the functions

$$\frac{1}{\sin^2(\pi\alpha(\lambda))} = -\frac{4X_\alpha}{(1 - X_\alpha)^2}, \qquad \cot(\pi\alpha(\lambda)) = i\frac{1 + X_\alpha}{1 - X_\alpha}, \tag{5}$$

belong to A. The algebra A is a subalgebra of $\hat{A} = \mathbb{C}[[X_1, \dots, X_r]]$

If $\xi \in \mathfrak{h}^*$, we introduce the A-module $A(\xi)$ and the $\hat{A}$-module $\hat{A}(\xi)$ of functions of the form $\exp(2\pi i\xi)f$ where $f \in A$, resp. $f \in \hat{A}$. These modules are preserved by derivatives with respect to λ. Therefore H preserves the spaces $A(\xi) \otimes U[0]$, $\hat{A}(\xi) \otimes U[0]$.

Thus any $\psi \in \hat{A}(\xi) \otimes U[0]$ has the form

$$\psi(\lambda) = \sum_{\beta \in Q} X_{\beta - \xi}(\lambda)\psi_\beta,$$

with $\psi_\beta \in U[0]$ vanishing if $\beta \notin Q_+$.

THEOREM 2.1. *For generic* $\xi \in \mathfrak{h}^*$, *and any non-zero* $u \in U[0]$, *there exists a unique* $\psi = \Sigma_\beta X_{\beta-\xi}\psi_\beta \in \hat{A}(\xi) \otimes U[0]$, *such that* $H\psi = \varepsilon\psi$, *for some* $\varepsilon \in \mathbb{C}$, *and such that* $\psi_0 = u$. *Moreover,* $\varepsilon = (2\pi)^2(\xi, \xi)$, *and* $\psi \in A(\xi) \otimes U[0]$.

The existence and uniqueness of ψ as a formal power series was proved in the scalar case by Heckman and Opdam (see Section 3 of [8]), who generalized a classical construction of Harish-Chandra (see [9], IV.5). The fact that the series converges to a meromorphic function seems to be new.

We next prove this theorem except for the statement that the formal power series ψ actually belongs to $A(\xi) \otimes U[0]$, which will follow from the explicit formulas for ψ given below.

Proof. The idea is that the eigenvalue equation $H\psi = \varepsilon\psi$ is a recursion relation for the coefficients ψ_β. We use the fact that α and $-\alpha$ give the same contribution to the sum in H, to apply (5). With the formula $\triangle X_{\beta-\xi}(\lambda) = (2\pi i)^2(\beta - \xi, \beta - \xi)X_{\beta-\xi}(\lambda)$, we see that $H\psi = \varepsilon\psi$ is equivalent to the recursion relation

$$[(\beta - \xi, \beta - \xi) - (2\pi)^{-2}\varepsilon]\psi_\beta = \sum_{j>0} 2j \sum_{\alpha\in\Delta_+} e_\alpha e_{-\alpha}\psi_{\beta-j\alpha}, \tag{6}$$

for the coefficients $\psi_\beta \in U[0]$. The sum on the right-hand side has only finitely many nonzero terms. The initial condition for this recursion is $\psi_0 = u \neq 0$. For $\beta = 0$, the equation reads then $(\xi, \xi) - (2\pi)^{-2}\varepsilon = 0$. Thus there is a solution only if $\varepsilon = (2\pi)^2(\xi, \xi)$. For generic ξ, the coefficient of ψ_β does not vanish if $\beta \neq 0$, so (6) gives ψ_β in terms of $\psi_{\beta'}$ with $\beta' < \beta$. We conclude that, for generic ξ, (6) has a solution if and only if $\varepsilon = (2\pi)^2(\xi, \xi)$, and this solution is unique. □

DEFINITION. We denote by $E(\xi)$ the vector space of functions $\psi \in A(\xi) \otimes U[0]$ such that $H\psi = (2\pi)^2(\xi, \xi)\psi$.

Remark 1. It looks as if the definition of $E(\xi)$ should depend on the choice of simple roots, but this is not so. From the explicit expressions given below it follows that, for any set R of simple roots, the functions in $E(\xi)$ can be analytically continued to functions in $A_R(\xi) \otimes U[0]$, where $A_R(\xi)$ is the space $A(\xi)$ defined using R (see Section 6).

Remark 2. H is part of a commutative r-dimensional algebra D of Weyl-invariant differential operators whose symbols are (Weyl-invariant) polynomials on $\mathfrak{h}^*$. These operators have coefficients which are polynomials in $\cot(\pi\alpha(\lambda))$, $\alpha \in \Delta_+$. It follows that $A(\xi)$ is preserved by D, see (5), and that the functions ψ of the theorem are common eigenfunctions of all operators in D. Indeed if $L \in D$ with symbol P, then L preserves $E(\xi)$ since it commutes with H. If $\psi \in E(\xi)$ has leading term $\mathrm{e}^{2\pi i\xi(\lambda)}u$, $u \in U[0]$, then $L\psi$ has leading term $P(2\pi i\xi)\mathrm{e}^{2\pi i\xi(\lambda)}u$. Thus $L\psi = P(2\pi i\xi)\psi$.

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Remark 3. If $\mathfrak{g}$ is a general Kac-Moody Lie algebra and U is a $\mathfrak{g}$-module with finite dimensional zero-weight space $U[0]$, H is still a well-defined endomorphism of $\hat{A}(\xi) \otimes U[0]$: the (possibly infinite) sum over Δ_+ gives only finitely many contributions in each fixed degree, as $\sin^{-2}(\pi\alpha(\lambda)) = O(X_\alpha)$, see (5). The above theorem holds (except for the statement that $\psi \in A(\xi) \otimes U[0]$) with the same proof.

Etingof and Kirillov [3] gave a representation theoretic construction of eigenfunctions: given ξ generic and $u \in U[0]$, let $M_{\xi-\rho}$ be the Verma module with highest weight $\xi - \rho$. Then there exists a unique homomorphism $\Phi_u \in \mathrm{Hom}_{\mathfrak{g}}(M_{\xi-\rho}, M_{\xi-\rho} \otimes U)$, such that the generating vector $v_{\xi-\rho} \in M_{\xi-\rho}$ is mapped to $v_{\xi-\rho} \otimes u + \cdots$, up to terms whose first factor is of lower weight. Then $\psi \in \hat{A}(\xi) \otimes U[0]$ is the ratio of formal power series

$$\psi_u(\lambda) = \frac{\mathrm{tr}_{M_{\xi-\rho}} \Phi_u \exp(2\pi i\lambda)}{\mathrm{tr}_{M_{-\rho}} \exp(2\pi i\lambda)}. \tag{7}$$

The traces are formal power series whose coefficients are traces over the finite dimensional weight spaces of the Verma modules. By definition, the trace of a map $M \to M \otimes U$ is the canonical map (for finite dimensional M)

$$\mathrm{tr}_M\colon \mathrm{Hom}_{\mathbb{C}}(M, M \otimes U) \simeq M^* \otimes M \otimes U \to U.$$

The numerator in (7) belongs to $\hat{A}(\xi - \rho) \otimes U[0]$, and the denominator to $\hat{A}(-\rho)$ with leading coefficient 1, so the ratio is a well-defined element of $\hat{A}(\xi) \otimes U[0]$. Combining this result with Theorem 2.1, we get

PROPOSITION 2.2. *For any generic $\xi \in \mathfrak{h}^*$, the map $u \to \psi_u$ defined by (7) is an isomorphism from $U[0]$ to $E(\xi)$.*

Remark. Etingof and Styrkas [6] showed that (7) viewed as a function of ξ, coincides the Chalykh–Veselov ψ-function (see [2], and [6] for the matrix case considered in this paper), which is defined in terms of its behavior as a function of ξ.

3. The first formula

Our first formula is an explicit calculation of the trace of the homomorphism $\Phi\colon M_{\xi-\rho} \to M_{\xi-\rho} \otimes U$ of the previous section. The image of the generating vector is a singular vector of weight $\xi - \rho$ (a vector of weight $\xi - \rho$ killed by $e_\alpha, \alpha \in \Delta_+$) and all singular vectors of weight $\xi - \rho$ correspond to some homomorphism. Let $f_i = e_{-\alpha_i}, i = 1, \ldots, r$ and set $f_I = f_{i_1} \ldots f_{i_m}$ for a multiindex $I = (i_1 \ldots, i_m)$. Let $v_{\xi-\rho} \otimes u + \sum_I f_I v_{\xi-\rho} \otimes u_I$ be a singular vector in $M_{\xi-\rho} \otimes U$ of weight $\xi - \rho$. Our formula gives an eigenfunction in terms of the coefficients u_I. Let us remark that, for generic ξ, such a singular vector is uniquely determined by its first coefficient $u \in U[0]$: the coefficients u_I can be given rather explicitly in terms of u and the inverse Shapovalov matrix (see [6]).

THEOREM 3.1. *Let* $\xi \in \mathfrak{h}^*$ *and let* $v_{\xi-\rho} \otimes u + \Sigma_L f_L v_{\xi-\rho} \otimes u_L$ *be a singular vector of weight* $\xi - \rho$ *in* $M_{\xi-\rho} \otimes U$. *Then the function* $\psi(\lambda) = \mathrm{e}^{2\pi i \xi(\lambda)}(u + \Sigma_L A_L(\lambda) u_L)$, *with*

$$A_{(l_1,\ldots,l_p)}(\lambda) = \sum_{\sigma \in S_p} \left(\prod_{j=1}^{p} \frac{X_{l_{\sigma(j)}}^{a_j+1}}{1 - X_{l_{\sigma(1)}} \ldots X_{l_{\sigma(j)}}} \right) f_{l_{\sigma(1)}} \ldots f_{l_{\sigma(p)}}, \tag{8}$$

where a_j *is the cardinality of the set of* $m \in \{j, \ldots, p-1\}$ *such that* $\sigma(m+1) < \sigma(m)$, *belongs to* $E(\xi)$, *and for generic* ξ *all functions in* $E(\xi)$ *are of this form.*

EXAMPLE Let $\mathfrak{g} = \mathrm{sl}_2$. If U is irreducible, $U[0]$ is one-dimensional if U has odd dimension and is zero otherwise. Let U be a $2s+1$-dimensional irreducible representation. Our formula reduces to

$$\psi(\lambda) = \mathrm{e}^{2\pi i \xi(\lambda)} \left(u + \sum_{l=1}^{s} \left(\frac{X}{1-X} \right)^l u_l \right), \quad (X = X_1).$$

The components u_l of the singular vector are easily computed, and we get the formula

$$\psi(\lambda) = \mathrm{e}^{2\pi i \xi(\lambda)} \sum_{l=0}^{s} (-1)^l \frac{(s+l)!\Gamma((\xi,\alpha)-l)}{l!(s-l)!\Gamma((\xi,\alpha))} \left(\frac{X}{1-X} \right)^l u. \tag{9}$$

We will prove the more general quantum version of Theorem 3.1 in Section 8.

Note that Theorem 3.1 completes the proof of Theorem 2.1: the coefficients are meromorphic functions. They seem to have poles on $\{X_\beta = 1\}$ for general $\beta \in Q$; however, these poles cancel, since the differential equation is regular there:

LEMMA 3.2. *Let* $\varepsilon \in \mathbb{C}$ *and suppose that* ψ *is a meromorphic solution of the differential equation* $H\psi = \varepsilon\psi$ *on* $\mathfrak{h}$, *whose poles belong to the union of the hyperplanes* $H_{\beta,m} = \{\lambda \mid \beta(\lambda) = m\}$, $\beta \in Q$, $m \in \mathbb{Z}$. *Then* ψ *is regular except possibly on the hyperplanes* $H_{\alpha,m}$, *with* $\alpha \in \Delta$, $m \in \mathbb{Z}$.

Proof. Let $\beta \in Q - \Delta$, $m \in \mathbb{Z}$, and choose a system of affine coordinates $z_1, \ldots, z_r$ on $\mathfrak{h}$ so that $z_r = \beta(\lambda) - m$. If ψ has a pole of order $p > 0$ on $H_{\beta,m}$, then, in the vicinity of a generic point of $H_{\beta,m}$, $\psi = z_r^{-p} f(z_1, \ldots, z_{r-1}) + \cdots$ with nonvanishing regular f. The leading term of $H\psi$ as $z_r \to 0$ comes from the Laplacian, as the potential term is regular on $H_{\beta,m}$ and is equal to $p(p+1)(\alpha,\alpha) z_r^{-p-2} f$ which is nonzero for $p > 0$. If $H\psi = \varepsilon\psi$, then it follows that ψ has a pole of order $p+2$, a contradiction. We have shown that ψ is regular on generic points of $H_{\beta,m}$. Therefore, on any bounded open subset V of $\mathfrak{h}$, the product of ψ by a suitable finite product of factors $\alpha(\lambda) - l$, $\alpha \in \Delta$, $l \in \mathbb{Z}$ is holomorphic on the complement of a set of codimension 2, and therefore everywhere on V by

Hartogs' theorem. □

This concludes the proof of Theorem 2.1.

