SEMI-PRIME MODULES

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1. Introduction. Properties and characterizations for prime and semi-prime rings have been provided by A. W. Goldie **(2, 3)**. In a previous paper **(1)**, the authors used the results of **(2)** to characterize prime and uniform prime modules. It is the aim of the present paper to generalize Goldie's work on semi-prime rings **(3)** to modules. In this setting certain new properties will appear.

Notationally, in the work to follow, the symbol R always denotes a ring and all R-modules will be right R-modules.

In the theory of rings an ideal C is said to be *prime* if and only if whenever $AB \subseteq C$ for ideals A and B, then either $A \subseteq C$ or $B \subseteq C$. A ring is prime if the zero ideal is prime. In order to avoid confusion in definitions for modules it was deemed necessary to introduce the term *closed prime submodule* (see (2.1)) and call a module *annihilator prime* provided the zero submodule is closed prime (see (2.2)).

In §2 the *prime radical* $P(M)$ of an R-module M is defined as the intersection of the closed prime submodules. It is shown that the prime radical of $M/P(M)$ is zero.

A module is termed *semi-prime* if the prime radical is zero. It is proved in §3 that if M is a faithful semi-prime R-module, then R is a semi-prime ring. In addition, (3.3) states that if an R-module M is semi-prime, then M is a subdirect sum of modules, each of which is a prime module over the difference ring R modulo the annihilator. This describes semi-prime modules in terms of prime modules as discussed by the authors in **(1)**.

In §4, torsion-free modules over Goldie semi-prime rings are investigated. Theorem (4.8) states that in this case M is a semi-prime module where the zero submodule is the intersection of maximal non-large submodules that are closed prime. Then M is a subdirect sum of uniform annihilator prime modules. If M is finite-dimensional, then a finite number of such submodules will suffice. This analysis enables one to express the zero ideal of a Goldie semi-prime ring as an intersection of an explicit set of prime ideals (see (4.13)).

The last section also includes a characterization for faithful semi-prime modules that generalizes a result obtained in **(1)** for prime modules.

2. The prime radical. If N is a submodule of an R-module M, then

$$\mathrm{cl}(N) = \{m \in M | mI \subseteq N \text{ for some large right ideal } I \text{ of } R\}.$$

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If $\mathrm{cl}(N) = N$, then N is said to be *closed*. The *singular submodule* $Z(M)$ is defined as $\mathrm{cl}(0)$. If N_1 and N_2 are submodules of M, then $(N_1 : N_2)$ is the ideal consisting of all $a \in R$ such that $N_2 a \subseteq N_1$.

(2.1) *Definition*. A submodule N of an R-module M is termed *closed-prime* provided the following two conditions are satisfied:

(i) if N' is a submodule such that $N \subset N' \subseteq M$, then $(N : N') \subseteq (N : M)$,

(ii) $\mathrm{cl}(N) = N$.

Here the symbol $\subset$ means proper inclusion.

Note that condition (i) is equivalent to the statement that for every non-zero submodule N'/N of the right $R/(N : M)$-module M/N, one has $(\bar{0} : N'/N) = \bar{0}$, where $\bar{0}$ denotes the zero coset.

The module M itself is closed-prime since condition (i) is (vacuously) true in the case $N = M$. It is also clear that maximal submodules that are not large are closed-prime. We shall see later that for torsion-free modules over semi-prime rings with chain conditions, the maximal elements in the set of non-large submodules are closed-prime.

(2.2) *Definition*. An R-module M is termed *annihilator-prime* provided (0) is a closed-prime submodule of M.

Thus the module M is annihilator-prime if $Z(M) = 0$ and $(0 : N') \subseteq (0 : M)$ for all non-zero submodules N' of M.

From the preceding definitions and remarks we obtain

(2.3) PROPOSITION. *A submodule N of an R-module M is closed prime if and only if the R-module M/N is annihilator-prime.*

If M is a faithful annihilator-prime module, then $Z(M) = 0$ and $(0 : N') = 0$ for all non-zero submodules N' of M. If, in addition, R is a ring with maximum condition, then M is a prime R-module in the sense of **(1)**. For modules over rings without chain conditions we have

(2.4) *Definition*. A faithful annihilator-prime module is termed a *prime module*.

