

# Canonical Resolution of a Quasi-ordinary Surface Singularity

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*Abstract.* We describe the embedded resolution of an irreducible quasi-ordinary surface singularity  $(V, p)$  which results from applying the canonical resolution of Bierstone-Milman to  $(V, p)$ . We show that this process depends solely on the characteristic pairs of  $(V, p)$ , as predicted by Lipman. We describe the process explicitly enough that a resolution graph for  $f$  could in principle be obtained by computer using only the characteristic pairs.

## Introduction

In this note, we study the embedded resolution of irreducible quasi-ordinary surface singularities in  $(\mathbb{C}^3, 0)$ .

A surface germ  $(V, p) \subset (\mathbb{C}^3, 0)$  is a *quasi-ordinary* singularity if it admits a finite projection  $\pi: (V, p) \rightarrow (\mathbb{C}^2, 0)$  such that the discriminant locus (*i.e.* the plane curve over which  $\pi$  ramifies) has only normal crossings. If  $(V, p)$  is irreducible, then it is equipped with a set of pairs  $\{(\lambda, \mu)\}$  of non-negative rational numbers, called *distinguished pairs*. These pairs determine much of the geometry and topology of the singularity. Lipman and others have studied this kind of singularity in detail (*cf.* [L1], [L2], [L3], and [G]).

One approach to understanding a quasi-ordinary singularity is to study its resolution. In [L1], Lipman described a definite procedure for resolving a quasi-ordinary surface singularity  $(V, p)$  embedded in  $\mathbb{C}^3$  (for an outline of this procedure, see [L2], pages 164–165 and pages 169–171). He shows that at each stage of the resolution, the singular locus remains quasi-ordinary, and the distinguished pairs at each stage are determined by those of the previous stage and the process (quadratic or monoidal transform) employed. However, this procedure does not give information about the exceptional surfaces, and it does not produce an *embedded* resolution of  $(V, p)$ .

Using Bierstone-Milman's resolution algorithm with Lipman's results, we study in this paper an embedded resolution of a quasi-ordinary surface singularity. Such a resolution is achieved by a sequence of blowings up. In this process, the exceptional surfaces can become very complicated, but they behave according to a pattern; we call it a *general configuration* (*cf.* Definition 2.5). This description of the exceptional surfaces allows us to compute the Bierstone-Milman invariant. At each stage of blowing up, we consider the *special directions* (*i.e.* special charts) of the transform. At each special direction the strict transform is again an irreducible quasi-ordinary surface germ, and it determines the next center to blow up (even though at some non-special direction, the strict transform may not be reduced). We

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show that at each stage of the resolution, the center to blow up is determined by the distinguished pairs and the process employed. Thus we establish a canonical embedded resolution for quasi-ordinary surfaces. The fact that the data of the resolution depend only on characteristic pairs will be used in a sequel to prove the existence of simultaneous embedded resolution for an equisingular family of quasi-ordinary surface singularities. As part of a much more general result, Villamayor [V2] constructs a canonical simultaneous embedded resolution of an equisingular family of quasi-ordinary hypersurfaces, which involves taking a normalization at an intermediate step. Our process is completely explicit and depends only on the characteristic monomials, but works only in dimension two. Although we deal only with the irreducible case, it would be desirable to understand the reducible case as well. An extension of our results to the reducible case would essentially carry out in a completely canonical way (in the two dimensional case) the program of Jung to resolve all singularities by first resolving the discriminant of a generic projection. The authors are thankful for several conversations and/or correspondences with E. Bierstone, G. Kennedy, J. Lipman, O. Villamayor and S. Encinas. A partial version of these results was obtained earlier in terms of the algorithm of [V1]. We ultimately found it easier to use the more explicit bottom-up approach of Bierstone and Milman's treatment of canonical resolution, but we learned this subject equally from E. Bierstone, O. Villamayor and S. Encinas. We gratefully acknowledge our debt to them.

## 1 Preliminaries

### 1.1 Quasi-Ordinary Singularities

A quasi-ordinary singularity is an analytic germ  $(V, p)$  of dimension  $d$  which admits a finite map of analytic germs  $\pi: (V, p) \rightarrow (\mathbb{C}^d, 0)$  whose discriminant locus  $\Delta$  (the hypersurface in  $\mathbb{C}^d$  over which  $\pi$  ramifies) has only normal crossings. We consider an irreducible quasi-ordinary surface singularity embedded in  $(\mathbb{C}^3, 0)$  here. In this case, we can choose local coordinates  $x, y, z$  such that  $\pi(x, y, z) = (x, y)$  and such that  $(V, p)$  is defined by an equation

$$f(x, y, z) = z^m + c_{m-1}(x, y)z^{m-1} + \cdots + c_0(x, y) = 0$$

where the  $c_i$  are power series. Then  $(V, p)$  being quasi-ordinary means that the *discriminant*  $D(x, y)$  of  $f$  (considered as a polynomial in  $z$ ) has the form

$$D(x, y) = x^a y^b \varepsilon(x, y), \quad \varepsilon(0, 0) \neq 0.$$

It is known that the roots of  $f$  are fractional power series. Let  $\zeta = H(x^{1/n}, y^{1/n})$  be one root of  $f$  ( $H(x, y)$  a power series). The other roots (the conjugates of  $\zeta$ ) can be expressed as

$$\zeta_i = H_i(x^{1/n}, y^{1/n}) = H(\omega_{i1}x^{1/n}, \omega_{i2}y^{1/n})$$

with the  $\omega_{ij}$  the  $n$ -th roots of unity. By letting  $\zeta_1 = \zeta$ , the polynomial  $f$  can be written as

$$\begin{aligned} f(x, y, z) &= z^m + c_{m-1}(x, y)z^{m-1} + \cdots + c_0(x, y) \\ &= \prod_{i=1}^m (z - \zeta_i), \end{aligned} \tag{1}$$

so that the  $c_i$  are the elementary symmetric functions of  $\zeta_i$ . We call  $\zeta$  a parametrization of  $(V, p)$ .

Since  $x^a y^b \varepsilon(x, y) = D(x, y) = \prod_{i \neq j} (\zeta_i - \zeta_j)$ , we have

$$\zeta_i - \zeta_j = x^{u/n} y^{v/n} \varepsilon_{ij}(x^{1/n}, y^{1/n}), \quad \varepsilon_{ij}(0, 0) \neq 0$$

for some  $u$  and  $v$  (depending on  $i, j$ ). The fractional monomials  $M_{ij} = x^{u/n} y^{v/n}$  so obtained are called the *characteristic monomials* of  $\zeta$ , and the exponents  $(u/n, v/n)$  are called the *characteristic pairs* of  $\zeta$ . These pairs satisfy certain conditions (cf. [L2], Proposition 1.5); for example, they are totally ordered by divisibility, where we define  $(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$  if and only if  $x^{\lambda_1} y^{\mu_1}$  divides  $x^{\lambda_2} y^{\mu_2}$  (i.e.  $\lambda_1 \leq \lambda_2$  and  $\mu_1 \leq \mu_2$ ). It turns out that these pairs determine quite a lot of the geometry and topology of  $(V, p)$  (cf. [L1], [L2], [L3], and [G]).