4. The second formula

Our second formula is an integral representation. It is obtained as the trigonometric limit of the integral representation of solutions of the Knizhnik–Zamolodchikov–Bernard (KZB) equations on elliptic curves with one marked point.

One ingredient in the integrand is the rational function of $2n$ variables $T_1, \dots, T_n$, $Y_1, \dots, Y_n$

$$W(T, y) = \prod_{j=1}^{n} \left(\frac{1}{T_j - T_{j+1}} - \frac{Y_1 \dots Y_j}{Y_1 \dots Y_j - 1} \frac{1}{T_j} \right), \quad T_{n+1} := 1.$$

Now suppose that U is an irreducible highest weight representation with non-trivial zero-weight space. The highest weight of U is then in Q_+, i.e., of the form $\Lambda = \Sigma_j n_j \alpha_j$, for some non-negative integers n_j. Set $n = \Sigma_j n_j$. It is convenient to introduce the associated 'color' function c: $\{1, \dots, n\} \to \{1, \dots, r\}$, the unique non-decreasing function with $|c^{-1}(\{j\})| = n_j, j = 1, \dots, n$. For each permutation $\sigma \in S_n$, let $W_{\sigma,c}$ be the rational function of $n + r$ variables $T_1, \dots, T_n, X_1, \dots, X_r$

$$W_{\sigma,c}(T, X) = W(T_{\sigma(1)}, \dots, T_{\sigma(n)}, X_{c(\sigma(1))}, \dots, X_{c(\sigma(n))}). \tag{10}$$

The other ingredient is the many-valued function of $T_1, \dots, T_n$

$$\Phi^{\kappa}_{\xi,\Lambda}(T) = \prod_{j<l} (T_j - T_l)^{(\alpha_{c(j)}, \alpha_{c(l)})/\kappa} \prod_{j=1}^{n} T_j^{-(\xi - \rho, \alpha_{c(j)})/\kappa} (T_j - 1)^{-(\Lambda, \alpha_{c(j)})/\kappa}.$$

We will consider integrals of $\Phi^{\kappa}_{\xi,\Lambda}(T) W_{\sigma,c}(T, X) \mathrm{d}T_1 \dots \mathrm{d}T_n$ over connected components γ of $\{t \in (0, 1)^n \mid t_i \neq t_j, (i \neq j)\}$. These 'hypergeometric integrals' are defined as (meromorphic) analytic continuation in the exponents of $T_i - T_j$, T_j, $T_j - 1$ in $\Phi^{\kappa}_{\xi,\Lambda}$ from a region in which the integral converges absolutely. We say that a hypergeometric integral *exists* if this analytic continuation is finite at the given value of the exponents.

THEOREM 4.1. *Suppose that U is an irreducible highest weight module with highest weight $\Lambda = \Sigma_j n_j \alpha_j$ and highest weight vector v_Λ. Set $n = \Sigma n_j$ and let c: $\{1, \dots, n\} \to \{1, \dots, r\}$ be the unique nondecreasing function such that $c^{-1}\{j\}$ has n_j elements, for all $j = 1, \dots, r$. Fix a generic complex number κ and $\xi \in \mathfrak{h}$ also generic. Then, for each connected component γ of $\{t \in (0, 1)^n \mid t_i \neq t_j, (i \neq$*

$j)\}$, the integral

$$\psi_\gamma(\lambda) = \mathrm{e}^{2\pi i\xi(\lambda)} \int_\gamma \Phi^\kappa_{\xi,\Lambda}(T) \times \sum_{\sigma\in S_n} W_{\sigma,c}(T, X(\lambda))\, \mathrm{d}T_1 \ldots \mathrm{d}T_n\; f_{c(\sigma(1))} \cdots f_{c(\sigma(n))} v_\Lambda, \tag{11}$$

exists and defines a function in $E(\xi)$. Moreover, for each generic κ and ξ, all functions in $E(\xi)$ can be represented in this way.

In the rest of this section we prove this theorem. The existence of the integral follows from [17], Theorem 10.7.12. Indeed, the coefficients in (11) are linear combinations of integrals considered in [17] in the context of the Knizhnik–Zamolodchikov equation with two points.

Clearly ψ_γ is of the form $\mathrm{e}^{2\pi i\xi(\lambda)}$ times a meromorphic function of $X_1, \ldots, X_r$ with poles on the divisors $X_\beta = 1$, $\beta \in Q$. To prove that ψ_γ belongs to $E(\xi)$ it is therefore sufficient, thanks to Lemma 3.2, to show that ψ_γ is an eigenfunction of H.

This follows from the results of [7], which we now recall.

Let us define, for $q = \exp(2\pi i\tau)$ in the open unit disk, $\theta(x) = \pi^{-1}\sin(\pi x)\Pi_1^\infty (1 - 2q^j\cos(\pi x) + q^{2j})$ (in the notation of [7], $\theta(x) = \theta_1(x)/\theta_1'(0)$) and $v(x) = -\mathrm{d}^2/\mathrm{d}x^2 \ln\,\theta(x)$. The function v is doubly periodic with periods 1 and τ. Then for any $\kappa \in \mathbb{C} - \{0\}$, the KZB equation is a partial differential equation for a function $u(\lambda, \tau)$ on $\mathfrak{h} \times \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$, with values in $U[0]$:

$$4\pi i\kappa \frac{\partial u}{\partial \tau} = \triangle u - \sum_{\alpha\in\Delta} v(\alpha(\lambda)) e_\alpha e_{-\alpha} u. \tag{12}$$

As $q \to 0$, $v(x) \to \pi^2 \sin^{-2}(\pi x)$, and the differential operator on the right-hand side converges to $-H$.

PROPOSITION 4.2. *Suppose that u is a meromorphic solution of the* KZB *equation* (12) *such that $q^{-a/2\kappa}u$ is a meromorphic function of λ and q, $|q| < 1$, and, as $q \to 0$,*

$$u(\lambda, \tau) = q^{a/2\kappa}(\psi(\lambda) + O(q)).$$

Then ψ obeys $H\psi = \varepsilon\psi$, with $\varepsilon = (2\pi)^2 a$.

The proof consists of comparing the leading coefficients in the expansion of both sides of the KZB equation in powers of q at $q = 0$.

A source of solutions of (12) with the property described in the Proposition is [7]. In that paper, we gave integral formulas for solutions.

A class of solutions with asymptotic behavior as above is constructed as follows (see [7] for more details). Let U be an irreducible highest weight module with highest weight $\Lambda \in Q_+$, and let as above $c\colon \{1,\dots,n\} \to \{1,\dots,r\}$ the nondecreasing function associated to Λ.

Solutions are labeled by an element ξ of $\mathfrak{h}$ and a connected component γ of $\{t \in (0,1)^n \mid t_i \neq t_j, (i \neq j)\}$. Let $\phi^{\kappa}_{\xi,\Lambda}$ be a choice of branch over γ of the many-valued function

$$\mathrm{e}^{\pi i(\xi,\xi)\tau/\kappa + 2\pi i(\xi,\lambda+\kappa^{-1}\Sigma_j t_j\alpha_{c(j)})} \prod_{i<j} \theta(t_i - t_j)^{(\alpha_{c(i)},\alpha_{c(j)})/\kappa}$$

$$\times \prod_{j=1}^{n} \theta(t_j)^{-(\alpha_{c(j)},\Lambda)/\kappa} \tag{13}$$

and let w be the meromorphic function of $t_1,\dots,t_n, y_1,\dots,y_n$

$$w(t,y) = \prod_{j=1}^{n} \frac{\theta(y_1+\cdots+y_j - t_j + t_{j+1})}{\theta(y_1+\cdots+y_j)\theta(t_j - t_{j+1})}$$

and for each permutation $\sigma \in S_n$, let $w_{\sigma,c}$ be the meromorphic function of $n+r$ variables $t_1,\dots,t_n, x_1,\dots,x_r$

$$w_{\sigma,c}(t,x) = w(t_{\sigma(1)},\dots,t_{\sigma(n)}, x_{c(\sigma(1))},\dots,x_{c(\sigma(n))}). \tag{14}$$

Consider the differential n-form with values in $U[0]$ depending on $\lambda \in \mathfrak{h}$,

$$\omega(\lambda,\tau) = \Sigma_{\sigma\in S_n} w_{\sigma,c}(t,\alpha_1(\lambda),\dots,\alpha_r(\lambda))\ \mathrm{d}t_1 \dots dt_n\ f_{c(\sigma(1))}\cdots f_{c(\sigma(n))} v_\Lambda.$$

Then the integral

$$\int_\gamma \phi^{\kappa}_{\xi,\Lambda}(t)\omega(\lambda,\tau)$$

exists (as analytic continuation in the exponents from a region of absolute convergence) and is a solution of the KZB equation. As $\tau \to i\infty$,

$$\theta(x) = \sin(\pi x)/\pi + O(q),$$

and the function (13) behaves as

$$q^{(\xi,\xi)/2\kappa}\mathrm{e}^{2\pi i\xi(\lambda)}[\Phi^{\kappa}_{\xi,\Lambda}(T) + O(q)],$$

in terms of the exponential variables $T_j = \exp(-2\pi i t_j)$. Let $X_j = \exp(-2\pi i\alpha_j(\lambda))$. The components of the differential form ω behave as

$$w_{\sigma,c}(t,x)\,\mathrm{d}t_1\dots\mathrm{d}t_n = W_{\sigma,c}(T,X)\,\mathrm{d}T_1\dots\mathrm{d}T_n + O(q).$$

Therefore, the solutions corresponding to cycles in $H_n(C_n^0(\tau), \mathcal{L}(\xi))$ have the properties of the Proposition, with $a = (\xi, \xi)$, and ψ is as stated in the claim of Theorem 4.1. We obtain in the limit an integral representation for eigenfunctions as in the Theorem but with an integration domain $\tilde{\gamma}_\sigma$ of the form $0 < \arg(T_{\sigma(1)}) < \cdots < \arg(T_{\sigma(n)}) < 2\pi$ for some $\sigma \in S_n$. This integration domain may be deformed, so that the corresponding integral can be written as linear combination of integrals over the domains $\gamma_{\sigma'}$ defined by $1 > T_{\sigma'(1)} > \cdots > T_{\sigma'(n)} > 0$. If we choose $\xi = ia\xi_0$, for some ξ_0 such that $(\xi_0, \alpha_j) > 0$ for all j, and let a tend to infinity, then the integral over $\tilde{\gamma}_\sigma$ is equal to the integral over γ_σ (for suitable choice of branch and orientation) plus terms that tend to zero. It follows that, for generic ξ and all $\sigma \in S_n$, the integral over γ_σ can be expressed as a linear combination of integrals over $\tilde{\gamma}_{\sigma'}$ and is therefore also an eigenfunction.

This completes the proof of the first part of Theorem 4.1.

We must still prove that, in the generic case, all eigenfunctions in $E(\xi)$ admit such an integral representation. In view of Theorem 2.1, it is sufficient to show that for every $u \in U[0]$, there exists a cycle γ such that $e^{-2\pi i\xi(\lambda)}\psi_\gamma \to u$ as $X_j \to 0$ $j = 1, \ldots, r$. In other words, one must show that all vectors in $U[0]$ are of the form

$$\int_\gamma \Phi^\kappa_{\xi,\Lambda}(T) \sum_{\sigma \in S_n} \left(\prod_{j=1}^{n-1} \frac{1}{T_{\sigma(j)} - T_{\sigma(j+1)}} \right) \frac{1}{T_{\sigma(n)} - 1}$$
$$\times f_{c(\sigma(1))} \cdots f_{c(\sigma(n))} v_\Lambda \, \mathrm{d}T_1 \ldots \mathrm{d}T_n$$

for some cycle γ. But this follows from the results in [17], Theorem 12.5.5.

5. The third formula

This is a formula of the Bethe ansatz type.

THEOREM 5.1. *Suppose U is an irreducible highest weight module with highest weight $\Lambda = \Sigma_j n_j \alpha_j$ and highest weight vector v_Λ. Set $n = \Sigma n_j$ and let c: $\{1, \ldots, n\} \to \{1, \ldots, r\}$ be the unique nondecreasing function such that $c^{-1}\{j\}$ has n_j elements, for all $j = 1, \ldots, r$. Then the function parametrized by $T \in \mathbb{C}^n$*

$$\psi(T, \lambda) = \mathrm{e}^{2\pi i \xi(\lambda)} \sum_{\sigma \in S_n} W_{\sigma,c}(T, X(\lambda)) f_{c(\sigma(1))} \cdots f_{c(\sigma(n))} v_\Lambda, \tag{15}$$

(see (10) for the definition of $W_{\sigma,c}$) belongs to $E(\xi)$ if the parameters $T_1, \ldots, T_n$ are a solution of the set of n algebraic equations (‘Bethe ansatz equations’)

$$\left(\sum_{l: l \neq j} \frac{\alpha_{c(l)}}{T_j - T_l} - \frac{\Lambda}{T_j - 1} - \frac{\xi - \rho}{T_j}, \alpha_{c(j)} \right) = 0, \quad j = 1, \ldots, n.$$

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Remark. These Bethe ansatz equations are the same as the Bethe ansatz equations of a special case of the Gaudin model (cf. [14]). The solutions are the critical points of the function

$$\prod_{1\leqslant i\leqslant j\leqslant n}(T_i-T_j)^{(\alpha_{c(i)},\alpha_{c(j)})}\prod_{j=1}^{n}T_j^{-(\xi-\rho,\alpha_{c(j)})}(T_j-1)^{-(\Lambda,\alpha_{c(j)})}.$$

Theorem 5.1 can be understood intuitively as a consequence of Theorem 4.1: one calculates the integrals, which up to normalization are independent of κ, in the limit $\kappa\to 0$, using the saddle point method.