(2.5) THEOREM. *If M is an annihilator-prime R-module, then M is prime as a module over $R/(0 : M)$.*

Proof. Since M is faithful over $R/(0 : M)$ we need only show that $\mathrm{cl}(0) = 0$. If $x \in \mathrm{cl}(0)$, then $x\bar{L} = 0$ where $\bar{L}$ is a large right ideal in $\bar{R} = R/(0 : M)$. If L is the inverse image of $\bar{L}$ in the natural homomorphism of R onto $\bar{R}$, then L is a large right ideal and $xL = 0$. Since M is annihilator-prime over R, we obtain $x = 0$ and the theorem is proved.

(2.6) COROLLARY. *If N is a closed-prime submodule of an R-module M, then M/N is a prime module over $R/(N : M)$.*

Proof. If N is closed-prime, then by (2.3) M/N is an annihilator-prime R-module. Now employ (2.5).

(2.7) PROPOSITION. *If N is a closed-prime submodule of an R-module M, then $(N : M)$ is a closed prime ideal in R.*

Remark. An ideal of R is closed if it is closed as a right R-module.

Proof. If N is closed-prime, then by (2.6) M/N is a prime module over $R/(N : M)$. We may employ the argument used to establish (**1**, (1.2)) to prove that $R/(N : M)$ is a prime ring, and hence that $(N : M)$ is a prime ideal. If $a \in \mathrm{cl}((N : M))$, then $MaI \subseteq N$ for some large right ideal I of R. This implies $Ma \subseteq \mathrm{cl}(N) = N$ or $a \in (N : M)$. Therefore $\mathrm{cl}((N : M)) = (N : M)$.

We now define a radical for modules that is analogous to the prime radical of a ring as given in (**7**).

(2.8) *Definition.* The *prime radical* $P(M)$ of an R-module M is the intersection of all the closed-prime submodules of M.

It follows from this definition that $Z(M) \subseteq P(M)$ and consequently when $P(M) = 0$, $Z(M) = 0$.

One can obtain a correspondence between closed-prime submodules of an R-module M and the closed-prime submodules of a homomorphic image.

(2.9) PROPOSITION. *Let θ be an R-homomorphism of an R-module M onto an R-module W with kernel K. If $\{N_i\}$ is the collection of closed-prime submodules of M that contain K, then the correspondence $N_i \to \theta N_i$ is a one–one mapping of $\{N_i\}$ onto the set of all closed-prime submodules of W.*

Proof. As in (**7**, (2.45)), the mapping $N \to \theta N$ is a one-to-one inclusion preserving correspondence between the set of submodules of M containing K onto the totality of submodules of W. Thus it suffices to show that under this mapping the closed-prime submodules correspond to one another.

Suppose that N is closed-prime in M with $N \supseteq K$. Let $a \in (\theta N : \theta N')$ where N' is a submodule of M such that $\theta N \subset \theta N'$ (and hence $N \subset N'$). Then $N'a \subseteq \theta^{-1}(\theta N) = N$, so $a \in (N : N')$. Since N is closed-prime, this implies that $a \in (N : M)$ and we have $(\theta M)a \subseteq \theta N$. Therefore,

$$(\theta N : \theta N') \subseteq (\theta N : W).$$

Now let $\theta m \in \mathrm{cl}(\theta N)$, where $m \in M$. Then $mI \subseteq \theta^{-1}(\theta N) = N$, where I is a large right ideal in R. Thus $m \in \mathrm{cl}(N) = N$, or $\theta m \in \theta N$. We have proved that $\mathrm{cl}(\theta N) \subseteq \theta N$ and hence θN is closed-prime in W.

We complete the proof by showing that if V is a closed-prime submodule of W, then $\theta^{-1}V$ is closed-prime in M. Let $a \in (\theta^{-1}V : N')$, where $\theta^{-1}V \subset N'$. Then $V = \theta(\theta^{-1}V) \subset \theta N'$ and since V is closed-prime,

$$a \in (V : W) = (V : \theta M) = (\theta^{-1}V : M),$$

so (2.1), (i) is true for $\theta^{-1}V$.