Although a quasi-ordinary singularity may have different characteristic pairs which depend on the choice of  $\zeta$ , the characteristic pairs of a *normalized* parametrization determine and are determined by the local topological type of  $(V, p)$  (cf. [G]); that is, there is a set of characteristic pairs which are independent of the choice of  $\zeta$ .

**Definition 1.1** A parametrization

$$\zeta = x^{a/n} y^{b/n} H(x^{1/n}, y^{1/n}),$$

where  $H(0, 0) \neq 0$  is *normalized* if

1.  $a$  and  $b$  are not both divisible by  $n$ ,
2. if  $a + b < n$ , then both  $a > 0$  and  $b > 0$ ,
3. labeling the characteristic pairs  $(\lambda_i, \mu_i)_{1 \leq i \leq s}$  of  $\zeta$  so that  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_s$  and  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_s$ , we have  $(\lambda_1, \lambda_2, \dots, \lambda_s) \geq (\mu_1, \mu_2, \dots, \mu_s)$  (lexicographically).
4. no term of  $\zeta$  has both exponents integral.

**Lemma 1.2** Any parametrization  $\zeta$  of a quasi-ordinary singularity  $(V, p)$  can be changed to a normalized one by the following type of change of coordinates:

$$\begin{array}{lll}
 \text{(i)} & \begin{array}{l} x' = x \\ y' = y \\ z' = z - p(x, y) \end{array} & \text{(ii)} \quad \begin{array}{l} x' = z \\ y' = y \\ z' = x \end{array} & \text{(iii)} \quad \begin{array}{l} x' = y \\ y' = x \\ z' = z \end{array} .
 \end{array}$$

**Proof** Let  $p(x, y)$  be the sum of the terms  $x^\alpha y^\beta$  in  $\zeta$  with  $\alpha, \beta$  both integers. By a change of type (i), we get a parametrization  $\zeta' = \zeta - p(x, y)$ . By the property of characteristic pairs (cf. [L2], Proposition 1.5), the smallest characteristic monomial divides every term in  $\zeta'$ . So,  $\zeta' = x^{\lambda_1} y^{\mu_1} H(x^{1/n}, y^{1/n})$  for some  $H$ . Therefore, we may assume that  $\zeta = x^{\lambda_1} y^{\mu_1} H(x^{1/n}, y^{1/n})$ , and 1 and 4 in (1.1) are satisfied. If 2 in (1.1) is not satisfied, from the proof of the *inversion formula* (cf. [G], Appendix) a change of type (ii) can take care of this. Upon a change of type (iii), 3 in (1.1) can be obtained. ■

### 1.2 Resolution of Quasi-ordinary Singularities

In [L1] J. Lipman described a definite procedure to resolve a quasi-ordinary singularity (for an outline of this procedure, see [L2], pages 164–165 and pages 169–171). This procedure determines a tree of *special transforms* of a quasi-ordinary singularity  $(V, p)$ . Each special transform in this tree is again a quasi-ordinary singularity. The center to blow up at each stage is determined by the characteristic pairs, and the characteristic pairs of the resulting special transform are determined by the pairs in the previous stage and the process employed (cf. [L2], Theorem 3.3). Let

$$(2) \quad \zeta = x^\lambda y^\mu H(x^{1/n}, y^{1/n}), \quad H(0, 0) \neq 0$$

be a normalized parametrization of a quasi-ordinary singularity  $(V, p)$ , where  $p = (0, 0, 0)$  is the origin. Roughly speaking, we blow up a *permissible curve* (an equi-multiple curve) whenever it is possible, and a point otherwise. The possible permissible curve is the plane section cut out by the coordinate plane  $x = 0$  (if  $\lambda \geq 1$ ) or  $y = 0$  (if  $\mu \geq 1$ ). By Bierstone-Milman’s algorithm, however, we will at some time blow up a point even though there is a permissible curve.

The following lemma by Lipman shows how the characteristic pairs change after blowing up the origin (quadratic transform) or a permissible curve (monoidal transform), or after an inversion is performed. (cf. [L2], Theorem 3.3.)

**Lemma 1.3** (omit  $i = 1$  if the corresponding pair consists of integers.)

Transformation	Pairs of resulting branch
INVERSION	$\frac{\lambda_i+1-\lambda_1}{\lambda_1}, \mu_i$
MONOIDAL TRANSFORMATION	
Center $(x, z)$	$\lambda_i - 1, \mu_i$
Center $(y, z)$	$\lambda_i, \mu_i - 1$
QUADRATIC TRANSFORMATION	
“Transversal Case” $(\lambda_1 + \mu_1 \geq 1)$	
Direction $(1 : 0 : 0)$	$\lambda_i + \mu_i - 1, \mu_i$
Direction $(0 : 1 : 0)$	$\lambda_i, \lambda_i + \mu_i - 1$
“Non-Transversal Case” $(\lambda_1 + \mu_1 < 1)$	
Direction $(1 : 0 : 0)$	$\lambda_i + \frac{(1+\mu_i)(1-\lambda_1)}{\lambda_1} - 2, \frac{1+\mu_i}{\lambda_1} - 1$
Direction $(0 : 1 : 0)$	$\mu_i + \frac{(1+\lambda_i)(1-\mu_1)}{\lambda_1} - 2, \frac{1+\lambda_i}{\lambda_1} - 1$
Direction $(0 : 0 : 1)$	$\frac{\lambda_i(1-\mu_1)+\mu_i\lambda_1}{1-\lambda_1-\mu_1}, \frac{\lambda_i\mu_1+\mu_i(1-\lambda_1)}{1-\lambda_1-\mu_1}$

For quadratic transformations, the above chart gives information at the “special” directions. The information at non-special directions can be determined by those at the special direction (cf. Lemma 2.4). In the case of quadratic transformations, we will also refer to the directions  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$ , and  $(0 : 0 : 1)$  as the  $x$ -,  $y$ -, and  $z$ -charts respectively.

## 2 Embedded Resolution

Let  $f(x, y, z)$  define a quasi-ordinary singularity germ at the origin, and let  $\{(\lambda_i, \mu_i)\}$  be the collection of characteristic pairs for  $f$ . We assume through-out that  $f$  is irreducible. We

describe the embedded resolution of  $f$  which results from applying the canonical resolution algorithm of [BM3] to  $f$ . We show that this process depends solely on the characteristic pairs of  $f$ . We also describe the process explicitly; from our description the characteristic data could be fed to a computer to output a resolution graph. To describe the process we divide the problem into two natural stages. First we show how to reduce the multiplicity of  $f$  for the first time; we call this “the base case.” Then we show how to reduce the multiplicity starting from a “general configuration.” For this we define a configuration of exceptional divisors arising after the multiplicity has decreased, and show that it is stable.