The proof of this theorem can be taken directly from [7] in the case of a degenerate elliptic curve.

The above result motivates the notion of Hermite–Bethe variety. Let $\Lambda = \Sigma_j n_j\alpha_j$. Without loss of generality, we may take $n_j>0$, for all $j=1,\dots,r$ (if this condition is not fulfilled, we may pass to a subalgebra). If $c\colon\{1,\dots,n\}\to\{1,\dots,r\}$ is the associated nondecreasing function, we let S_c be the product of symmetric groups $S_{n_1}\times\cdots\times S_{n_r}$. It acts on $\mathbb{C}^n$ by permutation of the variables T_j with same $c(j)$ and is a symmetry group of the system of Bethe ansatz equations. Moreover, we have, for all $\sigma\in S_c$,

$$\psi(\sigma T,\lambda)=\psi(T,\lambda).$$

Let us write the Bethe ansatz equations as $B_j(T)=0$, with

$$B_j(T)=\left(\sum_{l:l\neq j}\frac{\alpha_{c(l)}T_j}{T_j-T_l}-\frac{\Lambda T_j}{T_j-1}-\xi-\rho,\alpha_{c(j)}\right).$$

Subtracting pairs of equations we may eliminate ξ:

DEFINITION. Let $F_n=(\mathbb{C}-\{0,1\})^n-\cup_{i<j:(\alpha_{c(i)},\alpha_{c(j)})\neq 0}\{T\mid T_i=T_j\}$. The Hermite–Bethe variety $HB(c)$ is

$$\begin{aligned}HB(c) &= \{T\in F_n\mid B_j(T)\\ &= B_{j+1}(T), j\in\{1,\dots,n\}-\{n_1,n_1+n_2,\dots,n\}\}/S_c.\end{aligned}$$

The remaining equations define a regular map $p\colon HB(c)\to\mathfrak{h}$, $T\mapsto\xi$. The completeness hypothesis of Bethe states is in this case:

CONJECTURE 5.2. (cf. [18], [13] and conjectures in [14]). The map p has dense image, and the generic fiber consists of (at least) $\dim(U[0])$ points. The eigenfunctions corresponding to points in $p^{-1}(\xi)$ span $E(\xi)$.

In the case of sl_2 the conjecture holds and goes back to Hermite, see [19].

We prove this conjecture in two special situations, including the scalar case relevant for many-body systems.

THEOREM 5.3. *Conjecture 5.2 holds in the following cases:* (a) $\mathfrak{g} = \mathrm{sl}_N$, $U = S^{pN}\mathbb{C}^N$, $p = 1, 2, \dots$, (*see* (2)) (*b*) $\mathfrak{g} = \mathrm{sl}_N$, $U =$ *adjoint representation.*

Proof. The proof is by induction in N, viewing sl_N as a Lie subalgebra of sl_{N+1}. We choose simple roots $\alpha_1, \dots, \alpha_N$ of sl_N in such a way that $\alpha_1, \dots, \alpha_{N-1}$ are simple roots of sl_{N-1}. Also, it is convenient to replace ξ by $\zeta = \xi - \rho$.

In case (a), the highest weight of $U_N = S^{pN}\mathbb{C}^N$ is $\Lambda = p\Sigma_{j=1}^{N-1} j\alpha_j$, and $U_N[0]$ is one-dimensional. The fiber over $\xi = \zeta + \rho$ of the Hermite–Bethe variety consists of the critical points of

$$\Phi_N(\zeta, T) = \prod_{j<l} (T_j - T_l)^{(\alpha_{c(j)}, \alpha_{c(l)})} \prod_j T_j^{-(\zeta, \alpha_{c(j)})} (1 - T_j)^{-(\Lambda, \alpha_{c(j)})}$$

viewed as a function of $T = (T_1, \dots, T_{pN(N+1)/2})$. We have $c(j) = m$ if $pm(m-1)/2 < c(j) \leqslant pm(m+1)/2$, $1 \leqslant m \leqslant N$. We prove inductively that, for generic ζ, Φ_N has a nondegenerate critical point, and that the corresponding eigenfunction spans $E(\zeta + \rho)$. Let us consider critical points of $\Phi_{N+1}(\zeta, T)$ when $(\alpha_N, \zeta) = \varepsilon^{-1}$ tends to infinity, and (α_j, ζ), $j < N$ are kept fixed. Let $\bar{\xi}$ denote the orthogonal projection of ξ onto the Cartan subalgebra of sl_N. Let us replace the coordinates T_j indexed by j such that $c(j) = N$ by new coordinates a_j defined by $T_j = 1 - \varepsilon a_j$ ($c(j) = N$), and let $\bar{T} = (T_1, \dots, T_{pN(N+1)/2})$ denote the remaining coordinates. Then

$$\begin{aligned}
\varepsilon^{p(p+1)N} \Phi_{N+1}(\xi, T) \;=\; & \Phi_N(\bar{\zeta}, \bar{T}) \prod_{c(j)=N-1} (1 - T_j)^{pN} \\
& \times \prod_{c(j)=N-1, c(l)=N} (T_j - 1 + \varepsilon a_l)^{-1} \\
& \times \prod_{c(l)=N} (1 - \varepsilon a_l)^{-1/\varepsilon} a_l^{-p(N+1)} \\
& \times \prod_{j<l} (a_l - a_j)^2.
\end{aligned} \tag{16}$$

This function converges as $\varepsilon \to 0$ to a constant times

$$\Phi_N(\bar{\zeta}, \bar{T}) \prod_l \mathrm{e}^{a_l} a_l^{-p(N+1)} \prod_{j<l} (a_l - a_j)^2.$$

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The factor depending on the a_j's has a nondegenerate critical point a_* in the domain $a_j \neq a_k \neq 0$, $(j \neq k)$. This follows from the fact that this factor is obtained as a limit $M \to \infty$ of

$$\prod_j (1 - a_j/M)^{-M} a_j^{-p(N+1)} \prod_{j<l} (a_j - a_l)^2.$$

The critical points of the latter function are known explicitly (see (1.3.1) and (1.4.2) in [18]), and it can be easily checked that they have a limit as $M \to \infty$ in the domain $a_j \neq a_k \neq 0$, which is therefore a critical point of the limiting function. In fact, a_* is the set of zeros of the polynomial solutions of the differential equation $xy'' + (x - m)y' - Ny = 0$, where $m = p(N+1)$. By the induction hypothesis, $\Phi_N(\bar\zeta, \bar T)$ has a nondegenerate critical point $\bar T_*$. Moreover, in a neighborhood of $(a_*, \bar T_*, \varepsilon = 0)$, the right-hand side of (16) is holomorphic, which implies that the nondegenerate critical point at $\varepsilon = 0$ deforms to a nondegenerate critical point for generic ε.

It remains to show that the corresponding eigenfunction generically spans the one-dimensional vector space $E(\zeta + \rho)$, i.e., that it does not vanish. Assuming this inductively to hold for sl_N, we see that as $\varepsilon \to 0$, the leading contribution in the sum (15) is given by permutations such that $c(\sigma(j)) = N$, whenever $c(j) = N$. These permutations give terms proportional to $f_{c(\tau(1))} \cdots f_{c(\tau(pN(N+1)/2))} (f_N)^{pN} v_\Lambda$, for some $\tau \in S_{pN(N+1)/2}$. It follows that when $\varepsilon \to 0$ and $T(\varepsilon)$ is the above family of critical points, $\mathrm{e}^{-2\pi i(\xi,\lambda)} \psi_{\mathrm{sl}_{N+1}}(T(\varepsilon), \lambda)$ converges to $C(a_*)\mathrm{e}^{-2\pi i(\bar\xi,\bar\lambda)} \psi_{\mathrm{sl}_N}(\bar T_*, \bar\lambda)$, where the representation $U_N = S^{pN}\mathbb{C}^N$ of sl_N is viewed as the sl_N-submodule of U_{N+1} generated by the singular vector $(f_N)^{Np} v_\Lambda$. Using the identity

$$\sum_{\sigma \in S_N} \frac{1}{(a_{\sigma(1)} - a_{\sigma(2)}) \cdots (a_{\sigma(N-1)} - a_{\sigma(N)}) a_{\sigma(N)}} = \frac{1}{a_1 \ldots a_N},$$

we see that $C(a_*)$ is, up to a trivial nonzero factor, the inverse of the product of the components a_j of a_*, and therefore nonzero. This completes the proof of part (a) of the theorem.

The case (b) is treated in a similar way. The highest weight of the adjoint representation U_N of sl_N is $\Lambda = \Sigma\alpha_i$ and $c = \mathrm{Id}$. The function Φ for sl_N is

$$\Phi_N(\zeta, T) = \prod_{j=1}^{N-2} (T_j - T_{j+1})^{-1} \prod_{j=1}^{N-1} T_j^{-(\zeta,\alpha_j)} (1 - T_1)^{-1} (1 - T_{N-1})^{-1}.$$

As before we let $\varepsilon^{-1} = (\zeta, \alpha_N)$ go to infinity. If we set $T_N = 1 - \varepsilon a$, $\bar T = (T_1, \ldots, T_{N-1})$, $\bar\zeta =$ projection of ζ onto the Cartan subalgebra of sl_N, then

$$\begin{aligned} \varepsilon\Phi_{N+1}(\zeta, T) &= \Phi_N(\bar\zeta, \bar T) \frac{1 - T_{N-1}}{T_{N-1} - 1 + \varepsilon a} (1 - \varepsilon a)^{-1/\varepsilon} a^{-1} \\ &\times \xrightarrow{\varepsilon \to 0} -\Phi_N(\bar\zeta, \bar T) \frac{\mathrm{e}^a}{a}. \end{aligned}$$

Proceeding as before, we see that nondegenerate critical points $\bar{T}_*$, $a_* = 1$ of the limiting function deform to nondegenerate critical points for any generic ε. the corresponding eigenfunctions deform to eigenfunctions of sl_N taking values in $U_N[0]$, viewed as a subspace of $U_{N+1}[0]$ via the inclusion of sl_N in sl_{N+1}. In this way we get $N-1$ linearly independent eigenfunctions in the N-dimensional space $E(\zeta+\rho)$ associated to the adjoint representation of sl_{N+1}. To find the remaining eigenfunction, we let $T_j = 1 - \varepsilon a_j$ for all j. Then as $\varepsilon \to 0$, $\varepsilon^m \Phi_{N+1}$ for suitable m converges to

$$\frac{\mathrm{e}^{a_N}}{a_1 a_{N-1}} \prod_{j=1}^{N-1} (a_{j+1} - a_j)^{-1}$$

The critical point of this function can be computed explicitly: $a_j = j(1+1/N)$. It deforms to a nondegenerate critical point for ε generic. The corresponding eigenfunction converges, as $\varepsilon \to 0$, to a constant function. From its explicit expression it is clear that its value is not in $U_N = U(\mathrm{sl}_N) f_N v_\Lambda$ and is therefore linearly independent from the $N-1$ constructed before. $\square$

Remark. The proof of the preceding theorem indicates, at least in the examples considered, that the construction which to a pair $(\mathfrak{g}, U)$ consisting of a semisimple Lie algebra $\mathfrak{g}$ and a finite dimensional $\mathfrak{g}$-module associates the closure of the algebraic Bethe-Hermite variety X is *functorial*: to each homomorphism $(\mathfrak{g}, U) \to (\mathfrak{g}', U')$ preserving the Lie algebra and module structures there corresponds functorially a morphism of algebraic varieties $X \to X'$. This functor is compatible with the construction of eigenfunctions in a sense that should be made more precise.

6. Weyl group action, Jacobi polynomials, scattering matrices

In this section we study the Weyl group action on eigenfunctions, and discuss the relation with multivariable Jacobi polynomials [8]. The Weyl groups W acts on $U[0]$ (The normalizer N of the Cartan torus $T = \exp \mathfrak{h}$ of the simply connected group with Lie algebra $\mathfrak{g}$ acts on U, so $W = N/T$ acts on $U[0]$). Therefore we have a natural action of W on $U[0]$-valued functions: $w \in W$ acts as $(w\psi)(\lambda) = w \cdot \psi(w^{-1}\lambda)$. The Schrödinger operator H commutes with this action.

Let S be the set of $\xi \in \mathfrak{h}^*$ such that $(\xi, \beta) = (\beta, \beta)$ for some $\beta \in Q_+$. If $\xi \in \mathfrak{h}^* - S$, $E(\xi)$ is isomorphic to $U[0]$ and we have the explicit expression of the isomorphism in Theorem 3.1.