To show that (2.1), (ii) holds, let $m \in \mathrm{cl}(\theta^{-1}V)$. Then $(\theta m)I \subseteq V$ for a large right ideal I of R and hence $\theta m \in \mathrm{cl}(V) = V$, i.e. $m \in \theta^{-1}V$.

(2.10) PROPOSITION. *If N is a submodule of an R-module M and if L is a closed-prime submodule of M, then $N \cap L$ is a closed-prime submodule of N.*

Proof. We may assume that $N \cap L \neq N$. Let N' be a submodule such that $N \cap L \subset N' \subseteq N$. Then $L \subset L + N'$ and we have

$$(N \cap L : N') \subseteq (L : L + N') \subseteq (L : M),$$

whence $(N \cap L : N') \subseteq (N \cap L : M)$. It is evident that $\mathrm{cl}(N \cap L) = N \cap L$. Therefore $N \cap L$ is a closed-prime in N.

(2.11) THEOREM. *If M is an R-module, then $P(M/P(M)) = \bar{0}$.*

Proof. Let θ be the natural homomorphism of M onto $M/P(M)$. By (2.8), every closed-prime submodule of M contains the kernel of θ. Hence, from (2.9), the correspondence $N_i \rightarrow \theta N_i$ is a one–one mapping of the set $\{N_i\}$ of all closed-prime submodules of M onto the set of all closed-prime submodules of $M/P(M)$. If $\theta m \in P(M/P(M))$ for $m \in M$, then $m \in \theta^{-1}(\theta N_i) = N_i$ for all i and we have $m \in P(M)$. Thus $\theta m = \bar{0}$.

3. Semi-prime modules. As in (**7**, (4.25)), a ring R is semi-prime if and only if the prime radical of R is zero. We shall use this approach to define a semi-prime module.

(3.1) *Definition.* An R-module M is *semi-prime* if and only if $P(M) = 0$.

Notice that by (2.11), $M/P(M)$ is semi-prime for all R-modules M.

One can characterize semi-prime modules in terms of subdirect sums.

(3.2) THEOREM. *An R-module M is semi-prime if and only if M is a subdirect sum of annihilator-prime R-modules.*

Proof. If M is semi-prime, then $0 = \cap N_i$ where $\{N_i\}$ is the collection of closed-prime submodules of M. Then M is isomorphic to a subdirect sum of the modules M/N_i where, by (2.3), each M/N_i is an annihilator-prime R-module.

Conversely, if M is isomorphic to a subdirect sum of annihilator-prime modules $\{H_i\}$, then there exist submodules N_i of M such that $H_i \cong M/N_i$ and $\cap N_i = 0$. Since each N_i is closed-prime by (2.3), we have $P(M) = 0$ and (3.2) is proved.

(3.3) COROLLARY. *If M is a semi-prime R-module, then M is a subdirect sum of modules M_i, where each M_i is a prime module over $R/(0 : M_i)$.*

Proof. Apply (2.5) to the "only if" part of (3.2).

(3.4) PROPOSITION. *Let M be a semi-prime R-module and let $\{N_i\}$ be the collection of closed-prime submodules of M. Then M is faithful if and only if*

$$\cap (N_i : M) = 0.$$

Proof. If $a \in \bigcap (N_i : M)$, then $Ma \subseteq N_i$ for all i and hence $Ma = 0$. Consequently, if M is faithful, then $\bigcap(N_i : M) = 0$. Conversely, if $\bigcap(N_i : M) = 0$ and $Ma = 0$, then $a \in (N_i : M)$ for all i and hence $a = 0$.

(3.5) PROPOSITION. *If M is a faithful semi-prime R-module, then R is a semi-prime ring.*

Proof. By (3.4), we have $\bigcap(N_i : M) = 0$, where $\{N_i\}$ is the collection of closed-prime submodules of M. From (2.7), each $(N_i : M)$ is a prime ideal in R. Hence the intersection of these prime ideals is zero, which implies that R is a semi-prime ring.

(3.6) THEOREM. *A complete direct product of semi-prime R-modules is a semi-prime R-module.*

Proof. It is not difficult to show that given a direct sum $M = M_1 \oplus M_2$ of R-modules, if L_1 is a closed-prime submodule of M_1, then $L_1 \oplus M_2$ is a closed-prime submodule of M. The methods used in **(5**, p. 416**)** may now be used to complete the proof.

