Proceedings of the Edinburgh Mathematical Society (1985) **28**, 113–119 ©

CONGRUENCE-FREE REGULAR SEMIGROUPS

by W. D. MUNN

(Received 11th May 1984)

A semigroup is said to be congruence-free if and only if its only congruences are the universal relation and the identical relation. Congruence-free inverse semigroups were studied by Baird [**2**], Trotter [**19**], Munn [**15, 16**] and Reilly [**18**]. In addition, results on congruence-free regular semigroups have been obtained by Trotter [**20**], Hall [**4**] and Howie [**7**].

It was shown in [**15**] that if E is the semilattice of a congruence-free inverse semigroup, other than a simple group, then there is a congruence-free inverse semigroup T_E^* with semilattice (isomorphic to) E and such that each congruence-free inverse semigroup with semilattice E is isomorphic to a full inverse subsemigroup of T_E^*. For E as above, T_E^* can thus loosely be described as the "greatest" congruence-free inverse semigroup with semilattice E. The main purpose of the present paper is to extend this result from inverse semigroups to regular semigroups. Here we replace semilattices by regular biordered sets, as defined by Nambooripad [**17**], and make use of his work (equivalent to that of Hall [**3**]) on the analogue for regular semigroups of the inverse semigroup T_E introduced by the author in [**13**].

The first section provides a brief review of some properties of group congruences on a regular semigroup S and of the "hypercore" of S, a concept introduced by T. E. Hall and the author in [**5**]. In the second section, for a regular biordered set E, we define T_E^* to the hypercore of Nambooripad's semigroup T_E [**17**]. It is shown that if E is the biordered set of idempotents of a congruence-free regular semigroup which is not a simple group then T_E^* is a congruence-free regular semigroup with E as its biordered set of idempotents (to within isomorphism) and that each congruence-free regular semigroup with this biordered set of idempotents is isomorphic to a full regular subsemigroup of T_E^*. An example is given in the third section to show that there need not exist a "least" congruence-free inverse semigroup with semilattice E, where E is the semilattice of some congruence-free inverse semigroup. The paper ends with a short fourth section which provides sufficient conditions for A to be congruence-free, given that S and A are full regular subsemigroups of a regular semigroup T with $S \subseteq A \subseteq T$, that S and T are themselves congruence-free, and that T is not a simple group.

The principal results were announced by the author at a conference on semigroups held at Oberwolfach, W. Germany, in May 1981. In the version presented there, the hypercore of a regular semigroup was defined as the limit of its core series (see [**5**, Theorem 3]).

1. Preliminaries

The notation and terminology is based on that of Howie [**6**]. In particular, the set of idempotents of a semigroup S will be denoted by $E(S)$, the set of all inverses in S of an element a of S by $V_S(a)$, and the subsemigroup generated by a nonempty subset A of S by $\langle A\rangle$. By the *greatest* [*least*] subsemigroup of S with a given property $\mathscr{P}$ we mean a subsemigroup A with property $\mathscr{P}$ such that $A\supseteq B$ [$A\subseteq B$] for all subsemigroups B with property $\mathscr{P}$. If such an A exists then it is, of course, unique.

Now let S be a regular semigroup. It is well known and readily verified that a congruence ρ on S is a group congruence (that is, S/ρ is a group) if and only if $\rho\supseteq E(S)\times E(S)$. From this it follows that S possesses a least group congruence $\sigma(S)$, namely the congruence generated by $E(S)\times E(S)$ [**10**]. As in [**5**], the $\sigma(S)$-class containing $E(S)$ will be called the *core* of S and denoted by $\operatorname{core}(S)$. A subsemigroup T of S is termed

(i) *full* if and only if $E(S)\subseteq T$,

(ii) *unitary* if and only if

$$(\forall t\in T)(\forall x\in S)\quad [tx\in T\Rightarrow x\in T] \text{ and } [xt\in T\Rightarrow x\in T],$$

(iii) *self-conjugate* if and only if

$$(\forall t\in T)(\forall x\in S)\quad (\forall x'\in V_S(x))\quad x'tx\in T.$$

As is easily seen, there is a least self-conjugate subsemigroup of S containing a given subsemigroup T of S. It will be denoted by T^S. Thus S has a least full self-conjugate subsemigroup, namely $\langle E(S)\rangle^S$. Similarly, S has a least full unitary self-conjugate subsemigroup and, in [**8**], Feigenbaum showed that this coincides with $\operatorname{core}(S)$. Evidently $\langle E(S)\rangle^S\subseteq\operatorname{core}(S)$.