LEMMA 6.1. *Let $\xi \in \mathfrak{h}^* - S$. If $\psi \in E(\xi)$ and $w \in W$, then $\mathrm{e}^{-2\pi i \xi(w^{-1}\lambda)} w\psi$ is a rational function of $X_1, \ldots, X_r$ which is holomorphic on the complement in $\mathbb{C}^r$ of the root hypersurfaces $X_\alpha = 1$, $\alpha \in \Delta$.*

Proof. Suppose first that $w = 1$. From the explicit expression (8) it is clear that it is a rational function. By Lemma 3.2, the poles lie on the root hypersurfaces.

Let us assume that $w = s_k$ is a simple reflection corresponding to the simple root α_k. As root hypersurfaces are permuted under the action of the Weyl group, it suffices to show that $\mathrm{e}^{-2\pi i\xi(w^{-1}\lambda)}w\psi$ is regular at $X_j = 0$ for all j. This is obvious if $j \neq k$ since replacing λ by $s_k\lambda$ amounts to replacing X_j by $X_jX_k^{-a_{jk}}$, with $a_{jk} \leqslant 0$ if $j \neq k$. If $j = k$, X_k is replaced by X_k^{-1} and we have to check that the expression in parenthesis in (8) is regular when $X_k \to \infty$, but this follows easily by counting powers of X_k in the numerator and denominator, exploiting the fact that $a_j + 1 \leqslant p - j + 1$. The case of general w is reduced to this case by writing w as a product of simple reflections. □

LEMMA 6.2. *Let $\xi \in \mathfrak{h}^*$ be generic. For all $w \in W$, $\psi \mapsto w\psi$ is an isomorphism from $E(\xi)$ onto $E(w\xi)$.*

Proof. Let $\psi \in E(\xi)$. By the previous Lemma, $\mathrm{e}^{-2\pi i\xi(w^{-1}\lambda)}w\psi$ is holomorphic at $X = 0$. Also, since H is invariant, $w\psi$ is an eigenfunction with eigenvalue $4\pi^2(\xi, \xi)$ and therefore $w\psi \in E(w\xi)$. □

EXAMPLE. Let $\mathfrak{g} = \mathrm{sl}_2$, and U be the $2s+1$-dimensional irreducible representation. The Weyl reflection s_1 acts as $(-1)^s$ on $U[0]$. Fix a nonzero $u \in U[0]$ and let ψ_ξ be the eigenfunction (9). Then $(s_1\psi)(\lambda) = (-1)^s\psi_\xi(-\lambda) = S(\xi)\psi_{-\xi}(\lambda)$, where $S(\xi)$ can be computed in the limit $X \to 0$. This gives the 'two particle scattering matrix'

$$S(\xi) = (-1)^s \sum_{l=0}^{s} \frac{(s+l)!\Gamma((\xi,\alpha)-l)}{l!(s-l)!\Gamma((\xi,\alpha))} = \prod_{k=1}^{s} \frac{k+(\alpha,\xi)}{k-(\alpha,\xi)}. \tag{17}$$

We say that a function ψ is Weyl antiinvariant if $w\psi = \varepsilon(w)\psi$ for all $w \in W$.

PROPOSITION 6.3. (*cf. Proposition* 3 *in* [7]) *Let ψ be a meromorphic Weyl antiinvariant solution of the eigenvalue problem $H\psi = \varepsilon\psi$, regular on*

$$\mathfrak{h} - \bigcup_{m\in\mathbb{Z}} \bigcup_{\alpha\in\Delta} \{\lambda \in \mathfrak{h} \mid \alpha(\lambda) = m\},$$

such that $\psi(\lambda + p) = \chi(p)\psi(\lambda)$ for all p in the lattice $P^\vee = \{p \in \mathfrak{h} \mid \alpha(p) \in \mathbb{Z}\}$ and some character χ of $P^\vee$, then ψ extends to a holomorphic function on $\mathfrak{h}$. Moreover, for all $\alpha \in \Delta$ and $m \in \mathbb{Z}$,

$$\mathrm{e}_\alpha^l \psi = O((\alpha(\lambda) - m)^{l+1}) \tag{18}$$

as $\alpha(\lambda) \to m$.

Proof. By periodicity with respect to the coweight lattice $P^\vee$, we may limit our considerations to the hyperplanes through the origin. One proceeds as in Lemma 3.2 by approaching the singular hyperplane in a transversal direction. The leading

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term in the Laurent expansion in the transversal coordinate $x = \alpha(\lambda)$ is determined by the differential equation

$$(\alpha,\alpha)\frac{d^2}{dx^2}\psi_0 - \frac{e_\alpha e_{-\alpha} + e_{-\alpha}e_\alpha}{x^2}\psi_0 = 0.$$

Let us decompose U into irreducible representations of the subalgebra generated by $e_{\pm\alpha}$. Since ψ is of zero weight, we may replace $e_\alpha e_{-\alpha} + e_{-\alpha}e_\alpha$ by

$$C_\alpha = \frac{1}{(\alpha,\alpha)}h_\alpha h_\alpha + e_\alpha e_{-\alpha} + e_{-\alpha}e_\alpha,$$

where $h_\alpha = [e_\alpha, e_{-\alpha}]$. If ψ_0 belongs to a $2l+1$ dimensional irreducible representation, i.e., if $e_\alpha^l\psi \neq 0$ but $e_\alpha^{l+1}\psi = 0$, then the Casimir element C_α acts as $(\alpha,\alpha)l(l+1)$. Therefore either $\psi_0 \sim x^{l+1}$ or $\psi_0 \sim x^{-l}$. On the other hand, the Weyl reflection with respect to the hyperplane $\alpha = 0$ changes the sign of x and multiplies the value by $(-1)^l$. Since ψ is antiinvariant, it follows that the first possibility is realized and the function vanishes to order $l+1$. □

In particular, if ξ belongs to the weight lattice P, spanned by the fundamental weights $\omega_1,\dots,\omega_r$, and $\psi \in E(\xi)$, then the Weyl antiinvariant function $\psi^W = \Sigma_{w\in W}\varepsilon(w)w\psi$ is a rational function of $X_{\omega_1},\dots,X_{\omega_r}$ regular on $(\mathbb{C}-\{0\})^r$. Therefore, ψ is a Laurent polynomial. To apply the previous lemma, ξ should not be in S. This is true if $-\xi$ is dominant, and we have the following result.

COROLLARY 6.4. *Let $\xi = -\mu$ where μ is a dominant integral weight and $\psi \in E(\xi)$. Then $\psi^W = \Sigma_{w\in W}\varepsilon(w)w\psi$ is a Laurent polynomial in $X_{\omega_1},\dots,X_{\omega_r}$.*

EXAMPLE. Let $\mathfrak{g} = \mathrm{sl}_N$, $U = S^{pN}\mathbb{C}^N$, the scalar case, and let $\omega_1,\dots,\omega_{N-1}$ be the fundamental weights of sl_N. Let us fix an identification of $U[0]$ with $\mathbb{C}$. Let $\psi_{-\mu}$ be the eigenfunction of the previous corollary, normalized so that $\psi = \mathrm{e}^{-2\pi i\mu(\lambda)}(1+\cdots)$. By the previous corollary, it is a Laurent polynomial in the X_{ω_i}. Then the vanishing property (18) of Proposition 6.3 implies that the Weyl antiinvariant eigenfunction $\psi_{-\mu}^W$ is divisible in the ring $\mathbb{C}[X_{\omega_i}^{\pm1}]$ by Π^{p+1} where

$$\Pi = X_{-\rho}\prod_{\alpha\in\Delta_+}(1-X_\alpha).$$

The ratio $P = \psi_{-\mu}/\Pi^{p+1}$ is a Weyl invariant polynomial (since Π is antiinvariant) and obeys the differential equation

$$-\triangle P + (p+1)2\pi\sum_{\alpha\in\Delta_+}\cot(\pi\alpha(\lambda))\partial_\alpha P = \tilde{\varepsilon}P, \tag{19}$$

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(∂_α is the derivative in the direction of α), with $\tilde{\varepsilon} = 4\pi^2(\mu + (p+1)\rho, \mu - (p+1)\rho)$. This equation defines multivariable Jacobi polynomials (or Jack polynomials) associated to a dominant integral weight ν: The Jacobi polynomial P_ν is the unique solution in $\mathbb{C}[X_{\omega_1}^{\pm 1}, \ldots, X_{\omega_{N-1}}^{\pm 1}]$ of (19), normalized in such a way that $P_\nu = X_{-\nu} + \sum_{\beta \in Q_+} a_\beta X_{-\nu+\beta}$ for some coefficients $a_\alpha \in \mathbb{C}$. The leading term of $\psi_{-\mu}/\Pi^{p+1}$ is $X_{\mu+(p+1)\rho}$. After antisymmetrization, we obtain a leading term $cX_{w_0(\mu+(p+1)\rho)}$ where w_0 is the longest element of the Weyl group. Thus we obtain a function proportional to P_ν by choosing $\mu = w_0\nu - (p+1)\rho$ (w_0 is an involution). Therefore we have the corollary:

COROLLARY 6.5. *In the scalar case* $\mathfrak{g} = \mathrm{sl}_N$, $U = S^{pN}\mathbb{C}^N$, *with fixed identification of* $U[0]$ *with* $\mathbb{C}$, *let* ν *be a dominant integral weight, and for any* $\xi \in \mathfrak{h}^* - S$, *let* ψ_ξ *denote the eigenfunction in* $E(\xi)$ *given in Theorem* 3.1 *with* $u = 1$ *and* $\psi_\xi^W = \Sigma_{w \in W}\varepsilon(w)w\psi_\xi$ *its antisymmetrization. Then*

$$P_\nu = c_\nu \frac{\psi^W_{w_0\nu-(p+1)\rho}}{\Pi^{p+1}},$$

for some constant $c_\nu \neq 0$. *Here* w_0 *denotes the longest element of the Weyl group, i.e., the permutation* $j \mapsto N+1-j$ *of* S_N.

A formula for c_ν is given below.

We next study more closely the action of the Weyl group.

Let $\mathrm{sl}_2(j)$ be the subalgebra of $\mathfrak{g}$ generated by $e_{\pm\alpha_j}$, and for a $\mathfrak{g}$-module U, let $U = \oplus_s U_s^{(j)}$ be the decomposition of U viewed as $\mathrm{sl}_2(j)$-module into isotypic components: $U_s^{(j)}$ is isomorphic to a direct sum of $2s+1$-dimensional irreducible $\mathrm{sl}_2(j)$-modules.

THEOREM 6.6. *Let* ξ *be generic and denote by* $j(\xi)$ *the isomorphism* $U[0] \to E(\xi)$ *mapping* u *to the eigenfunction with leading term* $\mathrm{e}^{2\pi i\xi(\lambda)}u$. *Then there exists a family of maps* $S(\xi)\colon W \to \mathrm{GL}(U[0])$, $w \mapsto S_w(\xi)$ *such that if* $\psi = j(\xi)u$, *then* $w\psi = j(w\xi)S_w(\xi)u$. *These maps have the composition property*

$$S_{w_1w_2}(\xi) = S_{w_1}(w_2\xi)S_{w_2}(\xi).$$

If $w = s_j$ *is a simple reflection then* $S_w = S_j$ *has the form*

$$S_j(\xi) = P_0^{(j)} + \sum_{s\geqslant 1}\prod_{k=1}^{s}\frac{k+(\alpha_j,\xi)}{k-(\alpha_j,\xi)}P_s^{(j)}, \tag{20}$$

where $P_s^{(j)} \in \mathrm{End}(U[0])$ *is the projection onto* $U_s^{(j)} \cap U[0]$.

Proof. The first part of this theorem is just a rephrasing of the preceding lemma. The composition property follows from the definition:

$$\begin{aligned} j(w_1w_2\xi)S_{w_1w_2}(\xi) &= w_1w_2j(\xi) \\ &= w_1j(w_2\xi)S_{w_2}(\xi) \\ &= j(w_1w_2\xi)S_{w_1}(w_2\xi)S_{w_2}(\xi). \end{aligned}$$

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If $w = s_j$, we may compute S_j by letting all X_l, $l \neq j$ go to zero. Then the coefficients A_L in Theorem 3.1 tend to zero except if $L = (j, j, \dots, j)$. and the formula for $\exp(-2\pi i\xi(\lambda))\psi$ reduces to an sl_2 formula, and we can apply the previous example in each isotypic component. □

Remark. Note the similarity between this scattering matrix and the rational R-matrix associated with general representations of sl_2 [11].