## 2.1 Outline of Bierstone-Milman’s Canonical Resolution

The centers for blowing up an arbitrary germ  $f$  are chosen by a process that can be briefly outlined as follows. Using the Weierstrass Preparation Theorem,  $f$  is written as a pseudo-polynomial of degree  $m$  in a variable  $z$ , where  $m$  is the multiplicity of  $f$ . It can also be arranged that the coefficient of the  $z^{m-1}$  term vanishes. Then the surface  $N_1 = \{z = 0\}$  defines a “maximal contact space” for  $f$ . The coefficients of  $f$  are regarded as functions on  $N_1$ , one dimension lower than the ambient dimension. From this collection, a (weighted) “higher order multiplicity” is defined and a function is selected which achieves this multiplicity, which sets up an induction on decreasing dimension. As the procedure is iterated, the accumulated exceptional divisors are factored out or included in the coefficient set, depending on when they occurred in the history of the resolution. Because the codimension cannot increase indefinitely, the process must end. It does so in one of two ways:

1. At some stage, there are no more coefficient functions. Then the last maximal contact space is contained in the vanishing locus of the previous set of coefficient functions. The final multiplicity is infinity, the order of vanishing of 0. Since the contact space is automatically smooth, it becomes the natural choice of center for the next blow up.

2. A multiplicity of zero is obtained. This occurs when the set of coefficient functions is generated by a product of exceptional divisors (which are factored out before the multiplicity is computed; zero is the order of vanishing of 1). In the language of [A], [H], the coefficient ideal is principalized. Each component of the maximum locus of the invariant is an intersection of exceptional divisors, which is smooth by the inductive requirement of normal crossings. There is a rule, based on the order in the history of the exceptional divisors, which determines a component to select as the next center. (The invariant can be augmented by this rule so as to select a unique component.)

The details of this procedure are organized into an invariant  $\text{inv}_f$  which consists of a finite sequence of non-negative rational numbers. The terms of the sequence are just the weighted multiplicities  $\nu_i$  described above and the numbers  $s_i$  of exceptional planes occurring in stages of history between which portions of the invariant decreased. The numbers  $s_i$  reflect the “age” of exceptional planes; for example,  $s_1$  counts the oldest planes, those which occurred before the first multiplicity  $\nu_1$  decreased.

Ordered lexicographically,  $\text{inv}_f$  is an upper-semi-continuous function on the points of  $\{f = 0\}$ . Its maximum locus defines smooth components which can be ordered to define canonical centers for blowing up. All of these facts are explained and proved in [BM3]; see also [BM2], Section 4, Theorem B.

For quasi-ordinary singularities, we define a “general configuration of exceptional divisors”, and show that it is stable under every transformation that is needed in the resolution.

(The transformations we use are blow ups of points and curves, permuting coordinates, and collecting integral powers, *i.e.* the transformations used in Lemmas 1.2 and 1.3). We also show that whenever the multiplicity decreases and the strict transform is normalized, the resulting collection of exceptional divisors constitutes a general configuration.

The resolution will be presented in two stages; the first stage is the base where  $E$ , the collection of exceptional hyperplanes, is empty. The second stage is the case where the multiplicity has just dropped, and  $E$  is the general configuration. The first stage sets up the second; we will also use calculations in the first stage to introduce the machinery of Bierstone-Milman's canonical embedded resolution.

## 2.2 Stage One

Let  $f = z^m + \dots + c_0(x, y)$  be a quasi-ordinary germ in normalized form, and let  $(\lambda, \mu)$  be the first characteristic pair. Assume this is the beginning of the resolution and  $E = \emptyset$ . By a series of canonically-chosen centers, we blow up  $f$  until its multiplicity has been lowered. We distinguish two cases: (i)  $\lambda + \mu < 1$  (the “non-transverse case” according to Lipman) and (ii)  $\lambda + \mu \geq 1$  (the “transverse case”). It is easy to see that the multiplicity  $\nu_1$  of  $f$  is  $m(\lambda + \mu)$  in case (i) and  $m$  in case (ii).

**Case (i)**  $\lambda + \mu < 1$  Since  $f$  is normalized,  $\lambda\mu \neq 0$ . It follows immediately that the multiplicity of  $f$  at the origin is strictly greater than at any nearby point. Since  $\nu_1$  is the first entry in  $\text{inv}_f$ , the invariant is maximized at the origin. Thus in this case, the canonical center is immediately determined by the first characteristic pair: the center is the origin. In particular it's not necessary to compute the entire sequence  $\text{inv}_f$  to determine the center in this case.

We blow up the origin and examine three coordinate charts where the strict transforms are determined by  $x^{-\nu_1} f(x, xy, xz)$ ,  $y^{-\nu_1} f(xy, y, yz)$ , and  $z^{-\nu_1} f(xz, yz, z)$ , respectively. We will call them  $x$ -,  $y$ - and  $z$ -charts. (Lipman used the term  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  and  $(0 : 0 : 1)$  directions, respectively in [L2], p. 48.)

In the  $x$ -chart we have

$$\begin{aligned} x^{-\nu_1} f(x, xy, xz) &= x^{-\nu_1} (x^m z^m + \dots + c_0(\zeta_i(x, xy))) \\ &= x^{m-\nu_1} z^m + \dots + y^{\mu m} \cdot \text{unit}. \end{aligned}$$

Using Lipman's formula (Lemma 1.3), the multiplicity has been lowered to  $\nu'_1 = \mu m$ . To restore  $f$  to (perhaps non-normalized) quasi-ordinary form, exchange variables  $y$  and  $z$  and apply the preparation theorem. We obtain

$$f' = z^{\mu m} + \dots + c'_0(\zeta'_i),$$

where the new first characteristic pair is:

$$(\lambda', \mu') = \left( \frac{1 - \lambda - \mu}{\mu}, \frac{1}{\mu} \right)$$

and the new multiplicity is:

$$\nu'_1 = \mu m < (\lambda + \mu)m = \nu_1.$$

Note also that  $\mu' > 1$  and so  $f'$  is in the transverse case.  $f'$  is again a pseudo polynomial defining a quasi-ordinary singularity, but it may not be normalized: a branch may contain integer powers. To normalize  $f'$ , replace  $z$  by  $z - p(x, y)$  where  $p$  is the (convergent) power series of all terms in  $\zeta'$  with integral exponents. Assume for now that  $\lambda', \mu'$  are not both integral, *i.e.* that  $f'$  does not lose a characteristic pair. It follows from the characterization of a quasi-ordinary parametrization (*cf.* Definition 1.1) that  $x^{\lambda'} y^{\mu'} | p(x, y)$ . The final configuration is

$$(3) \quad f' = z^{m'} + \dots + (x^{\lambda'} y^{\mu'})^m \cdot u, \quad E = \{(y)\}$$

and the multiplicity has dropped.

