Can. J. Math., Vol. XXVI, No. 6, 1974, pp. 1380–1383

# SOME CRITERIA FOR HERMITE RINGS AND ELEMENTARY DIVISOR RINGS

THOMAS S. SHORES AND ROGER WIEGAND

Recall that a ring $R$ (all rings considered are commutative with unit) is an elementary divisor ring (respectively, a Hermite ring) provided every matrix over $R$ is equivalent to a diagonal matrix (respectively, a triangular matrix). Thus, every elementary divisor ring is Hermite, and it is easily seen that a Hermite ring is Bezout, that is, finitely generated ideals are principal. Examples have been given [**4**] to show that neither implication is reversible.

In [**5**], M. Henriksen asked whether every semilocal Bezout ring is Hermite. This question was answered affirmatively in [**8**], where it was shown there that every semilocal Bezout ring is an elementary divisor ring, and that every Bezout ring with only finitely many minimal primes is Hermite. In the present paper we extend these results by showing that every Bezout ring with noetherian maximal ideal spectrum is an elementary divisor ring, and that every Bezout ring with compact minimal prime spectrum and $T$-nilpotent nilradical is Hermite.

**1. Hermite rings.** It was shown in [**7**] that a ring $R$ is Hermite if and only if every 1 by 2 matrix over $R$ is equivalent to a triangular matrix. In other words, $R$ is Hermite if and only if every pair of elements $a, b$ in $R$ satisfies

(H) There exist $d, a', b'$ in $R$ such that
$a = a'd, b = b'd$ and $(a', b') = (1)$.

In this case $(d) = (a, b)$; conversely [**3**], if $(d') = (a, b)$ and the pair $a, b$ satisfies (H) then there exist $a'', b''$ such that $a = a''d'$, $b = b''d'$ and $(a'', b'') = (1)$.

1.1. Theorem. *Let $R$ be a Bezout ring with compact minimal prime spectrum. Then $R$ is Hermite if (and only if) every pair of nilpotent elements satisfies* (H).

*Proof.* We begin by showing that $R/N$ is semihereditary, where $N$ is the nilradical of $R$. Every localization of $R/N$ is a semiprime valuation ring, that is, a valuation domain, and it follows that every ideal of $R/N$ is flat. Thus it will suffice, by [**1**] to prove that the classical quotient ring of $R/N$ is von Neumann regular. But this follows from [**11**, Proposition 9], since every finitely generated faithful ideal of $R/N$ contains a non-zero-divisor (namely, its generator).

Received May 3, 1973 and in revised form, June 11, 1973. The research of the second named author was partially supported by an NSF grant.

Now let $a, b \in R$, and let $(d) = (a, b)$. We have to show that the pair $a, b$ satisfies (H). Denote passage to $\bar{R} = R/N$ by bars. Since $\bar{R}\bar{d}$ is a projective $\bar{R}$-module, the annihilator $\text{ann}_{\bar{R}}(\bar{d})$ is a direct summand of $\bar{R}$, say $\text{ann}_{\bar{R}}(\bar{d}) = \bar{R}\bar{e}$. Since idempotents can be lifted modulo $N$, we may assume $e = e^2$. Now $(ea, eb) = (ed) \subseteq N$, so by hypothesis the pair $ea, eb$ satisfies (H). The following lemma completes the proof:

1.2. LEMMA. *Let $R$ be a Bezout ring with Jacobson radical $J$. Suppose $(a, b) = (d)$, and $e$ is an idempotent such that* $(\text{ann}_R d) \cap (1 - e) \subseteq J$. *If the pair $ea, eb$ satisfies* (H), *so does the pair $a, b$.*

*Proof.* Write $d = \alpha a + \beta b$, $a = a_1 d$ and $b = b_1 d$ for suitable elements $\alpha, \beta, a_1, b_1 \in R$. Then $d(1 - \alpha a_1 - \beta b_1) = 0$, and, multiplying by $1 - e$, we see that $1 - e - \alpha a_1(1 - e) - \beta b_1(1 - e) \in (\text{ann}_R d) \cap (1 - e) \subseteq J$. It follows that $(1 - e) = (a_1(1 - e), b_1(1 - e))$. Since $(ea, eb) = (ed)$, the remark preceding (1.1) provides elements $a_2, b_2$ such that $ea = a_2 ed$, $eb = b_2 ed$ and $(a_2, b_2) = (1)$. Finally, set $a' = ea_2 + (1 - e)a_1$ and $b' = eb_2 + (1 - e)b_1$, and obtain that $a = a'd$, $b = b'd$ and $(a', b') = (e, 1 - e) = (1)$.