EXAMPLE. Let $\mathfrak{g} = \mathrm{sl}_N$, and identify $\mathfrak{h}^*$ with $\mathbb{C}^N/\mathbb{C}(1,\dots,1)$. The Weyl group is the symmetric group S_N and the simple reflections s_j act on $\mathbb{C}^N$ by transposition of the jth and $j+1$st coordinates. The corresponding S_j depends only on $(\alpha_j, \xi) = \xi_j - \xi_{j+1}$. The relations $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$ translate into 'unitarity'

$$S_j(\xi_{j+1} - \xi_j)S_j(\xi_j - \xi_{j+1}) = \mathrm{Id},$$

and the Yang–Baxter equation

$$\begin{aligned} &S_j(\xi_{j+1} - \xi_{j+2})S_{j+1}(\xi_j - \xi_{j+2})S_j(\xi_j - \xi_{j+1}) \\ &\quad = S_{j+1}(\xi_j - \xi_{j+1})S_j(\xi_j - \xi_{j+2})S_{j+1}(\xi_{j+1} - \xi_{j+2}). \end{aligned}$$

COROLLARY 6.7. *In the scalar case* ($\mathfrak{g} = \mathrm{sl}_N, U = S^{pN}\mathbb{C}^N$)

$$S_j(\xi) = \prod_{k=1}^{p} \frac{k + (\alpha_j, \xi)}{k - (\alpha_j, \xi)}.$$

Proof. The only isotypic component $U_s^{(j)}$ with non-trivial intersection with $U[0]$ has $s = p$ in this case. □

Let us conclude with a formula for the constant c_ν in Corollary 6.5. This constant can be computed from the leading term in ψ^W which appears in the summand indexed by w_0: $c_\nu^{-1} = \varepsilon(w_0)S_{w_0}(w_0(\nu + (p+1)\rho))$. If we identify $\nu \in \mathfrak{h}$ with the diagonal traceless matrix with diagonal entries $\nu_1, \dots, \nu_N$, a simple calculation gives

$$c_\nu^{-1} = \prod_{i>j} \prod_{k=0}^{p} \frac{k + \nu_i - \nu_j - (p+1)(i-j)}{k - \nu_i + \nu_j + (p+1)(i-j)}. \tag{21}$$

This constant is clearly different from zero since for dominant ν we have $\nu_i < \nu_j$ if $i > j$.

7. Bethe ansatz in the elliptic case

Let us fix τ in the upper half-plane, and denote E_τ the torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$. We first quote the result of [7] on eigenfunctions of the differential operator

$$H_e = -\Delta + \sum_{\alpha \in \Delta} v(\alpha(\lambda)) e_\alpha e_{-\alpha},$$

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on $U[0]$-valued functions on $\mathfrak{h}$. The elliptic function v is $v(x) = -\mathrm{d}^2/\mathrm{d}x^2 \ln\theta(x)$, $\theta(x) = \pi^{-1}\sin(\pi x)\Pi_1^\infty(1-2q^j\cos(\pi x)+q^{2j})$, $q = \mathrm{e}^{2\pi i\tau}$. It differs from the Weierstrass $\wp$-function by a constant.

THEOREM 7.1. ([7]) *Suppose U is an irreducible highest weight module with highest weight $\Lambda = \Sigma_j n_j\alpha_j$ and highest weight vector v_Λ. Set $n = \Sigma n_j$ and let $c\colon \{1,\dots,n\} \to \{1,\dots,r\}$ be the unique nondecreasing function such that $c^{-1}\{j\}$ has n_j elements, for all $j = 1,\dots,r$. Then the function parametrized by $t \in \mathbb{C}^n$*

$$\psi(t,\lambda) = \mathrm{e}^{2\pi i\xi(\lambda)}\sum_{\sigma\in S_n} w_{\sigma,c}(t,\alpha_1(\lambda),\dots,\alpha_r(\lambda))f_{c(\sigma(1))} \times \dots f_{c(\sigma(n))}v_\Lambda, \tag{22}$$

(*see* (14) *for the definition of $w_{\sigma,c}$*) *is an eigenfunction of H_e if the parameters t_1, $\dots$, t_n are a solution of the set of n equations ('Bethe ansatz equations')*

$$\left(\sum_{l:l\neq j}\frac{\theta'(t_j-t_l)}{\theta(t_j-t_l)}\alpha_{c(l)} - \frac{\theta'(t_j)}{\theta(t_j)}\Lambda + 2\pi i\xi, \alpha_{c(j)}\right) = 0, \quad j = 1,\dots,n. \tag{23}$$

The corresponding eigenvalue ε is

$$\varepsilon = 4\pi^2(\xi,\xi) - 4\pi i\frac{\partial}{\partial\tau}S(t_1,\dots,t_n,\tau),$$

$$S(t_1,\dots,t_m,\tau) = \sum_{i<j}(\alpha_{c(i)},\alpha_{c(j)})\ln\theta(t_i-t_j) - \sum_i(\Lambda,\alpha_{c(i)})\ln\theta(t_i).$$

Remark. Solutions of the Bethe ansatz equations are critical points of

$$\Phi_e(t) = \mathrm{e}^{2\pi i(\xi,\sum\alpha_{c(i)}t_i)}\prod_{i<j}\theta(t_i-t_j)^{(\alpha_{c(i)},\alpha_{c(j)})}\prod_i\theta(t_i)^{-(\Lambda,\alpha_{c(i)})},$$

in the domain $\Phi_e(t)\neq 0$.

As in the trigonometric case, we define the Hermite–Bethe variety by eliminating the spectral parameter ξ from the Bethe ansatz equations. The resulting equations are the $n-r$ equations

$$\left(\sum_{l:l\neq j,j+1}\left(\frac{\theta'(t_j-t_l)}{\theta(t_j-t_l)} - \frac{\theta'(t_{j+1}-t_l)}{\theta(t_{j+1}-t_l)}\right)\alpha_{c(l)}\right.$$

$$-\left(\left(\frac{\theta'(t_j)}{\theta(t_j)}-\frac{\theta'(t_{j+1})}{\theta(t_{j+1})}\right)\Lambda,\alpha_{c(j)}\right)=0, \tag{24}$$

where j runs over the set of indices such that $c(j)=c(j+1)$.

LEMMA 7.2. *The left-hand side of each of the equations* (24) *is a doubly periodic function of each of its arguments* t_i.

This lemma follows easily from the formulas

$$\frac{\theta'(t+1)}{\theta(t+1)}=\frac{\theta'(t)}{\theta(t)}, \qquad \frac{\theta'(t+\tau)}{\theta(t+\tau)}=\frac{\theta'(t)}{\theta(t)}-2\pi i,$$

taking into account the zero weight condition $\Lambda=\Sigma n_j\alpha_j$.

Therefore we may view the equations as algebraic equations on $E_\tau^n=E_\tau\times\cdots\times E_\tau$, a product of elliptic curves. Moreover, we have an action of $S_c=S_{n_1}\times\cdots\times S_{n_r}$ permuting t_i with the same $c(i)$, that maps solutions to solutions, and does not change ψ. Let $D(c)$ be the set of $t\in E_\tau^n$ such that $t_i=t_j$ for some $i\neq j$ with $(\alpha_{c(i)},\alpha_{c(j)})\neq 0$, or $t_i=0$ for some i with $(\Lambda,\alpha_{c(i)})\neq 0$. On this set the Bethe ansatz equations are singular.

DEFINITION. With the notation of the Theorem, assume that $n_j>0$ for all j. The Hermite–Bethe variety $HB(c)$ is the subvariety of $(E_\tau^n-D(c))/S_c$ defined by the equations (24).

The remaining r equations determine ξ as a function of a solution t of (24). They can be chosen to be the equations (23) with $j=n_1,n_1+n_2,\ldots,n$. We see from this formula that if t_j is replaced by $t_j+n+m\tau$, then ξ is shifted by $-m\alpha_{c(j)}$. It is easy to see that these replacements do not change the eigenfunction ψ. Therefore we have a map $\xi\colon HB(c)\to\mathfrak{h}^*/Q$ mapping t to ξ, and $HB(c)$ parametrizes eigenfunctions ψ such that $\psi(\lambda+\omega)=\mathrm{e}^{2\pi i\xi(\omega)}\psi(\lambda)$, $\omega\in P^\vee$.

One would like to prove that ‘all’ eigenfunctions are obtained in this way. In the general case not much is known, however in the scalar case and the case of the adjoint representation we have the following result:

THEOREM 7.3. *Let* $\mathfrak{g}=\mathrm{sl}_N$, $U=S^{pN}\mathbb{C}^N$ *or the adjoint representation. Then for each generic* $\xi\in\mathfrak{h}$ *there are* $\dim(U[0])$ *solutions* $t\in\mathbb{C}^n$ *of the Bethe ansatz equations* (23). *The corresponding eigenfunctions obey* $\psi(\lambda+\omega)=\mathrm{e}^{2\pi i\xi(\omega)}\psi(\lambda)$, $\omega\in P^\vee$, *and are linearly independent.*

The proof of this theorem is essentially the same as in the trigonometric case (Theorem 5.3): in the case of sl_2 one has Hermite’s result, and one proceeds by induction in N. The point is that one only has to use the asymptotic behavior of the theta functions involved, and this is the same as in the trigonometric case.

8. The q-deformed case

We give here the q-deformed version of our first formula. Our conventions for quantum groups are the following. Let q be a complex number different from 0, 1 or -1. We fix a logarithm of q, so that q^a is defined for all $a \in \mathbb{C}$. We normalize the invariant bilinear form on $\mathfrak{g}$ in such a way that $(\alpha_j, \alpha_l) \in \mathbb{Z}$, for all $j, l \in \{1, \ldots, r\}$. The Drinfeld–Jimbo quantum universal enveloping algebra $U_q\mathfrak{g}$, a Hopf algebra, is the algebra with unit over $\mathbb{C}$ generated by f_j, e_j, commuting elements k_j, and their inverses k_j^{-1}, $j = 1, \ldots, r$ with relations $k_j e_l k_j^{-1} = q^{(\alpha_j,\alpha_l)} e_l$, $k_j f_l k_j^{-1} = q^{-(\alpha_j,\alpha_l)} f_l$, $e_j f_l - f_l e_j = \delta_{jl}(k_j - k_j^{-1})/(q - q^{-1})$, and deformed Serre relations $s_a(q) = 0$, $a = 1, \ldots, 2m$, (see [1]). The coproduct is defined on generators to be $\Delta(f_j) = f_j \otimes 1 + k_j^{-1} \otimes f_j$, $\Delta(e_j) = e_j \otimes k_j + 1 \otimes e_j$ and $\Delta(k_j^{\pm 1}) = k_j^{\pm 1} \otimes k_j^{\pm 1}$. We consider modules M over $U_q\mathfrak{g}$ admitting a weight decomposition into finite dimensional weight spaces $M[\mu]$, $\mu \in \mathfrak{h}^*$, on which k_j acts as $q^{(\alpha_j,\mu)}$. In particular, the Verma module of weight μ is the quotient $M_\mu = U_q\mathfrak{g}/I(\mu)$ by the left ideal $I(\mu)$ generated by $k_j - q^{(\alpha_j,\mu)}1$, e_j, $j = 1, \ldots, r$. It is generated by its highest weight vector v_μ, the class of 1.

If U is a finite dimensional $U_q\mathfrak{g}$-module, and Φ a homomorphism of $U_q\mathfrak{g}$-modules $\Phi\colon M_{\xi-\rho} \to M_{\xi-\rho} \otimes U$, we may define, as in the classical case,

$$\psi(\lambda) = \frac{\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)} \mathrm{tr}_{M_{\xi-\rho}[\mu]} \Phi}{\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)} \mathrm{tr}_{M_{-\rho}[\mu]} 1} \in \mathrm{e}^{2\pi i\xi(\lambda)} \mathbb{C}[[X_1, \ldots, X_r]]. \tag{25}$$

Recall that $X_j = \mathrm{e}^{-2\pi i\alpha_j(\lambda)}$. As in the classical case, we have for generic ξ, q, and for each u in the zero-weight space $U[0]$ of a finite dimensional $U_q\mathfrak{g}$-module U, a unique homomorphism of $U_q\mathfrak{g}$-modules $\Phi\colon M_{\xi-\rho} \to M_{\xi-\rho} \otimes U$, sending the generating vector $v_{\xi-\rho}$ to $v_{\xi-\rho} \otimes u$ [4]. So, in the generic case, the ψ-function is uniquely determined by q, ξ and a vector $u \in U[0]$. It is a formal power series, but its explicit expression below shows that it is actually a meromorphic function of λ.

If $\mathfrak{g} = \mathrm{sl}_N$ and U is a deformation of the pNth symmetric power of $\mathbb{C}^N$, it was shown by Etingof and Kirillov [4] that $\psi(\lambda)$ is a common eigenfunction of a commuting family of difference operators (related by conjugation by a known function to the A_{N-1}-Macdonald operators).

As in the classical case, the image of the generating vector is a singular vector (a vector killed by e_i, $i = 1, \ldots, r$) of weight $\xi - \rho$ and all singular vectors of weight $\xi - \rho$ correspond to some homomorphism. Set $f_L = f_{l_1} \ldots f_{l_m}$ for a multiindex $L = (l_1 \ldots, l_m)$. Our formula gives, for each $u \in U[0]$, ψ in terms of the coefficients u_L in the formula of the the unique singular vector of the form $v \otimes u + \Sigma_L f_L v_{\xi-\rho} \otimes u_L$ These coefficients are given explicitly in terms of u and the inverse Shapovalov matrix, see [5].