The  $y$ -chart is exactly parallel. In the  $z$ -chart, several blow ups of the origin are required (their number depends only on  $(\lambda, \mu)$ ). Using Lemma 1.3 again,  $m$  and  $(\lambda, \mu)$  transform to  $m' = m - \nu_1$  and  $(\lambda', \mu') = (\frac{\lambda}{1-\lambda-\mu}, \frac{\mu}{1-\lambda-\mu})$ ; so long as the new  $(\lambda', \mu')$  satisfies  $\lambda' + \mu' < 1$ , the multiplicity remains unchanged. The strict transform is automatically in quasi-ordinary form and the multiplicity is still maximized at the origin, which must be the next center. Iterating this process, we continue to blow up origins in the new  $z$ -charts until  $\lambda' + \mu' \geq 1$ . This last condition occurs after  $k = \lceil \frac{1}{\lambda+\mu} \rceil$  blow ups (or one less, if  $1/(\lambda + \mu)$  is integral), where  $[x]$  is the greatest integer function. The final configuration is

$$f' = z^{m'} + \dots + x^{\lambda m} y^{\mu m} \cdot \text{unit}, \quad E = \{(z)\}$$

and  $m' = m(1 - k(\lambda + \mu)) = \nu'_1$ . By the definition of  $k$ ,  $\nu'_1 < \nu_1$ . Using Lipman's formula,

$$(\lambda', \mu') = \left( \frac{\lambda m}{m'}, \frac{\mu m}{m'} \right) = \left( \frac{\lambda}{1 - k(\lambda + \mu)}, \frac{\mu}{1 - k(\lambda + \mu)} \right).$$

So  $\lambda' + \mu' \geq 1$ , and  $f'$  is in the transverse case.

In  $x$ - or  $y$ -charts occurring after the first  $z$ -chart, the multiplicity drops after just one blow up. The final configuration is the same as (3) except  $E$  also contains  $(z)$ .

**Case (ii)**  $\lambda + \mu \geq 1$  We have in this case  $\nu_1 = m$ . We assume that  $\lambda\mu \neq 0$ . (The case where  $\lambda\mu = 0$  is simpler.) We assume that  $f$  is normalized and  $\zeta$  contains no terms with both its exponents integral. It follows that the coefficient of  $z^{m-1}$  in  $f$  is zero. Therefore in this case, a normalized quasi-ordinary germ has a smooth maximal contact space given by the hyperplane defined by  $z = 0$ . We will illustrate the calculation of  $\text{inv}_f$  and the determination of centers for a few steps, and then summarize the resolution for this case of stage one.

Proceeding inductively on dimension, let  $\mathcal{H}_1$  be the collection of coefficients of  $f$ , regarded as functions on  $N_1 = \{z = 0\}$ . More precisely,  $\mathcal{H}_1$  is the set of pairs  $(c_q, m - q)$  where  $c_q$  is the coefficient of  $z^q$ . The number  $m - q$  is the weighted multiplicity assigned to  $c_q$ . Following [BM2] (Definition 3.3), define

$$\mu_2 := \min \left\{ \frac{\mu(h)}{\mu_h} \mid (h, \mu_h) \in \mathcal{H}_1 \right\}$$

where  $\mu(h)$  is the order of  $h$  at  $O \in N_1$ . While  $E = \emptyset$  we define  $\nu_2 = \mu_2$ ;  $\nu_2$  is the “second multiplicity” at  $O$ . Now let

$$\mathcal{G}_2 = \{(h_1, \nu_2 \cdot \mu_h) \mid (h, \mu_h) \in \mathcal{H}_1\}.$$

Then there must be a pair  $(g_*, \mu_{g_*}) \in \mathcal{G}_2$  such that  $\mu(g_*) = \mu_{g_*}$ . From here we repeat the construction: we find a (linear) change of coordinates which leaves  $z$  unchanged such that a second variable defines a maximal contact space  $N_2 \subset N_1$  for the function  $g_*$ . Restricting the coefficients of  $g_*$  to  $N_2$ , define  $\mathcal{H}_2$  and  $\nu_3$ .

The salient points of this construction are: (see [BM2] and [BM3] for proofs and details)

- The numbers  $\nu_i$  are analytic invariants of the singularity.
- The construction at latter steps of the resolution is amended to reflect the presence of exceptional divisors.
- The collection  $\mathcal{G}_i$  can be replaced by an equivalent collection to make calculations simpler; an “equivalent” collection defines the same value for the invariant in a way that is stable after blowing up. If  $(g, \mu_g) \in \mathcal{G}_2$  satisfies  $\mu(g) = \mu_g$ , and  $g$  splits into factors  $g = \prod g_i^{m_i}$ , then  $(g, \mu_g)$  can be replaced by the pairs  $(g_i, \mu_{g_i})$ ,  $\mu_{g_i} = \mu(g_i)$ , to obtain an equivalent collection.

Now we complete the calculation of  $\text{inv}_f$  in the current setting of  $\lambda + \mu \geq 1$  and  $E = \emptyset$ . We have

$$f = z^m + c_{m-2}(x, y)z^{m-2} + \dots + c_0(x, y) = \prod_{i=1}^m (z - \zeta_i).$$

Each  $c_q$  is a symmetric function of the  $\zeta_i$ . We have

$$c_0(x, y) = \prod_{i=1}^m \zeta_i(x, y) = x^{m\lambda} y^{m\mu} \cdot \text{unit},$$

and

$$c_q(x, y) = (x^\lambda y^\mu)^{m-q} \cdot e_q(x, y),$$

where  $e_q$  is a fractional power series, possibly a non-unit. This is a general phenomenon: for any quasi-ordinary singularity,  $c_0$  “divides” the other coefficients in a weighted sense:  $h_1$  “divides”  $h_2$  if  $h_1^{\mu_{h_2}} \mid h_2^{\mu_{h_1}}$ . So,  $c_0$  determines the next weighted multiplicity. Therefore,

$$\nu_2 = \mu_2 = \frac{m(\lambda + \mu)}{m} = \lambda + \mu.$$

The pair  $(g, \mu_g) = (c_0, \nu_2 \cdot m) \in \mathcal{G}_2$  satisfies  $\mu(g) = \mu_g$ , and since  $c_0$  factors, we can replace it by the pairs  $(x, 1)$  and  $(y, 1)$ . (Here we use the assumption  $\lambda\mu \neq 0$ .) The function  $(y, 1)$  already defines a second maximal contact space:  $N_2 = (y, z) \subset N_1 = (z)$ . As before,  $\mathcal{H}_2$  is the collection of coefficient functions of the elements of  $\mathcal{G}_2$ , written as polynomials in  $y$ . Thus  $(x, 1) \in \mathcal{H}_2$ , and all other functions  $(h, \mu_h)$  of  $\mathcal{H}_2$  satisfy  $\mu(h) \geq \mu_h$ . Therefore,  $\nu_3 = \mu_3 = 1$ . There are no more variables left to define coefficient functions, and  $\nu_4$  is defined to be  $\infty$ . We define  $s_i = 0$  for  $i = 1, 2, 3$  and

$$\text{inv}_f = (\nu_1, s_1; \nu_2, s_2; \nu_3, s_3; \nu_4) = (m, 0; \lambda + \mu, 0; 1, 0; \infty).$$



When  $\text{inv}_f$  ends in the value  $\infty$ , the last contact space is chosen as the center. In this case,  $N_3 = (x, y, z)$  defines the origin. Note that the invariant depends only on the characteristic data of  $f$ , since  $m$  is determined by that data ( $m$  is the number of conjugates of  $\zeta$ ).