The following lemma [**8**; **9**, Corollary 2.2] provides a characterisation of $\sigma(S)$ in terms of the least full self-conjugate subsemigroup of S. It generalises an earlier result [**12**; **6**, Theorem V.3.1] on inverse semigroups.

Lemma 1. *Let S be a regular semigroup. Then, for all $a, b\in S$,*

$$(a,b)\in\sigma(S)\quad\Leftrightarrow\quad(\exists u,v\in\langle E(S)\rangle^S)\ au=vb. \qquad \square$$

In [**5**], T. E. Hall and the author introduced the concept of the *hypercore*, $\operatorname{hyp}(S)$, of a semigroup S. Let $\mathscr{S}_S$ denote the set of all subsemigroups A of S with no non-universal congruence ρ such that A/ρ is cancellative. Then, assuming that $\mathscr{S}_S\neq\varnothing$ (which is certainly true if $E(S)\neq\varnothing$), $\operatorname{hyp}(S)$ is the subsemigroup of S generated by the union of all A in $\mathscr{S}_S$. From [**5**, Theorems 1 and 2] we obtain a result that is the key to the developments in Section 2 below:

Lemma 2. *Let S be a regular semigroup. Then* $\operatorname{hyp}(S)$ *is a full unitary subsemigroup of S contained in* $\operatorname{core}(S)$ *and is the greatest regular subsemigroup A of S such that* $\operatorname{core}(A)=A$. $\square$

Another description of the hypercore of a regular semigroup is given in [**5**, Theorem 3].

2. The semigroup T_E^*

Recall that a semigroup S is termed "fundamental" if and only if the only congruence on S contained in $\mathscr{H}$ is the identity. For S regular, S is fundamental if and only if it has no idempotent-separating congruence other than the identity congruence [**10**; **6**, Proposition II.4.8].

For ease of reference we restate part of [**3**, Corollary 6] at this stage.

Lemma 3. *Every full regular subsemigroup of a fundamental regular semigroup is fundamental.* □

The set of idempotents of a regular semigroup is a regular biordered set, in the sense of Nambooripad [**17**]. Now let E be a regular biordered set. Then there exists a regular semigroup T_E with the following properties ([**17**]; see also Hall's earlier paper [**3**] for an equivalent result):

(i) $E(T_E) \cong E$ (*qua* biordered sets);

(ii) T_E is fundamental;

(iii) if S is a regular semigroup and $E = E(S)$ then there is a homomorphism $\theta: S \to T_E$ such that $E\theta = E(T_E)$ and $\theta \circ \theta^{-1} \subseteq \mathscr{H}$.

We denote hyp(T_E) *by* T_E^*.

If E is a semilattice then T_E and T_E^* coincide with the similarly-designated semigroups introduced by the author in [**13**] and [**14**]. (It should be noted that in [**17**] the notation T_E^* is used for an altogether different concept.)

We say that a regular semigroup S can be *fully embedded* in a regular semigroup T if and only if there is an injective homomorphism $\theta: S \to T$ such that $S\theta$ is a full subsemigroup of T.

Theorem 1. *Let E be a regular biordered set.*

(i) *T_E^* is a fundamental full unitary regular subsemigroup of T_E and* core$(T_E^*) = T_E^*$.

(ii) *If S is a fundamental regular semigroup with $E(S) \cong E$ and* core$(S) = S$ *then S can be fully embedded in T_E^*.*

Proof. (i) By Lemma 2, T_E^* is a full unitary regular subsemigroup of T_E and core$(T_E^*) = T_E^*$. But T_E is fundamental and so, by Lemma 3, T_E^* is fundamental.

(ii) Let S be a fundamental regular semigroup with $E(S) \cong E$ and core$(S) = S$. Then there exists a homomorphism $\theta: S \to T_E$ such that (a) $E(S)\theta = E(T_E)$ and (b) $\theta \circ \theta^{-1} \subseteq \mathscr{H}$. Evidently $S\theta$ is a regular subsemigroup of T_E. Moreover, by (b), since S is fundamental, θ is injective. Hence core$(S\theta) = S\theta$ and so, by Lemma 2, $S\theta \subseteq T_E^*$. Also, by (a), $E(S\theta) = E(T_E) = E(T_E^*)$. □

The principal result of this section (Theorem 2) will be obtained by combining Theorem 1 with the following lemma.

https://doi.org/10.1017/S0013091500003254 Published online by Cambridge University Press

Lemma 4. *Let A be a regular semigroup such that*

(i) *A is fundamental,*

(ii) $\operatorname{core}(A)=A$,

(iii) *A contains a congruence-free full regular subsemigroup.*

Then A is congruence-free.