4. Modules over semi-prime rings. A ring that satisfies the right quotient conditions of **(3)** is called a (right) *Goldie ring*. According to **(3**, (3.9)**)**, a right ideal of a semi-prime Goldie ring is large if and only if it contains a regular element. It follows directly that a module M over a Goldie semi-prime ring is torsion-free if and only if $Z(M) = 0$. (As in **(6)**, an R-module M is torsion-free if, whenever $mc = 0$, where $m \in M$ and c is regular in R, then $m = 0$.) The principal objective of the present section is to investigate the structure of such modules. Accordingly, in order to simplify our statements, we shall, *throughout this section, let M denote a torsion-free module over a Goldie semi-prime ring R and let Q denote the* (*semi-simple*) *quotient ring of R.*

If N is an R-submodule of M, then as in **(6**, (1.5)**)** N can be imbedded in the Q-module $N \otimes_R Q \cong NQ$. The elements of NQ may be written in the form nc^{-1} for $n \in N$, c regular in R, and we may assume that $N \subseteq NQ$ where $n \in N$ is identified with $n1 \in NQ$. Moreover, every submodule of MQ can be written in this way. Specifically, if T is a submodule of MQ, we have

$$T = (M \cap T)Q.$$

If N_1 and N_2 are R-submodules of M we write $N_1 \sim N_2$, and say that N_1 is related to N_2, if and only if N_1 and N_2 meet the same non-zero submodules of M. The following propositions are patterned after those given for rings by Goldie **(3)**.

(4.1) PROPOSITION. *If N_1 and N_2 are submodules of M, then the following conditions are equivalent:*

(i) $N_1 \sim N_2$,
(ii) $N_1 Q = N_2 Q$,
(iii) $\mathrm{cl}(N_1) = \mathrm{cl}(N_2)$.

Proof. Suppose that $N_1 \sim N_2$. For each $n \in N_1$ let

$$(N_2 : n) = \{a \in R \mid na \in N_2\}.$$

As in (**4**, p. 63) $(N_2 : n)$ is a large right ideal in R. Since R is semi-prime, $(N_2 : n)$ contains a regular element c and hence $N_1 \subseteq N_2 Q$. By symmetry, $N_2 \subseteq N_1 Q$, which proves that (i) implies (ii).

Conversely, suppose $N_1 Q = N_2 Q$ and let n be a non-zero element of N_1. Then $n = mc^{-1}$ for some $m \in N_2$ and c regular in R. Therefore $nR \cap N_2 \neq 0$, which implies that $N_1 \sim N_2$.

The equivalence of (i) and (iii) follows from the fact that if $N_1 \sim N_2$, then $N_1 \subseteq \mathrm{cl}(N_2)$.

(4.2) Corollary. *A submodule N of M is large if and only if $NQ = MQ$.*

(4.3) Proposition. *If N_1 and N_2 are submodules of M, then*

$$N_1 Q + N_2 Q = (N_1 + N_2)Q \quad \textit{and} \quad N_1 Q \cap N_2 Q = (N_1 \cap N_2)Q.$$

Proof. Use the method given in (**3**, p. 214).

(4.4) Proposition. *The minimal submodules of MQ are of the form NQ where N is a uniform submodule of M.*

Proof. If NQ is not minimal, there exist non-zero submodules N_1, N_2 of M such that $NQ = N_1 Q \oplus N_2 Q$. Then

$$N_1 Q \cap N_2 Q = (N_1 \cap N_2)Q = 0$$

or $N_1 \cap N_2 = 0$, so N is not uniform.

Conversely, if NQ is minimal, then N is uniform, for otherwise N contains non-trivial submodules N_1 and N_2 with $N_1 \cap N_2 = 0$. Then $N_1 Q \cap N_2 Q = 0$, contrary to the minimality of NQ.