The procedure for reducing the multiplicity falls into a handful of cases; we'll describe one case completely and then summarize the others.

**Year One** Blow-up the origin and examine the chart defined by  $x$ . The strict transform  $f'$  remains in quasi-ordinary form, assuming the multiplicity has not dropped. Let  $O'$  be the origin in this chart. Set  $E(O') = \{H_x\}$  where  $H_x = (x)$  is the exceptional divisor. Define  $E^1 \subset E(O')$  to be the collection of exceptional hyperplanes  $\{H' \mid H' \text{ is the strict transform of } H \in E(O)\}$  where  $O$  is the point that was blown up. Since  $E(O) = \emptyset$ ,  $E^1(O') = \emptyset$ . In general,  $E^1(O')$  are those exceptional hyperplanes at  $O'$  which are pulled back from the earliest stage of resolution in which  $f$  had the same multiplicity as it does now.  $E^{i+1}(O') \subset E \setminus \cup_{q=1}^i E^q$  is defined similarly, replacing the multiplicity by the first part of the invariant, namely the string  $\text{inv}_{i-\frac{1}{2}} = (\nu_1, s_1; \dots; \nu_i)$ . The numbers  $s_i$  are defined as the cardinalities of the sets  $E^i$  ([BM2], 1.12).

For clarity, we'll work through the machinery for calculating  $\text{inv}_f$  in the current chart; after this we give just the essential calculations.

Let  $\mathcal{G}_1 = \{(f', m)\}$  be the strict transform of  $f$ , which we've assumed retains the same multiplicity. Let  $\mathcal{F}_1 = \mathcal{G}_1 \cup (E^1, 1)$ , where  $(E^i, 1) = \{(g, 1) \mid g \in E^i\}$  (Currently,  $E^1 = \emptyset$  so  $\mathcal{G}_1 = \mathcal{F}_1$ ). From  $\mathcal{F}_1$  we draw a function with  $\mu(g) = \mu_g$ , and use it to define  $N_1$ . Of course,  $(f', m)$  is the only choice; since  $f'$  is already expressed as a normalized quasi-ordinary singularity, we may take  $N_1 = (z)$ . As before,  $\mathcal{H}_1 = \{(c_i', m - i)\}$  where the  $c_i'$  are the coefficients of  $f'$  as a polynomial in  $z$ . We see that  $\mu_2 = \frac{m(\lambda' + \mu')}{m} = \lambda' + \mu'$ , where  $(\lambda', \mu')$  is the first characteristic pair for  $f'$ , which can be obtained from Lipman's formula (1.3).

In general,  $\mu_2$  is modified to take exceptional divisors into account. Let  $(y_H) = H \in \mathcal{E}_1 = E \setminus E^1$  and define  $\mu_{2H} = \min_{h \in \mathcal{H}_1} \frac{\mu_H(h)}{\mu_h}$  where  $\mu_H(h)$  denotes the order to which  $y_H$  factors out of  $h$ . Then  $\nu_2 := \mu_2 - \sum_{H \in \mathcal{E}_1} \mu_{2H}$ . Define  $D_2 := \prod_{H \in \mathcal{E}_1} y_H^{\mu_{2H}}$ . By construction,  $D_2^m$  is the greatest common divisor of the elements of  $\mathcal{H}_1$  that is a monomial in exceptional coordinates  $y_H, H \in \mathcal{E}_1$ . For each  $h \in \mathcal{H}_1$ , write  $h = D_2^m \cdot g$  and  $\mu_g := \mu_h \cdot \nu_2$ . Then  $\mathcal{G}_2$  is defined to be the collection  $\{(g, \mu_g)\}$  for all  $(h, \mu_h) \in \mathcal{H}_1$  as above, together with the pair  $(D_2^m, (1 - \nu_2) \cdot m)$  if  $\nu_2 < 1$ . This completes a cycle in the definition of  $\text{inv}_f$ .

Now calculate explicitly:  $c_0' = (x^{\lambda'} y^{\mu'})^m \cdot \text{unit}$  is the principalizing element of  $\mathcal{H}_1$ , and  $\mathcal{E}_1 = E \setminus E^1 = \{H_x\}$  where  $H_x = (x)$ . Clearly  $\mu_{H_x} = \lambda'$ , so  $\nu_2 = \mu'$  and  $D_2 = x^{\lambda'}$ . We will assume that  $\lambda \geq 1$  and  $\mu \geq 1$ ; using Lipman's formula,  $\mu = \mu'$ , so  $\nu_2 \geq 1$ , and  $D_2$  is not included in  $\mathcal{G}_2$  ([BM2], Section 3, year-one construction). We may exclude functions from  $\mathcal{G}_2$  which are divisible by some other function in  $\mathcal{G}_2$ . We have  $\mathcal{G}_2 = \{(y^{\mu'} m, \mu' m)\} \sim \{(y, 1)\}$ . The first stage of the invariant,  $\text{inv}_{1\frac{1}{2}}$ , is  $(m, 0; \mu') = (m, 0; \mu)$ . Before blowing up,  $\text{inv}_{1\frac{1}{2}} = (m, 0; \lambda + \mu)$ , so  $\text{inv}_{1\frac{1}{2}}$  has decreased. Therefore  $E^2 = \{H_x\}$ , and  $\mathcal{E}_2$  is defined to be  $\mathcal{E}_1 \setminus E^2 = \emptyset$ . We have  $s_2 = 1$ , and  $\mathcal{F}_2 = \mathcal{G}_2 \cup (E^2, 1) = \{(x, 1), (y, 1)\}$ . It's now clear that we may take  $(y, z)$  and  $(x, y, z)$  to successively define maximal contact spaces, and  $\nu_3 = 1, s_3 = 0, \nu_4 = \infty$ . We have  $\text{inv}_f = (m, 0; \mu', 1; 1, 0; \infty)$ . The last maximal contact space is the origin, which must be the next center to blow up.