Proof. Let ρ be a congruence on A other than the identity congruence. Since A is fundamental there exist $e, f \in E(A)$ such that $(e,f) \in \rho$ and $e \neq f$. By hypothesis, A contains a congruence-free full regular subsemigroup B. Write $\tau = \rho \cap (B \times B)$. Then τ is a congruence on B; also τ is not the identity congruence, since it contains (e,f). Consequently $\tau = B \times B$ and so $B \times B \subseteq \rho$. In particular, $E(A) \times E(A) \subseteq \rho$, which shows that ρ is a group congruence. But $\operatorname{core}(A)=A$; that is $\sigma(A)=A \times A$. Hence $\rho = A \times A$. Thus A is congruence-free. □

Theorem 2. *Let S be a congruence-free regular semigroup which is not a simple group and let E denote $E(S)$. Then T_E^* is a congruence-free regular semigroup and S can be fully embedded in T_E^*.*

Proof. By Theorem 1(i), T_E^* is a fundamental regular semigroup with $\operatorname{core}(T_E^*) = T_E^*$. Since S is not a group, $|E|>1$. Hence S is fundamental and $\operatorname{core}(S)=S$. Thus, by Theorem 1(ii), S can be fully embedded in T_E^*. The hypotheses of Lemma 4 are therefore satisfied when $A = T_E^*$ and so T_E^* is congruence-free. □

This theorem generalises an earlier result on inverse semigroups [**15**, Corollary 1.8]. Note that if $S=S^0$ above then T_E^* coincides with T_E.

3. A counterexample to a conjecture

Let E be the regular biordered set of idempotents of some congruence-free regular semigroup which is not a simple group. Theorem 2 can be interpreted as showing that there is a greatest congruence-free regular semigroup S with $E(S)=E$, namely T_E^*. It is natural to conjecture that there is also a least such semigroup.

For certain choices of E a least congruence-free regular semigroup S with $E(S)=E$ does indeed exist. It can be shown, for instance, that if S is a congruence-free (or, more generally, fundamental) completely 0-simple semigroup then $S \cong T_{E(S)}$ and so S is uniquely determined by $E(S)$. Again, consider a congruence-free idempotent-generated regular semigroup S. (Examples of such semigroups have been given by Howie [**7**]; see also Hall [**4**].) Let $E=E(S)$. Then, since S can be fully embedded in T_E^*, $S \cong \langle E(T_E^*) \rangle$. Now let A be any congruence-free regular semigroup with $E(A) \cong E$. Then A is isomorphic to a full regular subsemigroup A' of T_E^* and hence $\langle E(T_E^*) \rangle \subseteq A'$. Thus S can be regarded as the least congruence-free regular semigroup with regular biordered set E.

The conjecture is false, however, even in the inverse case, as will be demonstrated by the example below. First, we recall some definitions [**2, 14**] concerning a semilattice $E=E^0$. We say that E is *disjunctive* if and only if for all $e, f \in E$ with $e \neq f$ there exists

https://doi.org/10.1017/S0013091500003254 Published online by Cambridge University Press

$g \in E$ such that exactly one of eg and fg is 0. The domain and codomain of an element α of the inverse semigroup T_E will be denoted by $\Delta(\alpha)$ and $\nabla(\alpha)$ respectively. An inverse subsemigroup A of T_E is said to be 0-*subtransitive* if and only if for all $e, f \in E \backslash 0$ there exists $\alpha \in A$ such that $\Delta(\alpha) = Ee$ and $\nabla(\alpha) \subseteq Ef$. Further, E is termed 0-*subuniform* if and only if T_E is itself 0-subtransitive. Baird [**2**] and Trotter [**19**] have shown that E is the semilattice of a congruence-free inverse semigroup (with zero) if and only if E is 0-subuniform and disjunctive; moreover, if S is any congruence-free inverse semigroup with semilattice E then S is isomorphic to a 0-subtransitive inverse subsemigroup of T_E and every 0-subtransitive inverse subsemigroup of T_E (including, of course, T_E itself) is congruence-free. Some analogous results for congruence-free regular semigroups with zero were subsequently obtained by Trotter [**20**].