THEOREM 8.1. *Let $\xi \in \mathfrak{h}^*$ be generic and let $v_{\xi-\rho} \otimes u + \Sigma_L f_L v_{\xi-\rho} \otimes u_L$ be a singular vector of weight $\xi - \rho$ in $M_{\xi-\rho} \otimes U$. Let $\Phi \in \mathrm{Hom}_{U_q\mathfrak{g}}(M_{\xi-\rho}, M_{\xi-\rho} \otimes U)$*

be the corresponding homomorphism, and ψ *the* ψ*-function* (25). *Then* $\psi(\lambda) = \mathrm{e}^{2\pi i\xi(\lambda)}(u + \Sigma_L A_L(\lambda)u_L)$, *with*

$$A_{(l_1,\ldots,l_p)}(\lambda) = \sum_{\sigma\in S_p} q^{a(L,\sigma)} \left(\prod_{j=1}^{p} \frac{X_{l_{\sigma(j)}}^{a_j(\sigma)+1}}{1 - X_{l_{\sigma(1)}}\ldots X_{l_{\sigma(j)}} q^{|\alpha_{l_{\sigma(1)}}+\cdots+\alpha_{l_{\sigma(j)}}|^2}}\right) \times f_{l_{\sigma(1)}}\ldots f_{l_{\sigma(p)}},$$

where $a_j(\sigma)$ *is the cardinality of the set of* $m \in \{j,\ldots,p-1\}$ *such that* $\sigma(m+1) < \sigma(m)$, $X_j = \exp(-2\pi i\alpha_j(\lambda))$, *and*

$$a(L,\sigma) = \sum_{j=1}^{p}(\alpha_{l_j}, \rho - \xi) + \sum_{k<j,\sigma(k)>\sigma(j)} (\alpha_{l_{\sigma(j)}}, \alpha_{l_{\sigma(k)}}) + \sum_{m\in S}\left(\sum_{j=1}^{m} \alpha_{l_{\sigma(j)}}, \sum_{j=1}^{m} \alpha_{l_{\sigma(j)}}\right),$$

$$S = \{m \in \{1,\ldots,p-1\} \mid \sigma(m) > \sigma(m+1)\} \cup \{p\}.$$

We conclude this section by giving a conjectural formula for Macdonald polynomials, which is a q-deformation of Corollary 6.5. The A_{N-1} Macdonald polynomials [12] are symmetric polynomials $P_\nu(x,q,t)$ (we use the notation of [4]) in N variables $x_1,\ldots x_N$, depending on parameters q and t. They are labeled by dominant integral weights ν of gl_N, i.e., decreasing sequence $\nu_1 \geqslant \cdots \geqslant \nu_N$ of non-negative integers. We consider the case where $t = q^k$ for some positive integer k as in [4]. The symmetric polynomials $P(x,q,q^k)$ are uniquely characterized by having leading term $x_1^{\nu_1}\ldots x_N^{\nu_N}$ (as $x_i/x_{i+1} \to \infty$, $i = 1,\ldots,N-1$) with unit coefficient and by being orthogonal with respect to the inner product $(f,g) =$ constant term of $f(x_1,\ldots,x_N)g(x_1^{-1},\ldots x_N^{-1})\Delta(x)$, where

$$\Delta(x) = \prod_{i\neq j}\prod_{m=0}^{k}(1 - q^{2m}x_i/x_j).$$

Obviously $P_{\nu+(1,\ldots,1)} = x_1\ldots x_N P_\nu$ so it is sufficient to evaluate Macdonald polynomials on the hypersurface $x_1\ldots x_N = 1$. Then ν may be considered modulo $\mathbb{Z}(1,\ldots,1)$, i.e., as sl_N dominant weight. The q-analogue of Corollary 6.5 is the following conjecture, which can be proved with the same method as in the classical case if $N = 2$ or 3, but is open in the general case.

CONJECTURE 8.2. Consider the scalar case $\mathfrak{g} = \mathrm{sl}_N$, with $\mathfrak{h} = \{\lambda \in \mathbb{C}^N \mid \Sigma\lambda_i = 0\}$, $U = S^{pN}\mathbb{C}^N$, and fix an identification of $U[0]$ with $\mathbb{C}$. Set $x_j = \mathrm{e}^{2\pi i\lambda_j}$.

Let $\nu \in \mathfrak{h}^*$ be a dominant integral weight, and for $\xi \in \mathfrak{h}^*$, let ψ_ξ denote the function given in Theorem 8.1 with $u = 1$. Then

$$P_\nu(x, q, q^{p+1}) = c_\nu \sum_{w \in S_N} \frac{\psi_{w_0\nu-(p+1)\rho}(w\lambda)}{\delta(w\lambda)},$$

where

$$\delta(\lambda) = \prod_{m=0}^{p} \prod_{j<l} (\mathrm{e}^{\pi i(\lambda_j - \lambda_l)} - q^{-2m} \mathrm{e}^{\pi i(\lambda_l - \lambda_j)}),$$

for some constant $c_\nu \neq 0$. Here w_0 denotes the longest element of the Weyl group, i.e., the permutation $j \mapsto N + 1 - j$ of S_N.

As before, the constant c_ν may be computed in terms of the q-analogue of the scattering matrix (21)

$$c_\nu^{-1} = \prod_{i>j} \prod_{m=0}^{p} \frac{[m + \nu_i - \nu_j - (p+1)(i-j)]_q}{[m - \nu_i + \nu_j + (p+1)(i-j)]_q}, \qquad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

9. Proof of Theorems 3.1 and 8.1

We give the proof of Theorem 8.1. The proof of Theorem 3.1 is essentially a special case.

The proof is in two parts. We first prove that the calculation can be done in the algebra without Serre relation (Proposition 9.2), and then do this calculation, reducing it to a combinatorial problem.

We denote by $U_q\mathfrak{b}$ the subalgebra of $U_q\mathfrak{g}$ generated by $f_j, k_j^{\pm 1}$, $j = 1, \ldots, r$ and by $U_q\mathfrak{n}$ the subalgebra generated by f_j, $j = 1, \ldots, r$. We have $\Delta(U_q\mathfrak{n}) \subset U_q\mathfrak{b} \otimes U_q\mathfrak{n}$. Let us introduce a Q-grading of these algebras by setting $\deg(f_j) = \alpha_j$, $\deg(k_j) = 0$. What we need to know about the deformed Serre relations is that (i) $U_q\mathfrak{n}$ is the quotient of the free algebra generated by $f_1, \ldots, f_r$ by the ideal J_q generated by the deformed Serre relations $s_1(q)$,...,$s_m(q)$, (ii) $s_1(q), \ldots, s_m(q)$ are homogeneous polynomials in the f_j with coefficients in $\mathbb{Z}[q, q^{-1}]$ reducing to the (classical) Serre relations at $q = 1$, and (iii) $\Delta(s_a(q)) = s_a(q) \otimes 1 + K \otimes s_a(q)$ for some $K = \prod k_j^{-m_j}$.

We will also need the fact that the dimensions of the spaces of fixed degree in $U_q\mathfrak{n}$ are independent of q and coincide with the dimensions of the classical enveloping algebra $U\mathfrak{n}$ of the Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$ generated by $f_j = e_{-\alpha_j}$, $j = 1, \ldots, r$ (see, e.g., [1]).

The first observation is that the computation of the trace, given the components of the singular vector is a computation in $U_q\mathfrak{n}$:

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LEMMA 9.1. *Let for* $w \in M_{\xi-\rho}$, $\Phi^{\mathfrak{n}}(w)\colon M_{\xi-\rho} \to M_{\xi-\rho} \otimes U_q\mathfrak{n}$ *be the unique homomorphism of* $U_q\mathfrak{n}$*-modules mapping* $v_{\xi-\rho}$ *to* $w \otimes 1$. *Then, in the notation of Theorem 8.1,*

$$A_L(\lambda) = \frac{\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)} \mathrm{tr}_{M_{\xi-\rho}[\mu]} \Phi^{\mathfrak{n}}(f_L v_{\xi-\rho})}{\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)} \mathrm{tr}_{M_{-\rho}[\mu]} 1}.$$

This Lemma follows immediately from the fact that Verma modules are free over $U_q\mathfrak{n}$.

Let us introduce algebras without Serre relations.

DEFINITION. Let $U_q\widetilde{\mathfrak{b}}$ be the algebra over $\mathbb{C}$ generated by $f_1, \dots, f_r, k_1^{\pm 1}, \dots, k_r^{\pm 1}$ with relations $k_j f_k k_j^{-1} = q^{-(\alpha_j, \alpha_k)} f_k$, $k_j k_l = k_l k_j$, $k_j k_j^{-1} = 1$ and with coproduct as in $U_q\mathfrak{g}$. Let $U_q\widetilde{\mathfrak{n}} \subset U_q\widetilde{\mathfrak{b}}$ be the free algebra on generators $f_1, \dots, f_r$. The *Verma module* $\widetilde{M}_\mu$ over $U_q\widetilde{\mathfrak{b}}$ of highest weight $\mu \in \mathfrak{h}^*$ is the left module $U_q\widetilde{\mathfrak{b}}/I(\mu)$ where $I(\mu)$ is the left ideal generated by $k_j - q^{(\mu, \alpha_j)}1$, $j = 1, \dots, r$. The image of 1 in $\widetilde{M}_\mu$ is denoted by $\widetilde{v}_\mu$.

The Verma module $\widetilde{M}_\mu$ has a weight decomposition with finite dimensional weight spaces, and is freely generated by $\widetilde{v}_\mu$ as a $U_q\widetilde{\mathfrak{n}}$-module. In particular, $M_\mu = \widetilde{M}_\mu / J_q \widetilde{v}_\mu$. Moreover, $\Delta(U_q\widetilde{\mathfrak{n}}) \subset U_q\widetilde{\mathfrak{b}} \otimes U_q\widetilde{\mathfrak{n}}$. Therefore the construction of the homomorphisms $\Phi^{\mathfrak{n}}(\widetilde{w})\colon \widetilde{M}_{\xi-\rho} \to \widetilde{M}_{\xi-\rho} \otimes U_q\widetilde{\mathfrak{n}}$ works also in this case and traces make sense (as a formal power series).

PROPOSITION 9.2. *Let for* $w \in M_{\xi-\rho}$, $\Phi^{\mathfrak{n}}(w)\colon M_{\xi-\rho} \to M_{\xi-\rho} \otimes U_q\mathfrak{n}$ *be the unique homomorphism of* $U_q\mathfrak{n}$*-modules mapping* $v_{\xi-\rho}$ *to* $w \otimes 1$, *and for* $\widetilde{w} \in \widetilde{M}_{\xi-\rho}$ *let* $\Phi^{\mathfrak{n}}(\widetilde{w})$ *be the same object for the algebra without Serre relations. Then for any* $\widetilde{w}$ *projecting to* w *under the canonical projection* $\widetilde{M}_{\xi-\rho} \to M_{\xi-\rho}$,

$$\frac{\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)}\, \mathrm{tr}_{M_{\xi-\rho}[\mu]} \Phi^{\mathfrak{n}}(w)}{\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)}\, \mathrm{tr}_{M_{-\rho}[\mu]} 1} = \frac{\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)}\, \mathrm{tr}_{\widetilde{M}_{\xi-\rho}[\mu]} \Phi^{\mathfrak{n}}(\widetilde{w})}{\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)}\, \mathrm{tr}_{\widetilde{M}_{-\rho}[\mu]} 1}. \tag{26}$$

The proof of this proposition requires some preparation, and will be completed after Lemma 9.7 below.

LEMMA 9.3. *If* $x \in U_q\mathfrak{n}$ *is homogeneous of degree* $\alpha = \Sigma_j m_j \alpha_j \in Q_+$, *then* $\Delta(x)$ *can be written as* $\Delta(x) = \prod_j k_j^{-m_j} \otimes x + \Sigma x_j' \otimes x_j''$, *where* x_j'' *are homogeneous of degree* $< \alpha$.

This lemma is an easy consequence of the form of the coproduct of the generators f_j.

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DEFINITION. Let $\tilde{\mathfrak{n}}$ be the free Lie algebra on r generators $f_1, \ldots, f_r$ (see [10]). An *iterated bracket of length one* is an element f_j of this set. An *iterated bracket of length* $l > 1$ is defined recursively as an expression of the form $[a, b]$, where a, and b are iterated brackets of length l_1, $l - l_1$ respectively, for some $l_1 \in \{1, \ldots, l-1\}$. An iterated bracket of length l is called *simple* if it is of the form $[f_{i_1}, [f_{i_2}, [\ldots, f_{i_l}]]]$.

