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A NOTE ON GENERALIZED DIRECT PRODUCTS OF GROUPS

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In [1] Tang proved that the generalized direct product of a finite set of cyclic groups amalgamating subgroups which satisfy certain compatibility conditions always exists. In the proof, Theorem 4.1 is made use of. However, this theorem is not correct since we can construct examples of groups which satisfy the conditions of Theorem 4.1, but whose generalized direct product does not exist. Therefore, a modification of this result as pointed out by Professor Tang is given here, together with the resulting modification of the proof of the result stated above.

The terminology and notation are identical to those of [1] and we freely refer to the results of that paper.

THEOREM 1 (Modification of Theorem 4.1 of [1]). *Given a group amalgam consisting of groups* G_i, $1 \leqq i \leqq n$, *subgroups* $H_{ij} \subseteq Z(G_i)$ *and isomorphisms* $\Theta_{ij} : H_{ij} \to H_{ji}$, *if the GDPI of this amalgam exists so does the GDPII of*

$$\left(\prod_{i=1}^{n-1} G_i\right)_{H_{ij}=H_{ji}} = G$$

and G_n *amalgamating* H_{in} *with* H_{ni} *for* $i = 1, 2, \ldots, n-1$ *under isomorphisms* $\Theta_{in} : H_{in} \to H_{ni}$. *Conversely, if* $G = (\prod_{i=1}^{n-1} G_i)_{H_{ij}=H_{ji}}$ *exists and the GDPII* P *of* G *and* G_n *amalgamating* H_{in} *with* H_{ni} *by isomorphisms* Θ_{in} *exists such that for each* $i = 1, 2, \ldots, n-1$ $G_i \cap \{H_{jn}, 1 \leqq j \leqq n-1\} = H_{in}$, *then* P *is the GDPI of the given group amalgam.*

Proof. The first part of the theorem follows as in the proof of Theorem 4.1 of [1].

Conversely, P contains subgroups G_i, $1 \leqq i \leqq n$ and $[G_i, G_j] = 1$ for each $i \neq j$, $1 \leqq i, j \leqq n$ by the definition of P. Also, $G_i \cap G_j = H_{ij}$ for $i \neq j$, $1 \leqq i, j \leqq n-1$ by the definition of $G \subseteq P$ as a GDPI. Since it was assumed that $G_i \cap \{H_{jn}, 1 \leqq j \leqq n-1\} = H_{in}$ in P, we have $G_i \cap G_n = H_{in}$ for all $i = 1, 2, \ldots, n$ and thus P is the GDPI of the given group amalgam.

THEOREM 2 (Theorem 5.4 of [1]). *The GDPI of cyclic groups* G_i, $1 \leqq i \leqq n$, *amalgamating subgroups* $H_{ij} \subseteq Z(G_i)$ *by the isomorphisms* $\Theta_{ij} : H_{ij} \to H_{ji}$ *always exists if the compatibility conditions given in* [1] *are satisfied.*

Proof. In view of the proof of Theorem 5.4 of [1] and Theorem 1 above we need only show that $G_i \cap \{H_{jn}, 1 \leqq j \leqq n-1\} = H_{in}$. Since G_n is cyclic

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and isomorphic to $\{H_{jn}, 1 \leqq j \leqq n-1\}$ in

$$K = \left(\prod_{i=1}^{n-1} G_i\right)_{H_{ij}=H_{ji}},$$

we may assume that $\{H_{jn}, 1 \leqq j \leqq n\} = \{y\}$. Then $H_{jn} = \{y^{t_j}\}$ for $1 \leqq j \leqq n-1$ and there exist s_j such that $y = y^{t_1s_1+t_2s_2+\ldots+t_{n-1}s_{n-1}}$. Clearly $H_{jn} \subseteq G_i \cap \{H_{jn}, 1 \leqq j \leqq n\}$. If $y^k \in G_i$ then $y^{t_1s_1k+\ldots\ldots+t_{n-1}s_{n-1}k} = y^k$. This implies that $G_i \cap \{H_{jn}, 1 \leqq j \leqq n-1\} \subseteq \{G_i \cap H_{jn}, 1 \leqq j \leqq n-1\}$ and thus $G_i \cap \{H_{jn}, 1 \leqq j \leqq n-1\} = \{G_i \cap H_{jn}, 1 \leqq j \leqq n-1\}$. Now we note that $G_i \cap H_{jn} \subseteq G_i \cap G_j$. The induction hypothesis asserts that the GDPI K of $G_1, G_2, \ldots, G_{n-1}$ exists. Therefore $G_i \cap H_{jn} \subseteq H_{ij} = H_{ji}$. Applying the compatibility condition (ii) of [**1**, Definition 2.2], $H_{ji} \cap H_{jn} = H_{ij} \cap H_{in} \subset H_{in}$ for $1 \leqq j \leqq n-1$. It follows immediately that $\{G_i \cap H_{jn}, 1 \leqq j \leqq n-1\} = H_{in}$. This completes the proof.

It may be noted that a similar result no longer holds for slightly more complicated group amalgams. In particular it is possible to construct an example of 4 free abelian groups of rank 2 with cyclic subgroups satisfying the compatibility conditions whose GDPI does not exist.

Reference

1. C. Y. Tang, *An existence theorem for generalized direct products with amalgamated subgroups*, Can. J. Math. *18* (1966), 75–82.

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