Example. Let F_2 denote the free monoid generated by two symbols a, b: thus F_2 consists of all words in a, b together with the empty word 1. The length of a word $w \in F_2$ will be denoted by $l(w)$. We define a partial order $\leqq$ on F_2 by

$$u \leqq v \Leftrightarrow v \text{ is an initial segment of } u$$

and extend it to a partial order $\leqq$ of F_2^0 by requiring that $0 \leqq u$ for all $u \in F_2^0$. Then $(F_2^0, \leqq)$ is a lower semilattice, with least element 0 and greatest element 1. We denote it by E. The principal ideal of E generated by $w \in F_2$ will be written as $[w]$. As noted by Baird [**2**], E is 0-subuniform and disjunctive; thus T_E is a congruence-free inverse semigroup with zero.

To show that there is no least congruence-free inverse semigroup with semilattice E it suffices to show that T_E contains no least 0-subtransitive inverse subsemigroup. Suppose that T_E *does* contain a least such subsemigroup S. Choose $\gamma \in S$ such that $\Delta(\gamma) = E$ and $\nabla(\gamma) \neq E$. Then $\nabla(\gamma) = [w]$ for some $w \neq 1$. Let $n = l(w) + 1$ and let $u_1, u_2, \ldots, u_{2^n}$ denote the elements of F_2 of length n. Now define $\alpha_i \in T_E$ $(i = 1, 2, \ldots, 2^n)$ by

$$\Delta(\alpha_i) = E, \quad \nabla(\alpha_i) = [u_i], \quad x\alpha_i = \begin{cases} u_i x & \text{if } x \in F_2, \\ 0 & \text{if } x = 0, \end{cases}$$

and let A denote the inverse subsemigroup of T_E generated by $E(T_E)$ and the elements $\alpha_1, \alpha_2, \ldots, \alpha_{2^n}$. Note that, for all $\alpha \in A \backslash 0$ and all $x \in \Delta(\alpha) \backslash 0$,

$$l(x) \equiv l(x\alpha) \pmod{n}.$$

Hence, in particular, $\gamma \notin A$: for $1 \in \Delta(\gamma) \backslash 0$, $l(1) = 0$ and $l(1\gamma) = l(w) = n - 1 > 0$.

We now show that A is 0-subtransitive. Let $p, q \in E \backslash 0$ $(= F_2)$. Choose $r \in F_2$ such that $l(qr) \equiv 0 \pmod{n}$. Then there exist $m \in \mathbb{N}$ and $i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, 2^n\}$ such that $qr = u_{i_1} u_{i_2} \ldots u_{i_m}$. Hence $1\alpha_{i_m}\alpha_{i_{m-1}} \ldots \alpha_{i_1} = qr$. Let ε_p denote the identity mapping on $[p]$ and write $\beta = \varepsilon_p \alpha_{i_m} \alpha_{i_{m-1}} \ldots \alpha_{i_1}$. Evidently $\beta \in A$ and

$$\Delta(\beta) = [p], \quad \nabla(\beta) = [qrp] \subseteq [q].$$

Thus A is 0-subtransitive. But $\gamma \notin A$ and so $S \nsubseteq A$. This contradicts our assumption that S is the least 0-subtransitive inverse subsemigroup of T_E. Hence no such subsemigroup exists.

4. A convexity question

In this section we concern ourselves with the following question. Suppose that T is a congruence-free regular semigroup, other than a simple group, and that S, A are full regular subsemigroups with $S \subseteq A \subseteq T$ and S congruence-free. Under what conditions is A congruence-free?

We show that A is always congruence-free when $T = T^0$ and provide a sufficient condition for this to hold when $T \neq T^0$.

Theorem 3. *Let T be a congruence-free regular semigroup, other than a simple group, and let S, A be full regular subsemigroups such that*

(i) $S \subseteq A \subseteq T$,

(ii) *S is congruence-free,*

(iii) $\langle E \rangle^S = \langle E \rangle^T$ *if* $T \neq T^0$, *where* $E = E(S) = E(T)$.

Then A is congruence-free.

Proof. Since T is fundamental, so also is A, by Lemma 3. We proceed to show that $\operatorname{core}(A) = A$. This is immediate if $T = T^0$; for then $A = A^0$. Hence suppose that $T \neq T^0$.