LEMMA 9.4. *Let $a, b \in \tilde{\mathfrak{n}}$, and a be an iterated bracket. Then $[a, b]$ is a linear combination of elements of the form*

$$[f_{i_1}, [f_{i_2}, [\ldots, [f_{i_l}, b]]]]. \tag{27}$$

Proof. We use induction in the length of a. If the length of a is one there is nothing to prove. If $a = [a_1, a_2]$ is of length l, with a_i of length $l_i < l$, the Jacobi identity gives $[a, b] = [a_1, [a_2, b]] - [a_2, [a_1, b]]$. By the induction hypothesis, $b' = [a_1, b]$ and $b'' = [a_2, b]$ are a linear combination of elements of the desired form (27). Then we use the induction hypothesis once more with b replaced by b' and b''. □

We apply this lemma to the following situation. The Lie algebra $\mathfrak{n} \subset \mathfrak{g}$ is the quotient of the free Lie algebra $\tilde{\mathfrak{n}}$ on r generators by the Lie ideal $\mathfrak{s}$ generated by Serre relations $s_1, \ldots, s_m$, which are simple iterated brackets in $\tilde{\mathfrak{n}}$.

LEMMA 9.5. *Denote by $\tilde{\mathfrak{n}}$ the free Lie algebra on r generators $f_1, \ldots, f_r$, and let $s_1, \ldots, s_m$ be simple iterated brackets in $\tilde{\mathfrak{n}}$. Let $\mathfrak{s}$ be the Lie ideal generated by $s_1, \ldots, s_m$, and assume that $d = \dim(\tilde{\mathfrak{n}}/\mathfrak{s}) < \infty$. Then there exists a basis $b_1, b_2, \ldots$ of $\tilde{\mathfrak{n}}$ consisting of simple iterated brackets such that $b_{d+1}, b_{d+2}, \ldots$ is a basis of $\mathfrak{s}$, and such that, for $j > d$, b_j is of the form*

$$[f_{i_1}, [f_{i_2}, [\ldots, [f_{i_l}, s_k]]]].$$

Proof. By definition every element in $\tilde{\mathfrak{n}}$ is a linear combination of iterated brackets. The ideal $\mathfrak{s}$ is spanned by iterated brackets containing at least one of the s_j. Using the skew-symmetry of the bracket, we see that every element of $\mathfrak{s}$ is a linear combination of elements of the form $[a_1, [a_2, [\ldots, s_k]]]$ for some iterated brackets a_j. The claim then follows from the preceding lemma. □

By the Poincaré–Birkhoff–Witt theorem, the universal enveloping algebra $U\tilde{\mathfrak{n}}$ has a basis $b^J = b_k^{j_k} \cdots b_2^{j_2} b_1^{j_1}$ labeled by the set $\mathcal{J}$ of sequences $J = (j_1, j_2, \ldots)$ of non-negative integers with $j_l = 0$ for all sufficiently large l. As the Verma module $\widetilde{M}_\mu$ is freely generated by the highest weight vector $\tilde{v}_\mu$ as a $U\tilde{\mathfrak{n}}$-module, $(b^J \tilde{v}_\mu)_{J \in \mathcal{J}}$ is a basis of $\widetilde{M}_\mu$.

Let us extend this to the q-deformed case. Note that $U_q\mathfrak{n}$ as an algebra is the same for all q.

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LEMMA 9.6. *There exists a sequence* $b_1(q), b_2(q), \ldots$ *of homogeneous elements in* $U\tilde{\mathfrak{n}} \otimes \mathbb{C}[q, q^{-1}]$ *such that*

(i) $b_j(q) = b_j \operatorname{mod}(q-1)\mathbb{C}[q, q^{-1}]$

(ii) *If* q *is a transcendental number, the elements* $b^J(q) = \cdots b_2(q)^{j_2} b_1(q)^{j_1}$ *form a basis of* $U_q\tilde{\mathfrak{n}}$.

(iii) *For* $j \geqslant d+1$, $\Delta(b_j(q)) = b_j(q) \otimes 1 \operatorname{mod} U_q\tilde{\mathfrak{b}} \otimes J_q$.

Proof. We may take $b_j(q) = b_j$ if $j \leqslant d$. If $a \in U_q\tilde{\mathfrak{n}}$ is a homogeneous element of degree $\beta \in Q$, set $\mathrm{ad}_q(f_j)a = f_j a - q^{(\beta,\alpha_j)} a f_j$. If $j > d$ and $b_j = \mathrm{ad}(f_{j_1}) \ldots \mathrm{ad}(f_{j_l}) s_k$, set

$$b_j(q) = \mathrm{ad}_q(f_{j_1}) \ldots \mathrm{ad}_q(f_{j_l}) s_k(q).$$

It is clear that (i) holds. (ii) holds too, since the determinant of the matrix expressing, in each homogeneous component of $U\tilde{n} \otimes \mathbb{C}[q, q^{-1}]$, $b^J(q)$ in terms of b^L has a determinant in $\mathbb{Q}[q, q^{-1}]$ with the value 1 at $q = 1$, and is therefore invertible if q is transcendental. As for (iii), we know that $s_k(q)$ has the required property since $\Delta(s_k(q)) = s_k(q) \otimes 1 + K \otimes s_k(q)$ for some K. Moreover if $a \in U_q\mathfrak{n}$ of degree β obeys $\Delta(a) = a \otimes 1 \operatorname{mod} U_q\mathfrak{b} \otimes J_q$, then

$$\begin{aligned} \Delta(\mathrm{ad}_q(f_j)a) &= f_j a \otimes 1 + k_j^{-1} a \otimes f_j - q^{(\beta,\alpha_j)} \\ &\qquad \times (a f_j \otimes 1 + a k_j^{-1} \otimes f_j) \operatorname{mod} U_q\tilde{\mathfrak{b}} \otimes J_q \\ &= \mathrm{ad}_q(f_j)a \otimes 1 \operatorname{mod} U_q\tilde{\mathfrak{b}} \otimes J_q, \end{aligned}$$

since $k_j^{-1} a = q^{(\beta,\alpha_j)} a k_j^{-1}$. Therefore, by induction, $\Delta b_j(q) = b_j(q) \otimes 1 \operatorname{mod} U_q\tilde{b} \otimes J_q$. $\square$

Suppose that q is transcendental, and fix a sequence $b_j(q)$, as in Lemma 9.6 and thus a basis $(b^J(q))$ of $U_q\tilde{\mathfrak{n}}$. To simplify the notation, we will write, for any $J \in \mathcal{J}$, $\deg(J) = \deg(b^J(q))$. Let $\mathcal{J}''$, be the set of $J = (j_1, j_2, \ldots) \in \mathcal{J}$ such that $j_l = 0$ if $l > d$, and $\mathcal{J}'$ be the set of J such that $j_l = 0$ if $l \leqslant d$. Then $\mathcal{J} = \mathcal{J}' \times \mathcal{J}''$ canonically, and we may write the basis as $b^{J'}(q) b^{J''}(q)$, $(J', J'') \in \mathcal{J}' \times \mathcal{J}''$. Then $b^{J'}(q) b^{J''}(q) \in J_q$ if J' is nontrivial. Since the dimensions of the homogeneous components of $U_q\mathfrak{n}$ are the same as in the classical case, the basis elements with J' nontrivial form a basis of J_q and the classes of $b^{J''}(q)$ form a basis of $U_q\mathfrak{n} = U_q\tilde{\mathfrak{n}}/J_q$.

LEMMA 9.7. *Suppose that* $q \in \mathbb{C}$ *is transcendental. Then the elements* $b^J(q)$, $J \in \mathcal{J}'$ *form a basis of the subalgebra of* $U_q\tilde{\mathfrak{n}}$ *consisting of elements* x *such that* $\Delta(x) = x \otimes 1 \operatorname{mod} U_q\tilde{\mathfrak{b}} \otimes J_q$

Proof. Call B this subalgebra. By Lemma 9.6(iii), the linearly independent elements $b^J(q)$, $J \in \mathcal{J}'$ belong to B. What is left to prove is that any $x \in B$ can be written as a linear combination of these elements. Write $x = \Sigma_{\mathcal{J}' \times \mathcal{J}''} a_{J',J''} b^{J'}(q) b^{J''}(q)$,

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with complex coefficients $a_{J',J''}$. Let $Z \subset Q_+$ be the set of β such that there exist J', J'' with $\deg(J'') = \beta$ and $a_{J',J''} \neq 0$. An element $\beta \in Z$ is called maximal if $\beta' > \beta$ implies $\beta' \notin Z$. Obviously, every element in Z is $\leqslant$ some maximal element. Therefore, if we show that the only maximal element is 0, we have proved the Lemma.

By Lemma 9.6,

$$\Delta(x) = \sum a_{J',J''}(b^{J'}(q) \otimes 1)\Delta(b^{J''}(q)) \bmod U_q\tilde{\mathfrak{b}} \otimes J_q.$$

Let β be maximal. The terms whose second factor has degree β are, by Lemma 9.3,

$$\Delta(x) = + \cdots + \sum_{\deg(J'')=\beta} a_{J',J''}b^{J'}(q) \otimes b^{J''}(q) + \cdots \bmod U_q\tilde{\mathfrak{b}} \otimes J_q.$$

These terms must vanish if $\beta \neq 0$ since $x \in B$. But the elements $b^{J''}$ are linearly independent, even mod J_q. And, by construction, $a_{J',J''} \neq 0$ for some J', J'' with $\deg(J'') = \beta$. Therefore $\beta = 0$. $\square$

The proof of Proposition 9.2 is a calculation of the traces using the basis $b^J(q)\tilde{v}_{\xi-\rho}$, $J \in \mathcal{J}$ of $\widetilde{M}_{\xi-\rho}$ and $b^J(q)v_{\xi-\rho}$, $J \in \mathcal{J}''$ of $M_{\xi-\rho}$. It is sufficient to prove Proposition free in the case where q is transcendental, since all traces over weight spaces are clearly rational functions of q. Let $\tilde{w}$ and w be as in Proposition 9.2. If $J \in \mathcal{J}''$, $\Delta b^J(q)\tilde{w} = \Sigma_{L\in\mathcal{J}} b^L(q) \otimes A_{JL}$, for some $A_{JL} \in U_q\tilde{\mathfrak{n}}$.

The numerator of the left-hand side of (26) is then

$$\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)} \operatorname{tr}_{M_{\xi-\rho}[\mu]} \Phi^{\mathfrak{n}}(w) = \sum_{J\in\mathcal{J}''} A_{JJ}X_{\xi-\rho+\deg(J)} \bmod J_q.$$

On the other hand, the calculation of the numerator on the right-hand side of (26) involves (see Lemma 9.6(iii))

$$\Delta(b^{J'}(q)b^{J''}(q))w \otimes 1 = \sum_{L\in\mathcal{J}} b^{J'}(q)b^L(q) \otimes A_{J''L} \bmod U_q\mathfrak{b} \otimes J_q.$$

Now we claim that the only terms in this summation that may give a nontrivial contribution to the trace are those with $L \in \mathcal{J}''$. Let indeed $b^L(q) = b^{L'}(q)b^{L''}(q)$, with $L' \in \mathcal{J}'$ and $L'' \in \mathcal{J}''$. Since both $b^{J'}(q)$ and $b^{L'}(q)$ are in the algebra defined in Lemma 9.7, their product is also in this algebra and is thus a linear combination of $b^{M'}(q)$, $M' \in \mathcal{J}'$, $\deg(M') = \deg(J') + \deg(L')$. If $L' \neq 0$, this degree is strictly larger than $\deg(J')$ and therefore does not contribute to the trace. The trace is therefore

$$\sum_\mu \mathrm{e}^{2\pi i\mu(\lambda)} \operatorname{tr}_{\widetilde{M}_{\xi-\rho}[\mu]} \Phi^{\mathfrak{n}}(\tilde{w})$$

$$= \sum_{J'\in\mathcal{J}'} \sum_{J''\in\mathcal{J}''} A_{J''J''}X_{\rho-\xi+\deg(J'')}X_{\deg(J')} \bmod J_q$$

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$$= \left(\sum_{J' \in \mathcal{J}'} X_{\deg(J')} \right) \sum_{\mu} \mathrm{e}^{2\pi i \mu(\lambda)} \operatorname{tr}_{M_{\xi-\rho}[\mu]} \Phi^{\mathfrak{n}}(w) \bmod J_q.$$

On the other hand,

$$\sum_{\mu} \mathrm{e}^{2\pi i \mu(\lambda)} \operatorname{tr}_{\widetilde{M}_{-\rho}[\mu]} 1 = \sum_{J' \in \mathcal{J}'} \sum_{J'' \in \mathcal{J}''} X_{\rho + \deg(J'') + \deg(J')}$$

$$= \left(\sum_{J' \in \mathcal{J}'} X_{\deg(J')} \right) \sum_{\mu} \mathrm{e}^{2\pi i \mu(\lambda)} \operatorname{tr}_{M_{-\rho}[\mu]} 1.$$

Taking the ratio of the last two expression completes the proof of Proposition 9.2.