One may easily prove

(4.5) Proposition. *If N is a submodule of M, then*

$$\mathrm{cl}(N) = NQ \cap M \quad \textit{and} \quad \mathrm{cl}(NQ \cap M) = NQ \cap M.$$

In (**3**, p. 202), Goldie defined the notion of *complement submodule* of an R-module M. If X denotes the set of *non-large* submodules of M, then a direct argument proves that the maximal complement submodules are precisely the maximal elements of X. We shall term the latter *maximal non-large submodules.*

(4.6) Theorem. *If N^* is a maximal submodule of MQ, then $N^* \cap M$ is a maximal non-large submodule of M. If N is a maximal non-large submodule of M, then NQ is a maximal submodule of MQ. The correspondence $N_i^* \to N_i^* \cap M$ is a one-to-one mapping of the set $\{N_i^*\}$ of maximal submodules of MQ onto the set of maximal non-large submodules of M.*

Proof. Suppose N^* is maximal in MQ and let N be a submodule of M such that $N \supseteq N^* \cap M$ and $NQ \neq MQ$. Then $N^* = (N^* \cap M)Q \subseteq NQ \subset MQ$ and we have $N^* = NQ$. Hence $N \subseteq N^* \cap M$, which proves that $N^* \cap M$ is a maximal non-large submodule of M.

Now suppose that N is a maximal non-large submodule of M. If N^* is a Q-submodule of MQ such that $NQ \subset N^* \subset MQ$, then $NQ \cap M \subset N^* \cap M$ and hence $N \subset N^* \cap M$. However, $N^* \cap M$ is non-large since

$$(N^* \cap M)Q = N^* \neq MQ.$$

This contradicts the maximality of N, so NQ is maximal in MQ.

The last statement follows from the fact that if N is a maximal non-large submodule, then $N = NQ \cap M$.

(4.7) THEOREM. *If N is a maximal non-large submodule of M, then N is closed-prime and intersection-irreducible. Consequently, M/N is a uniform annihilator-prime R module.*

Remark. A submodule N is intersection-irreducible if it cannot be written as an intersection of two submodules that properly contain N.

Proof. From (4.6) we may write $N = N^* \cap M$ where N^* is a maximal submodule of MQ. If N' is a submodule such that $N \subset N' \subseteq M$, then by the maximal nature of N and (4.2), $N'Q = MQ$. Hence

$$N'(N : N')Q \subseteq NQ = N^*.$$

As in **(3**, (5.2)**)**, $(N : N')Q$ is an ideal of Q so that

$$N'Q(N : N') \subseteq N'(N : N')Q \subseteq N^*.$$

Therefore $M(N : N') \subseteq N^* \cap M = N$, or $(N : N') \subseteq (N : M)$. Thus condition (2.1), (i) is satisfied by N. The fact that $N = \mathrm{cl}(N)$ is clear since $N \sim \mathrm{cl}(N)$.

If N is maximal non-large, it follows readily that N is intersection-irreducible; see **(4**, p.60**)**. It is evident that this implies uniformity for M/N.

(4.8) THEOREM. *The R-module M is semi-prime. Specifically, the zero submodule is the intersection of maximal non-large closed-prime submodules, and hence M is a subdirect sum of uniform annihilator-prime R-modules.*

Proof. Since Q is semi-simple, MQ can be expressed as a direct sum of simple submodules $\{L_i\}$. For each i let $N_i^* = \sum_{j\neq i} L_j$. (If there is only one L_i, then MQ is simple and, by convention, we take $N_i^* = 0$.) Then each N_i^* is a maximal submodule of MQ and $\cap N_i^* = 0$. By (4.6), $N_i = N_i^* \cap M$ is a maximal non-large submodule and is closed-prime by (4.7). Then

$$\cap N_i \subseteq \cap N_i^* = 0,$$

which implies $P(M) = 0$. The last statement follows from (2.3) and (4.7).

The next result characterizes faithful semi-prime modules over Goldie rings.

(4.9) THEOREM. *Let L be a faithful module over a Goldie ring R. Then L is semi-prime if and only if L is contained in a unitary right Q-module where Q is a semi-simple Artinian ring and a right quotient ring for R.*

Proof. If L is semi-prime, then by (3.5) R is a semi-prime ring. Moreover, since $Z(L) \subseteq P(L) = 0$, L is torsion-free over R. Then the module LQ fulfils the conditions cited in the statement of the theorem.