**Year Two** Blow up the origin, and again examine the  $x$ -chart. As before, the strict trans-

form is a normalized quasi-ordinary polynomial. The assumption  $\lambda \geq 1, \mu \geq 1$  implies that the multiplicity cannot decrease by blowing up the origin, so  $\nu_1 = m$ . It follows that  $E^1 = \emptyset$ , and  $\mathcal{F}_1 = \mathcal{G}_1$  contains only the strict transform, which we will again denote as  $f$ . Then as before  $N_1 = (z)$ , and  $\mathcal{H}_1 = \{(c_0, m)\}$  (we ignore all higher coefficient functions, which are divisible by  $c_0$ ). Again we have  $\mu_2 = \lambda'' + \mu''$ , where  $(\lambda'', \mu'')$  is the current first characteristic pair, and  $\mathcal{E}_1 = \{H_x\}$  leads to  $\mu_{2H_x} = \lambda''$  and  $\nu_2 = \mu''$ . Using Lipman's formula (Lemma 1.3, transversal case, direction  $(1 : 0 : 0)$ ),  $\mu'' = \mu$  and  $\text{inv}_{1\frac{1}{2}} = (m, 0; \mu)$ . This time  $\text{inv}_{1\frac{1}{2}}$  does not drop, so  $E^2 = \emptyset$  (the exceptional divisor from the previous stage is blown away), and  $\mathcal{E}_2 = \mathcal{E}_1 \setminus E^2 = \{H_x\}$ . Then  $\mathcal{F}_2 = \mathcal{G}_2 = \{(y^{\mu m}, \mu m)\} \sim \{(y, 1)\}$ . The next maximal contact space  $N_2$  is defined by  $y$  restricted to  $N_1$ , i.e.  $N_2 = (y, z)$ . There are no coefficient functions contributed to  $\mathcal{H}_2$  by the only function  $(y, 1)$  belonging to  $\mathcal{F}_2$ , so  $\nu_3 = \infty$ . We have  $\text{inv}_f = (m, 0; \mu, 0; \infty)$  at this stage. The next center is defined by  $N_2$ , which is the  $x$ -axis. Note that by assuming  $\mu \geq 1$ , the  $x$ -axis is equimultiple.

**Year Three** Blow-up the  $x$ -axis. In this case there is only one chart to consider, the  $y$ -chart. (The strict transform of  $f$  is smooth in the other chart.) The strict transform  $f'$  of  $f$  is  $y^{-m}f(x, y, yz)$ .  $f'$  is (up to multiplication by a suitable unit) a normalized quasi-ordinary polynomial, with first characteristic pair  $(\lambda', \mu') = (\lambda, \mu - 1)$  where  $(\lambda, \mu)$  was the corresponding pair in year two (Lemma 1.3, monoidal case). To see the general pattern, we continue to assume  $\lambda', \mu' \geq 1$ . Therefore the multiplicity  $m$  has remained constant. We see that  $E = \{H_x, H_y\}$  and  $E^1 = \emptyset$ . Clearly  $\mathcal{F}_1 = \mathcal{G}_1 = \{(f', m)\}$ , and we take  $N_1 = (z)$ . The coefficients  $\mathcal{H}_1$  are generated again by  $((x^{\lambda'} y^{\mu'})^m, m)$ , but now  $\mathcal{E}_2 = E \setminus E^1 = \{H_x, H_y\}$  implies  $D_2 = x^{\lambda'} y^{\mu'}$ . Then  $\mu_2 = \lambda' + \mu'$  and  $\nu_2 = 0$ . The invariant sequence terminates if  $\nu_i = 0$  or  $\infty$  ([BM2], Definition 3.3). Thus  $\text{inv}_f = (m, 0; 0)$ .

When  $\nu_i = 0$ , the center for the next blow up is constructed out of the exceptional locus. List the elements of  $E$  by the order in which each arose in history; this can be done globally, not just for the current chart. (Thus we consider  $E$  to contain  $n$  elements after  $n$  blow ups, most of which play no role in the calculation of  $\text{inv}_f$  at a given point). Any subset  $I$  of  $E$  can be ordered via the  $n$ -tuple  $(\delta_1, \delta_2, \dots, \delta_n)$  where  $\delta_i = 0$  if the  $i$ -th element of  $E$  is not in  $I$ ;  $n$ -tuples are ordered lexicographically.

Returning to the determination of the center of blowing up, when  $\nu_2 = 0$ , define  $\mathcal{G}_2 = \{(D_2, 1)\}$  and  $S_{\text{inv}} = \{p \in N_1 \mid \mu_p(D_2) \geq 1\}$ .

In the current setting,

$$S_{\text{inv}} = \{p \in N_1 \mid \mu_p(x^{\lambda'} y^{\mu'}) \geq 1\} = (H_x \cap N_1) \cup (H_y \cap N_1),$$

the union of the  $x$ -axis and  $y$ -axis. The order of  $H_x$  is  $(0, 1, 0)$  and the order of  $H_y$  is  $(0, 0, 1)$ . Thus  $H_x$  precedes  $H_y$  and the next center is  $C = S_{\text{inv}} \cap H_x =$  the  $y$ -axis.

The procedure now takes on a stable pattern. So long as  $\lambda$  and  $\mu$  both exceed 1, we have  $\text{inv}_f = (m, 0; 0)$ . We are forced to continue blowing up axes, which lowers either  $\lambda$  or  $\mu$  by one each time. The corresponding divisor is blown away and replaced by a newer one, causing the choice of axes to alternate each time. Eventually one of  $\lambda$  or  $\mu$  is driven smaller than one, and the requirement  $\mu_p(D_2) \geq 1$  automatically selects the other axis ever after, until both  $\lambda < 1$  and  $\mu < 1$ . If at this point we still have  $\lambda + \mu \geq 1$ , then the multiplicity still remains unchanged. But the condition  $\mu_p(D_2) = \mu_p(x^\lambda y^\mu) \geq 1$  now selects the origin

as the center, and in fact the origin is the maximum locus for the multiplicity  $\nu_1$ . We are forced to blow up the origin repeatedly until  $\lambda + \mu < 1$ . (Only  $x$ -charts and  $y$ -charts need to be considered.) The multiplicity drops precisely when this occurs. Finally, it is clear that all these decisions are completely determined by the arithmetic of the original first pair  $(\lambda, \mu)$ .

Except for which charts  $(x$  or  $y)$  we choose to consider in the first two years and the assumptions on  $(\lambda, \mu)$ , the preceding completely describes how to reduce the multiplicity starting from year zero in the transversal case. Other starting assumptions (e.g.,  $\lambda < 1, \mu \geq 1$ ) are entirely similar.

### 2.3 Stage Two

Once the multiplicity drops, the algorithm begins again; all previous history is forgotten. All exceptional divisors passing through the current point of interest are placed in  $E^1$  (the set of “oldest” divisors). When the multiplicity drops for the first time, all the divisors are given by coordinate functions. In later stages this may not be true. The general phenomena occurring in later stages is completely illustrated by three examples.

**Example 2.1** Suppose  $f$  is normalized,  $\lambda + \mu < 1$ , and  $E = E^1 = \{H_x, H_y\}$ . This is the usual situation the first time the multiplicity is lowered when we begin in the transversal case. Also, note that  $\lambda\mu \neq 0$  since  $f$  is normalized.