Recall that an ideal $I$ of $R$ is said to be $T$-nilpotent if for each sequence $\{a_k\} \subseteq I$ there is an integer $n$ such that $a_1 \ldots a_n = 0$.

1.3. COROLLARY. *Let $R$ be a Bezout ring with compact minimal prime spectrum. If the nilradical of $R$ is $T$-nilpotent then $R$ is Hermite.*

*Proof.* Let $N$ be the nilradical of $R$. For each $R$-module $A$, define a Loewy series $\{L_\alpha(A)\}$ as follows: $L_0(A) = 0$, $L_{\alpha+1} = \{x \in A \mid Nx \subseteq L_\alpha\}$, and $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ if $\lambda$ is a limit ordinal. By [**12**, Proposition 3.3], the $T$-nilpotence of $N$ is equivalent to the condition that $L_\alpha(R) = R$ for some ordinal $\alpha$. The least such ordinal is called the $N$-length of $R$. (The results in [**12**] are stated for "$I$-Loewy series" where $I$ is a maximal ideal of $R$. The proofs, however, can easily be altered to fit our set-up.) We proceed by induction on the $N$-length of $R$, which, we note, cannot be a limit ordinal.

If the $N$-length of $R$ is 1, then $N = 0$, and $R$ is Hermite by Theorem 1.1. Assume $R$ has $N$-length $\alpha > 1$. Let $a, b \in N$ and let $(a, b) = (d)$. By Theorem 1.1 we need only verify that $a$ and $b$ satisfy condition (H). Let $I$ be the annihilator of $d$, so that $R/I \cong Rd$; we claim $L_{\alpha-1}(R/(N \cap I)) = R/(N \cap I)$. We have $R/(N \cap I) \subseteq R/N \oplus Rd$, and

$$L_{\alpha-1}((R/N) \oplus Rd) = L_{\alpha-1}(R/N) \oplus L_{\alpha-1}(Rd) = R/N \oplus (Rd \cap L_{\alpha-1}(R)),$$

by [**12**, 3.1 and 3.2]. But $L_{\alpha-1}(R) \supseteq N$, so $L_{\alpha-1}(R/N \oplus Rd) = R/N \oplus Rd$, and another application of [**12**, Proposition 3.1] verifies the claim. It follows that the ring $\bar{R} = R/(N \cap I)$ has $(N/(N \cap I))$-length at most $\alpha - 1$. Since $\bar{R}$ is Bezout and has compact minimal prime spectrum, $\bar{R}$ is Hermite, by induction.

Choose $u, v \in R$ such that $a = ud$ and $b = vd$. Applying condition (H) to $\bar{u}, \bar{v} \in \bar{R}$ and lifting back to $R$, we obtain elements $h$, $a'$, $b'$ such that $u \in a'h + (N \cap I)$, $v \in b'h + (N \cap I)$, and $(a', b') + (N \cap I) = (1)$. Then $a = ud = a'(hd)$, $b = vd = b'(hd)$, and $(a', b') = (1)$.

Theorem 1.1 was triggered by a letter from M. Henriksen, asking whether every Bezout ring with compact minimal prime spectrum is Hermite. This question remains unsettled, although we suspect counterexamples exist. Another unanswered question arises in connection with Theorem 1.1. Suppose $R$ is a Bezout ring with nilradical $N$. If $R/N$ is Hermite and elements of $N$ satisfy condition (H), is $R$ necessarily Hermite?

**2. Elementary divisor rings.** By a $j$-ideal of $R$ we mean an intersection of maximal ideals of $R$; similarly, a $j$-prime is a prime $j$-ideal. A ring $R$ is $j$-noetherian provided $R$ has maximum condition on $j$-ideals.