We now turn to the actual calculation of the ratio of traces on the right-hand side of (26). We use the basis $(f_J = f_{j_1} \dots f_{j_m})$ of $U_q \tilde{\mathfrak{n}}$, labeled by finite sequences J in $\{1, \dots, r\}$ of arbitrary length $m \geqslant 0$ (if $m = 0$, we set $f^{(\,)} = 1$), and we may assume that $\tilde{w} = f_L \tilde{v}_{\xi-\rho}$, with $L = (l_1, \dots, l_p)$. By definition, the trace in the numerator of (26) is the sum over $J = (j_1, \dots, j_m)$ of the diagonal elements B_{JJ} in

$$\Delta(f_J) f_L \tilde{v}_{\xi-\rho} \otimes 1 = \sum_M f_M \tilde{v}_{\xi-\rho} \otimes B_{MJ} \tag{28}$$

weighted by a factor $X_{\rho-\xi} \prod_{k=1}^m X_{j_k}$. Expanding the coproduct on the left-hand side of (28) gives 2^m terms, each of them of the form

$$q^a f_{j_{b_1}} \dots f_{j_{b_s}} f_L \tilde{v}_{\xi-\rho} \otimes f_{j_{d_1}} \dots f_{j_{d_{m-s}}}$$

labeled by subsets $B = \{b_1 < \dots < b_s\}$ of $\{1, \dots, m\}$ with complement $D = \{d_1 < \dots < d_{m-s}\}$. The exponent of q is

$$a = \left(\xi - \rho + \sum_j \alpha_{l_j}, \sum_{d \in D} \alpha_{j_d} \right) + \sum_{B \ni b > d \in D} (\alpha_{j_b}, \alpha_{j_d}). \tag{29}$$

This term gives a contribution to the trace if and only if

$$(j_{b_1}, \dots, j_{b_s}, l_1, \dots, l_p) = (j_1, \dots, j_m). \tag{30}$$

If it does, the contribution is

$$q^a X_{\xi-\rho} X_{j_1} \dots X_{j_m} f_{j_{d_1}} \dots f_{j_{d_{m-s}}}.$$

Note that the j_{d_i} are necessarily equal to the l_i up to ordering. In particular $m - s = p$. The terms contributing to the trace in the numerator are therefore in one-to-one correspondence with pairs consisting of a finite sequence $(j_1, \dots, j_m)$ and a subset

$B \subset \{1, \ldots, m\}$ obeying (30). Let $B = \{1 \leqslant b_1 < b_2 < \cdots < b_s\} \subset \mathbb{N}$ be given. If $b_j = j$ for all j, set $t = s+1$. Otherwise let t be the smallest integer so that $b_t > t$. Then the condition (30) on J determines uniquely $j_t, j_{t+1}, \ldots, j_m$ in terms of L, and gives no constraint on $j_1, \ldots, j_{t-1}$. Therefore we can restrict our attention to the case $t = 1$: the general solution (J, B) of (30) is $J = (j_1, \ldots, j_{t-1}, j'_1, \ldots, j'_m)$, $B = \{1, \ldots, t-1, b'_1 + t - 1, \ldots, b'_s + t - 1\}$ where (J', B') is a solution with $b'_1 > 1$, and $j_1, \ldots, j_{t-1}$ are arbitrary. The contribution to the trace of such a solution (J, B) is $X_{j_1} \ldots X_{j_{t-1}}$ times the contribution of (J', B'). The sum over all such solutions with fixed (J', B') gives a factor that cancels with the trace in the denominator (except for X_ρ).

$$\sum_\mu \mathrm{e}^{2\pi i \mu(\lambda)} \operatorname{tr}_{\widetilde{M}_{-\rho}[\mu]} 1 = X_\rho \sum_{m=0}^{\infty} \sum_{j_1, \ldots, j_m} X_{j_1} \ldots X_{j_m}.$$

This implies

LEMMA 9.8. *Let* $L = (l_1, \ldots, l_p)$.

$$\frac{\sum_\mu \mathrm{e}^{2\pi i \mu(\lambda)} \operatorname{tr}_{\widetilde{M}_{\xi-\rho}[\mu]} \Phi^{\mathfrak{n}}(f_L \widetilde{v}_{\xi-\rho})}{\sum_\mu \mathrm{e}^{2\pi i \mu(\lambda)} \operatorname{tr}_{\widetilde{M}_{-\rho}[\mu]} 1}$$
$$= \mathrm{e}^{2\pi i \xi(\lambda)} \sum_{(J,B)} q^a X_{j_1} \ldots X_{j_m} f_{j_{d_1}} \ldots f_{j_{d_p}}, \tag{31}$$

where (J, B) *runs over all solutions of* (30) *such that* $b_1 > 1$. *The exponent* $a = a(J, B)$ *is* (29), *and* $\{d_1 < \cdots < d_p\}$ *is the complement of* B.

Our problem is therefore reduced to the combinatorial problem of finding all solutions (J, B) of (30) with $b_1 > 1$ and computing their contribution to the trace. The problem can be reformulated as follows: consider a circle with p distinct marked points numbered counterclockwise from 1 to p starting from a base point $*$ distinct from the marked points. The marked points are assigned labels: the jth point is assigned the label l_j. Solutions (J, B) are in one-to-one correspondence with games that a player can play erasing marked point on this circle according to the following rules. The player walks around the circle in *clockwise* direction starting from $*$ a finite number of times to return to $*$. Each move consists of proceeding to the next (not yet erased) marked point. When the player meets a marked point, he or she has the option of erasing it. If he or she does not erase it, the score is X_l where l is the label of the point, and we say that the player 'visited' the point. The score for erasing the point is $X_l q^{(\alpha_l, \beta)}$, where $\beta = \sum_j m_j \alpha_{l_j}$ and m_j is the number of times the jth point has been visited previously. The game continues until all marked points have been erased. The score of the game is the

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product of scores collected during the game. For instance, if $L = (1, 1, 2)$, one game consists of the sequence VEVEE of visits (V) and erasures (E). Its score is

$$X_2 X_1 q^{(\alpha_1,\alpha_2)} X_1 X_2 q^{(\alpha_2,\alpha_1+\alpha_2)} X_1 q^{(\alpha_1,\alpha_1+\alpha_2)} = X_1^3 X_2^2 q^{(\alpha_1,\alpha_2)+|\alpha_1+\alpha_2|^2}.$$

The relation with the previous description is that $j_m, \ldots, j_2, j_1$ is the list of labels of marked points met (visited or erased) at each move of the game, and B indicates at which moves points are visited. The contribution of a solution (J, B) to (31) is the overall factor

$$q^{-(\Sigma_j \alpha_{l_j}, \xi - \rho - \Sigma_j \alpha_{l_j})} \mathrm{e}^{2\pi i \xi(\lambda)}, \tag{32}$$

times the score of the corresponding game times $f_{i_p} \ldots f_{i_1}$ where $i_1, \ldots, i_p$ are the labels of the erased points, in order of erasure.

The next step is a reduction to the classification of ‘minimal’ games. An *empty round* in a game is a sequence of of subsequent visits of all marked points exactly once without any erasures. A game without empty rounds is called *minimal*. Out of a game we can construct a new game by inserting an empty round just before an erasure. The score of the new game is $\prod_{s\in E} X_{l_s} q^b$ times the score of of the old game, where the product is taken over the set E of marked points yet to be erased, and $b = |\Sigma_{s\in E} \alpha_{l_s}|^2$. Any game can be obtained from a unique minimal game by doing this construction sufficiently many times. The games obtained this way have all the same sequence of erased points, and give thus proportional contributions to the trace.

Minimal games are in one-to-one correspondence with permutation $\sigma \in S_p$: we erase first $\sigma(p)$, then $\sigma(p-1)$, and so on, always at the earliest opportunity. The example above gives the minimal game corresponding to the permutation (231). If σ is the identity, the score of the corresponding minimal game is $X_{l_1} \ldots X_{l_p}$. For general σ, we must calculate the numbers b_{jk} of times the point $\sigma(k)$ has been visited before $\sigma(j)$ is erased. The score of the minimal game corresponding to σ is then

$$q^c \prod_{j=1}^{p} X_{l_{\sigma(j)}}^{b_{jj}+1}, \qquad c = \sum_{j,k=1}^{p} b_{jk} (\alpha_{l_{\sigma(j)}}, \alpha_{l_{\sigma(k)}}). \tag{33}$$

The sum of the scores of all games obtained from this game by inserting empty rounds is then

$$\frac{q^c \prod_{j=1}^{p} X_{l_{\sigma(j)}}^{b_{jj}+1}}{\prod_{j=1}^{p} (1 - X_{l_{\sigma(1)}} \ldots X_{l_{\sigma(j)}} q^{|\alpha_{l_{\sigma(1)}} + \cdots + \alpha_{l_{\sigma(j)}}|^2})}. \tag{34}$$

We proceed to compute the numbers b_{jk}. Roughly speaking, b_{jk} is the number of times one goes around the circle before erasing either $\sigma(j)$ or $\sigma(k)$. More precisely, let

$$S^\times = \{m \in \{1, \ldots, p-1\} \mid \sigma(m) > \sigma(m+1)\}.$$

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If $k < j$ and $\sigma(k) > \sigma(j)$, $\sigma(j)$ is erased first, and b_{jk} is the number of times the player passes through the base point $*$, including at the beginning of the game, before erasing $\sigma(j)$. This number is $b_{jk} = |\{m \in S^\times | m \geqslant j\}\ | +1$. If $k < j$, and $\sigma(k) < \sigma(j)$, b_{jk} is the number of passages through $*$ before erasing $\sigma(j)$, minus 1: $b_{jk} = |\{m \in S^\times \mid m \geqslant j\}|$. If $k \geqslant j$, we get in all cases the number of passages through $*$ before erasing $\sigma(k)$, minus 1: $b_{jk} = |\{m \in S^\times \mid m \geqslant k\}|$. In particular, $b_{jj} = |\{m \in \{j, \ldots, p-1\} | \sigma(m) > \sigma(m+1)\}$. The exponent c of q in (33) is then

$$c = \sum_{k<j, \sigma(k)>\sigma(j)} (\alpha_{l_{\sigma(j)}}, \alpha_{l_{\sigma(k)}}) + \sum_{m \in S^\times} \left(\sum_{j=1}^{m} \alpha_{l_{\sigma(j)}}, \sum_{j=1}^{m} \alpha_{l_{\sigma(j)}} \right).$$

The formula in the Theorem is then obtained by combining this formula with (34), (32).

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References

1. Chari, V. and Pressley, A.: *A Guide to Quantum Groups*, Cambridge University Press, 1994.
2. Chalykh, O. and Veselov, A.: Commutative rings of partial differential operators and Lie alrebras, *Commun. Math. Phys.* 126 (1990), 597–611.
3. Etingof, P. and Kirillov, A. Jr.: A unified representation-theoretic approach to special functions, *Funct. Anal. Appl.* 28(1) (1994) 91–94.
4. Etingof, P. and Kirillov, A. Jr.: Macdonald's polynomials and representations of quantum groups, *Math. Res. Lett.* 1 (1994), 279–296.
5. Etingof, P. and Kirillov, A. Jr.: Representation-theoretic proof of inner product and symmetry identities for Macdonald's polynomials, hep-th/9410169, to appear in *Comp. Math.*
6. Etingof, P. and Styrkas, K.: Algebraic integrability of Schrödinger operators and representations of Lie algebras, hep-th 9403135 (1994), to appear in *Comp. Math.*
7. Felder, G. and Varchenko, A.: Integral representation of solutions of the elliptic Knizhnik–Zamolodchikov–Bernard equations, *Int. Math. Res. Notices* No. 5 (1995), 221–133, and paper in preparation.
8. Heckman, G. and Opdam, E.: Root systems and hypergeometric functions I, *Comp. Math.* 64 (1987), 329–352.
9. Helgason, S.: *Groups and Geometric Analysis*, Associated Press, 1984.
10. Humphreys, J.: *Introduction to Lie Algebras and Representation Theory*, Springer, 1972.
11. Kulish, P., Reshetikhin, N. and Sklyanin, E.: Yang–Baxter equation and representation theory I, *Lett. Math. Phys.* 5 (1981), 393–403.
12. Macdonald, I.: A new class of symmetric functions, Publ. IRMA Strasbourg, 372/S-20, *Séminaire Lotharingien* (1988), 131–171.
13. Orlik, P. and Terao, H.: The number of critical points of a product of powers of linear factors, *Inv. Math.* 120 (1995), 1–14.
14. Reshetikhin, N. and Varchenko, A.: Quasiclassical asymptotics of solutions to the KZ equations, in *'Geometry, Topology, and Physics, for Raoul Bott'*, S.-T. Yau (ed.), International Press, 1995, pp. 293–322.
15. Schechtman, V. and Varchenko, A.: Arrangements of hyperplanes and Lie algebra homology, *Inv. Math.* 106 (1991), 139–194.

16. Sutherland, B.: Exact results for a quantum many-body problem in one dimension, *Phys. Rev.* A4 (1971), 2019–2021; *Phys. Rev.* A5 (1972), 1372–1376.
17. Varchenko, A.: Multidimensional hypergeometric functions and representation theory of quantum groups, *Advanced Series in Mathematical Physics*, Vol. 21, World Scientific, 1995.
18. Varchenko, A.: Critical points of the product of powers of linear functions and families of bases of singular vectors, *Comp. Math.* 97 (1995), 385–401.
19. Whittaker, E. T. and Watson, G. N.: *Modern Analysis*, Cambridge University Press, 1927.