As before,  $\lambda + \mu < 1$  immediately implies that the origin has higher multiplicity than nearby points. We are required to blow up the origin. Examine the  $x$ -chart. As before, the multiplicity drops immediately. The new first characteristic pair is

$$(\lambda', \mu') = \left( \frac{1 - \lambda - \mu}{\mu}, \frac{1}{\mu} \right),$$

and the strict transform of  $f$  has the form

$$f' = z^m x^{m-\nu} + \dots + y^{\mu m} \cdot \text{unit}.$$

Note that  $\mu' > 1$  so we are again in the transversal case. We have  $E = E^1 = \{H_x, H_y\}$ . To normalize  $f'$ , we first permute  $y$  and  $z$ , and, up to a unit factor, get the configuration

$$(4) \quad f' = z^{\mu m} + \dots + x^{m-\nu_1} y^m \cdot \text{unit}, \quad E = \{H_x, H_z\}.$$

But now  $f'$  may not be normalized; a quasi-ordinary branch  $\zeta'$  may contain terms with integer exponents. Let  $p(x, y)$  be the power series consisting of these terms, and apply transformation (i) of Lemma 1.2. It follows that  $p(x, y)$  is divisible by  $x^{\lambda'} y^{\mu'}$ . We now have  $H_{z-p(x,y)} \in E^1 = E$ . Since  $N_1 = \{z = 0\}$  can be chosen as the first maximal contact space, both  $(x^{\lambda' m'} y^{\mu' m'}, m')$  and  $(p(x, y), 1)$  will belong to the coefficient set  $\mathcal{H}_1$ , and by the divisibility relation between them, the latter function will have no effect on the calculation of  $\text{inv}_f$ . This relationship is preserved under blowing up. Thus the divisor  $H_{z-p(x,y)}$  behaves as though  $p(x, y) = 0$ .

**Example 2.2** Suppose again that  $f$  is normalized and  $\lambda + \mu < 1$  with  $E = E^1 = \{H_x, H_y, H_{z-p(x,y)}\}$  where  $x^\lambda y^\mu \mid p(x, y)$ . This case may occur when Example 2.1 has lowered its multiplicity a second time.

Since  $f$  is in the non-transversal case, we must blow up the origin. In the  $z$ -chart, the divisor  $H_{z-p(x,y)}$  is blown away and plays no further role. In the  $x$ -chart, the multiplicity drops immediately. As before, the strict transform  $f'$  requires two steps to be normalized: permute  $y$  and  $z$ , and then absorb integral powers in the quasi-ordinary branch. In the second step, we replace  $z$  by  $z - q(x, y)$  where  $x^{\lambda'} y^{\mu'} \mid q(x, y)$  and  $(\lambda', \mu') = (\frac{1-\lambda-\mu}{\mu}, \frac{1}{\mu})$ . We need to see the effect of these changes on the elements of  $E^1$ . Of course  $H_x$  is unaffected, and  $H_y$  becomes  $H_{z-q(x,y)}$ , which behaves as in Example 2.1. On the other hand,  $H_{z-p(x,y)}$  becomes  $H_{y-p(x,z-q(x,y))}$ . By the divisibility properties of  $p$  and  $q$ ,  $y^\mu \mid p(x, y)$  and  $x^{\lambda'} y^{\mu'} \mid q(x, y)$ . Since  $N_1 = \{z = 0\}$  will again be the first contact space, it follows that the function  $y - p(x, z - q(x, y))$  is equal to  $y$  up to a unit when restricted to  $N_1$ . Thus  $H_{y-p(x,z-q)}$  behaves like  $H_y$  through-out the next stage of the resolution.

In Examples 2.1 and 2.2, we implicitly assumed that Lipman's formula for  $(\lambda', \mu')$  results in a new first characteristic pair which is not entirely integral.

**Example 2.3 (Losing a characteristic pair)** Suppose  $f$  is normalized,  $\lambda + \mu < 1$ ,  $E = E^1 = \{H_x, H_y, H_{z-p(x,y)}\}$  with  $x^\lambda y^\mu \mid p(x, y)$ , and  $(\lambda, \mu) = (\frac{a}{m}, \frac{1}{m})$  where  $1 < a < m$  and  $a \in \mathbb{N}$ .

As in Example 2.2, blow up the origin and consider the direction defined by  $x$ . The multiplicity of  $f'$  drops, but this time  $(\lambda', \mu') = (m - a - 1, m)$ . (The new pair becomes integral.) When  $f'$  is normalized, the "first" characteristic pair  $(\lambda', \mu')$  has to be absorbed in the transformation  $z \rightarrow z - q(x, y)$ . Thus  $q(x, y) = x^{\lambda'} y^{\mu'} \cdot \text{unit}$ . The strict transform has one fewer characteristic pairs. (Eventually  $f$  has to lose all its characteristic pairs to become smooth.)

Just as before,  $H_z$  becomes  $H_{y-p(x,z-q(x,y))}$  and the latter is generated by  $(y)$  once we restrict to  $N_1 = \{z = 0\}$ . But while the transform  $H_y \rightarrow H_{z-q(x,y)}$  is as before, we now have  $f' = z^{m'} + \dots + (x^{\lambda_2'} y^{\mu_2'})^{m'}$  unit. The degree-zero coefficient of  $f'$ ,  $c_0' = (x^{\lambda_2'} y^{\mu_2'})^{m'}$  unit, no longer divides  $q(x, y)$ . In fact the weighted "divisibility" relation is reversed:  $q(x, y)$  "divides"  $c_0'$ . When we construct  $\mathcal{H}_1$ , it is  $q$  that principalizes the functions and determines the invariant, until  $H_{z-q}$  is blown away. Since the role of  $q$  in the process is determined by  $(\lambda, \mu)$ , although  $(\lambda', \mu')$  is integral and no longer counts as a characteristic pair, the process is still determined by the original characteristic data. We sketch the next phase of the resolution. The invariant for the configuration  $(f', E)$  is  $(m', 3; 1, 0; 1, 0; \infty)$  and the next center is the origin. Blow-up the origin and consider the  $x$  direction; the multiplicity can't drop but  $H_x$  is blown away and replaced by a new divisor. The invariant records this as  $(m', 2; 1, 1; 1, 0; \infty)$ , and the origin is the center again. Blow up the origin and consider the  $y$ -direction; now  $H_x$  and  $H_y$  are "new" and belong to  $\mathcal{E}_1$ . The calculation now gives  $\mathcal{H}_1 = \{(c_0, m'), (q, 1)\}$  and  $D_2 = x^{\lambda'} y^{\mu'}$ , where  $q(x, y) = x^{\lambda'} y^{\mu'} \cdot \text{unit}$ . We have  $\nu_2 = 0$ , and  $\text{inv}_f = (m', 1; 0)$ .  $D_2$  now determines the centers for blowing up by  $S = \{p \mid \mu_p(D_2) \geq 1\}$  and the ordering of the divisors in  $\mathcal{E}_1 = \{H_x, H_y\}$ . So we are required to blow up the  $x$ - and  $y$ -axis alternately until  $\lambda' < 1$  and  $\mu' < 1$ ; since  $(\lambda', \mu')$  are integers, we continue until  $\lambda' = \mu' = 0$  and  $H_{z-q(x,y)}$  is blown away. The relation  $q(x, y) \mid c_0(x, y)$  is maintained through-out, so when  $H_{z-q(x,y)}$  is blown away we have  $\mathcal{H}_1 =$

$\{(c_0, m')\}$ . Then since  $c_0(x, y) = x^{\lambda_2} y^{\mu_2} \cdot \text{unit}$ , the second characteristic pair  $(\lambda_2, \mu_2)$  takes over the process.