Note first that $\langle E \rangle^S = \bigcup_{i=1}^{\infty} S_i$, where $S_1 = \langle E \rangle$ and, for $i \in \mathbb{N}$, S_{i+1} is defined inductively as the subsemigroup of S generated by all elements of the form $x'ax$, with $a \in S_i$, $x \in S^1$ and $x' \in V_{S^1}(x)$ [**11**, p. 493]. Using this, and the corresponding results for A and T, we see from (i) that $\langle E \rangle^S \subseteq \langle E \rangle^A \subseteq \langle E \rangle^T$. Hence, by (iii), $\langle E \rangle^A = \langle E \rangle^T$. Now let $(a, b) \in A \times A$. Then, since $\operatorname{core}(T) = T$, we have that $(a, b) \in \sigma(T)$ and so, by Lemma 1, there exist $u, v \in \langle E \rangle^T$ such that $au = vb$. Since $\langle E \rangle^A = \langle E \rangle^T$ it follows from Lemma 1 that $(a, b) \in \sigma(A)$. Thus $\sigma(A) = A \times A$; that is, $\operatorname{core}(A) = A$.

The result now follows from Lemma 4. □

It is easily verified that if, in addition, S is [0-]bisimple then A is [0-]bisimple.

Since condition (iii) is automatically satisfied if T is an inverse semigroup ($\langle E \rangle^T$ then reducing to E), we obtain the following corollary—which can also be deduced from the results in [**15**, Section 1].

Corollary. *Let T be a congruence-free inverse semigroup, other than a simple group, and let S, A be full inverse subsemigroups such that*

(i) $S \subseteq A \subseteq T$,

(ii) *S is congruence-free.*

Then A is congruence-free. □

Finally, we remark that nothing new arises if we replace the adjective "inverse" by "orthodox" in the hypotheses of the corollary: for, as shown by Bailes [**1**, p. 498], every congruence-free orthodox semigroup S with $|S| > 2$ is an inverse semigroup.

REFERENCES

1. G. L. Bailes, Jr., Right inverse semigroups, *J. Algebra* **26** (1973), 492–507.

2. G. R. Baird, Congruence-free inverse semigroups with zero, *J. Austral. Math. Soc.* (A) **20** (1975), 110–114.

3. T. E. Hall, On regular semigroups, *J. Algebra* **24** (1973), 1–24.

4. T. E. Hall, Amalgamation and inverse and regular semigroups: a brief survey, *Proc. conference on regular semigroups* (Northern Illinois University, 1979).

5. T. E. Hall and W. D. Munn, The hypercore of a semigroup, *Proc. Edinburgh Math. Soc.* **28** (1985), 107–112.

6. J. M. Howie, *An introduction to semigroup theory* (Academic Press, New York, 1976).

7. J. M. Howie, A class of bisimple, idempotent-generated congruence-free semigroups, *Proc. Roy. Soc. Edinburgh* (A) **88** (1981), 169–184.

8. R. Feigenbaum, *Kernels of regular semigroup homomorphisms* (Ph.D. dissertation, University of South Carolina, 1975).

9. R. Feigenbaum, Regular semigroup congruences, *Semigroup Forum* **17** (1979), 373–377.

10. G. Lallement, Congruences et équivalences de Green sur un demi-groupe régulier, *C.R. Acad. Sci. Paris Sér. A* **262** (1966), 613–616.

11. D. B. McAlister, Regular semigroups, fundamental semigroups and groups, *J. Austral. Math. Soc.* (A) **29** (1980), 475–503.

12. W. D. Munn, A class of irreducible matrix representations of an arbitrary inverse semigroup, *Proc. Glasgow Math. Assoc.* **5** (1961), 41–48.

13. W. D. Munn, Uniform semilattices and bisimple inverse semigroups, *Quart. J. Math. (Oxford) (2)* **17** (1966), 151–159.

14. W. D. Munn, Fundamental inverse semigroups, *Quart. J. Math. (Oxford) (2)* **21** (1970), 157–170.

15. W. D. Munn, Congruence-free inverse semigroups, *Quart. J. Math. (Oxford) (2)* **25** (1974), 463–484.

16. W. D. Munn, A note on congruence-free inverse semigroups, *Quart. J. Math. (Oxford) (2)* **26** (1975), 385–387.

17. K. S. S. Nambooripad, Structure of regular semigroups I, *Mem. Amer. Math. Soc.* **224** (1979).

18. N. R. Reilly, Congruence-free semigroups, *Proc. London Math. Soc. (3)* **33** (1976), 497–514.

19. P. G. Trotter, Congruence-free inverse semigroups, *Semigroup Forum* **9** (1974), 109–116.

20. P. G. Trotter, Congruence-free regular semigroups with zero, *Semigroup Forum* **12** (1976), 1–5.

Department of Mathematics
University of Glasgow
Glasgow G12 8QW
Scotland