Conversely, if an R-module L is contained in a unitary right Q-module L' and if Q is a right quotient ring for R which is semi-simple Artinian, then by **(3)**, R is a Goldie semi-prime ring. Suppose $xc = 0$ where $x \in L$ and c is regular in R. Then c^{-1} exists in Q and, since L' is unitary, $x = x1 = (xc)c^{-1} = 0$. Hence L is torsion-free over R and, by (4.8), L is semi-prime.

Goldie **(3)** has called a module *finite-dimensional* if every direct sum of non-zero submodules has only a finite number of terms. In our case notice that M is finite-dimensional if and only if MQ is finite-dimensional. This follows from the fact that given a direct sum of submodules $T_1 \oplus \ldots \oplus T_k$ in MQ, the sum $(T_1 \cap M) \oplus \ldots \oplus (T_k \cap M)$ is direct in M. Likewise a direct sum $L_1 \oplus \ldots \oplus L_k$ in M induces a direct sum $L_1 Q \oplus \ldots \oplus L_k Q$ in MQ; cf. the proof of **(3**, (4.3)**)**.

(4.10) PROPOSITION. *If M is finitely generated, then M is finite-dimensional.*

Proof. If $x_1, x_2, \ldots, x_n$ are generators for M, then $x_1 1, x_2 1, \ldots, x_n 1$ are generators for MQ. Since Q is semi-simple, this implies that MQ is finite-dimensional. Then, from the preceding paragraph, M is finite-dimensional.

(4.11) PROPOSITION. *Let M be finite-dimensional and let N be a non-large submodule of M. Then N is maximal non-large if and only if N is intersection-irreducible.*

Proof. If N is intersection-irreducible, then by an argument similar to **(1**, (2.2)**)**, N is closed. From **(4**, 3.9)**)**, $\mathrm{cl}(N)$ is a finite intersection of maximal non-large submodules and, because of its intersection-irreducibility, N must equal one of these submodules. The converse follows from (4.7).

(4.12) THEOREM. *If M is finite-dimensional, then M is a finite subdirect sum of uniform annihilator-prime submodules.*

First proof. By our previous remarks, MQ is finite-dimensional. The proof of (4.8) now provides a *finite* number of maximal non-large closed-prime submodules $\{N_i\}$ with $\cap N_i = 0$. Then M is a subdirect sum of the modules $M_i = M/N_i$. Applying (2.3) and (4.7), we have the theorem.

Second proof. By **(4**, (3.7)**)** we may write $0 = \cap N_i$ where the N_i are maximal non-large submodules. The remainder of the proof is the same as that given above.

Let M and N_i be as in the proof of (4.12). Then

$$(0 : M) = (\cap N_i : M) = \cap (N_i : M)$$

where, by (2.7), each $(N_i : M)$ is a closed-prime ideal in R. If, in addition, M is faithful, then the zero ideal is a finite intersection of closed-prime ideals of the form $(N_i : M)$. Applying these remarks to a Goldie semi-prime ring R, considered as a module over itself, we obtain the following

(4.13) THEOREM. *If R is a Goldie semi-prime ring, there exist a finite number of prime ideals $\{P_i\}$ such that $0 = \cap P_i$. Moreover, each P_i is of the form $(J_i : R)$ where J_i is a maximal non-large right ideal which is closed-prime as a right R-module.*

We shall conclude with a characterization for Noetherian semi-prime modules.

(4.14) THEOREM. *Let R be a Goldie semi-prime ring. An R-module L is Noetherian semi-prime if and only if L is a finite subdirect sum of uniform Noetherian annihilator-prime R-modules.*

Proof. If L is Noetherian, it is finite-dimensional. In addition, if L is semi-prime and R is a Goldie semi-prime ring, then L is torsion-free. Using (4.12) we see that L is a finite subdirect sum of uniform annihilator-prime modules of the form L/N_i. It is evident that each such module is Noetherian.

Conversely, if L is a subdirect sum of uniform Noetherian annihilator-prime modules $L_1, \ldots, L_k$, then there exist submodules $N_1, \ldots, N_k$ in M such that $L_i \cong L/N_i$ and $\cap N_i = 0$. Since the N_i are closed-prime by (2.3), we have $P(L) = 0$. The fact that L is Noetherian follows as in (**1**, (3.3)).

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