The above computation shows how the distinguished pairs determine the invariants at the special directions. The following lemma claims that the computation at the special directions is enough to determine the center to blow up.

**Lemma 2.4** *At each stage of the resolution, the center of blowing up is determined by the invariant evaluated at special directions.*

**Proof** Because  $\text{inv}_f$  is upper-semi-continuous, its value at a special direction exceeds or equals its value at nearby “general” directions. On the other hand, Lipman shows that the non-special directions along an exceptional curve are equisingular (cf. [L1] pages 86–87). It is also clear that each new component of the singular locus arising from a blow up is contained in the new exceptional divisor. It follows that  $\text{inv}_f$  is constant on the non-special points of each component. Thus, if  $\text{inv}_f$  is the same at a special direction and at nearby non-special points, it is determined at all non-special points on a given axis. Otherwise it exceeds the value at all nearby points, in which case the special direction must be chosen as the center. ■

**Definition 2.5 (General Configuration)** Let  $f$  be a normalized quasi-ordinary singularity, and let  $(\lambda, \mu)$  be the first characteristic pair. Furthermore, suppose  $f$  was obtained by applying the Bierstone-Milman algorithm to  $\tilde{f}$  with characteristic pairs  $\Lambda = \{(\tilde{\lambda}_i, \tilde{\mu}_i)\}$ . Let

$$E = E^1 \subset \left\{ (z - q(x, y)), (y - p(x, z - q(x, y))), (x) \right\}$$

where

$$y - p(x, z - q(x, y)) \mid_z = y \cdot \text{unit},$$

and  $x^\lambda y^\mu \mid q(x, y)$ , or  $q(x, y) = x^a y^b \cdot \text{unit}$  and  $x^a y^b \mid x^\lambda y^\mu$ , and where  $(a, b)$  depends on  $\Lambda$ . Then  $E$  is called a *general configuration*.

**Theorem 2.6** *At any stage in the resolution when the multiplicity of  $f(x, y, z)$  has dropped, and  $f$  is expressed as a normalized quasi-ordinary singularity, the exceptional divisors constitute a general configuration.*

**Proof** The theorem is trivial in the base case. We have shown it to be true at the end of the base case, and in representative examples of the general case. A complete proof involves checking the calculations for each possible subset of the general configuration, and seeing that in each case the final set of divisors (once  $f$  is normalized) is again a general configuration. The calculations are similar in all cases to the three examples we presented above. ■

Since the controlling function in every case depends only on characteristic data, we have

**Theorem 2.7** *The Bierstone-Milman canonical resolution algorithm for a quasi-ordinary singularity  $(V, p)$  depends only on the (normalized) characteristic pairs.*

**Example 2.8** We will find locally an embedded resolution for  $(V, p)$  defined by  $f(x, y, z) = z^3 + x^2y^4$ . The function  $f$  has one characteristic pair  $(2/3, 4/3)$ .

**Year 0** In the beginning,  $\text{inf}_f(O) = (3, 0; 2, 0; 1, 0; \infty)$  and the center is the origin  $O$ . Blow up  $O$  and consider the  $x$ -chart.

**Year 1**  $f = z^3 + x^3y^4$ ,  $\text{inf}_f(O) = (3, 0; 4/3, 1; 1, 0; \infty)$  and  $E = \{H_x\}$ ; the center is again the origin. Blow up the origin and consider the  $y$ -chart.

**Year 2**  $f = z^3 + x^3y^4$  and  $\text{inf}_f(O) = (3, 0; 0)$ . The centers are now determined by  $D_2 = xy^{4/3}$  and the ordering of the divisors  $E = \{H_x, H_y\}$ . Since  $H_x$  is older, we first blow up  $H_x \cap N_1$  which is the  $y$ -axis.

**Year 3**  $f = z^3 + y^4$ ,  $\text{inf}_f(O) = (3, 0; 0)$ ,  $D_2 = y^{4/3}$ . Blow up the  $x$ -axis, and consider the  $y$ -chart.

**Year 4**  $f = z^3 + y$ . The multiplicity has dropped, and the strict transform is now smooth, but the divisors do not have normal crossings with  $f$ . If we “normalize”  $f$  as a quasi-ordinary singularity, we see that  $f$  has lost its only characteristic pair: the previous characteristic pair becomes  $(0, 1/3)$ , which becomes integral when we normalize by the transformation  $y \leftrightarrow z$ ,  $z \rightarrow z + y^3$ . We have  $f = z$ , and  $E = E^1 = \{H_x, H_{z-y^3}\}$ . Since the multiplicity just dropped, we begin again, regarding the current state as year 0. We have  $\text{inf}_f(O) = (1, 2; 1, 0; 3, 0; \infty)$  and the center to blow up is the origin  $O$ . Blow up and consider the  $x$ -chart.

**Year 1** In the  $x$ -chart we have  $f = z$  and  $E = \{H_x, H_{z-x^2y^3}\}$ . Since of course the multiplicity cannot drop, the new divisor  $H_x$  does not belong to  $E^1$ . We have  $\text{inf}_f(O) = (1, 1; 3, 1; 1, 0; \infty)$  and the next center is the origin. Blow up and consider the  $y$ -chart.

**Year 2**  $f = z$  and  $E = \{H_x, H_y, H_{z-x^2y^4}\}$ . Now we have  $E^1 = \{H_{z-x^2y^4}\}$  and  $\mathcal{E}_1 = \{H_x, H_y\}$ , which leads to  $\text{inf}_f(O) = (1, 1; 0)$  and  $D_2 = x^2y^4$ . All remaining centers are determined by  $D_2$ , the exponents of which came from the integral pair  $(0, 3)$  and the transformations of Lemma 1.2 occurring in the last two steps. As in the last cycle, we blow up the  $y$ -axis.

**Year 3**  $f = z$  and  $E^1 = \{H_{z-xy^4}\}$ , and  $\text{inv}_f(O) = (1, 1; 0)$ . Blow up the  $x$ -axis and consider the  $y$ -chart.

**Year 4**  $f = z$  and  $E^1 = \{H_{z-xy^3}\}$ . Blow up the  $y$ -axis and consider the  $x$ -chart.

**Year 5–8** In year 5,  $f = z$  and  $E^1 = \{H_{z-y^3}\}$ . Now  $D_2 = y^3$  and only the  $x$ -axis is selected. Blow up the  $x$ -axis three more times. In year 8,  $f = z$  and  $E = \{H_x, H_y, H_{z-1}\}$ . We have  $\text{inf}_f(O) = (1, 0; 0)$ . The invariant  $\text{inf}_f$  is now locally constant everywhere on  $f$ , and we have achieved an embedded resolution.

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