2.1. THEOREM. *Every j-noetherian Bezout ring is an elementary divisor ring.*

*Proof.* We will appeal to the following facts about Bezout rings: (1) If $R$ is a Bezout ring with only finitely many minimal primes then $R$ is Hermite. (2) If $R$ is a Hermite ring with Jacobson radical $J(R)$, and if $R/J(R)$ is an elementary divisor ring, then $R$ is an elementary divisor ring. (3) A ring $R$ is an elementary divisor ring if and only if every $R$-module presented by a $2 \times 2$ triangular matrix is a direct sum of cyclics. Statement (1) is Theorem 2.2 of [**8**], (2) is Theorem 3 of [**5**], while (3) is clear from the proof of [**8**, Theorem 3.8]. One more detail must be checked before we proceed to the proof of the theorem:

2.2. LEMMA. *Let R be a ring such that for each prime P, the set of primes contained in P is totally ordered by inclusion. Then R is j-noetherian if and only if every ideal of R has only finitely many minimal prime divisors.*

*Proof.* The sufficiency of the latter condition is valid in any ring, by [**9**, 1.4] and [**10**]. To prove necessity, assume $R$ is $j$-noetherian. Choose, for each minimal prime $P$ of $R$, a $j$-prime $P'$ minimal over $P$. (This is possible by Zorn's lemma.) We claim $P'$ is in fact a minimal $j$-prime. For, suppose $Q_1$ is a $j$-prime properly contained in $P'$. Then $Q_1$ contains a minimal prime $Q_2$. Since $P$ and $Q_2$ are minimal primes contained in $P'$, we must have $P = Q_2$, contradicting the minimality of $P'$ over $P$. Now the map $P \to P'$ is clearly one-to-one, and since $R$ has only finitely many minimal $j$-primes [**9**], it follows that $R$ has only finitely many minimal primes. Now, for each ideal $I$, $R/I$ is a $j$-noetherian ring satisfying the hypothesis of the lemma. By what we have shown, $R/I$ has only finitely many minimal primes.

Now let $R$ be a Bezout ring with noetherian maximal ideal spectrum. By 2.2 and (1), $R$ is Hermite. Therefore by (2) we may assume $J(R) = 0$.

Suppose, first of all, that $R$ is a domain. Then 0 is a $j$-prime, and we can

https://doi.org/10.4153/CJM-1974-131-8 Published online by Cambridge University Press

assume, by "noetherian induction", that $R/P$ is an elementary divisor ring for every non-zero $j$-prime $P$. Let $M$ be the $R$-module presented by the matrix

$$A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}.$$

By (3) it will suffice to prove that $M$ is a direct sum of cyclics. Note that $acM = 0$, so that if $M$ is faithful, either $a = 0$ or $c = 0$. Since $R$ is Hermite, it follows easily that $A$ is equivalent to a diagonal matrix, and hence $M$ is a direct sum of cyclics. If $M$ is *not* faithful, let $I$ be the annihilator of $M$, let $P_1, \ldots, P_k$ be the $j$-primes minimal over $I$, and let $J = P_1 \cap \ldots \cap P_k$. Then $R/J = R/P_1 \times \ldots \times R/P_k$, so $R/J$ is an elementary divisor ring. But $J/I$ is the Jacobson radical of $R/I$, and it follows from (2) that $R/I$ is an elementary divisor ring. Since $M$ is a finitely presented $(R/I)$-module, $M$ is a direct sum of cyclics, and hence $R$ is an elementary divisor ring.

Finally, if $R$ is not necessarily a domain, let $Q_1, \ldots, Q_m$ be the minimal $j$-primes of $R$. Since $J(R) = 0$, $R = R/Q_1 \times \ldots \times R/Q_m$. But we have just seen that each $R/Q_i$ is an elementary divisor ring; therefore so is $R$.

We remark that there exist $j$-noetherian Bezout domains with arbitrary $j$-dimension [**6**]. (The $j$-dimension of $R$ is the supremum of lengths of chains of $j$-primes in $R$.) The referee has pointed out, however, that *every* Bezout domain with 1 in the stable range is an elementary divisor ring, by [**2**, Proposition 5.1], whether or not it is $j$-noetherian.

## References


**1.** S. Endo, *On semi-hereditary rings*, J. Math. Soc. Japan *13* (1961), 109–119.

**2.** D. Estes and J. Ohm, *Stable range in commutative rings*, J. Algebra *7* (1967), 343–362.

**3.** L. Gillman and M. Henriksen, *Some remarks about elementary divisor rings*, Trans. Amer. Math. Soc. *82* (1956), 362–365.

**4.** ——— *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. *82* (1956), 366–391.

**5.** M. Henriksen, *Some remarks on elementary divisor rings, II*, Michigan Math. J. *3* (1955–56), 159–163.

**6.** W. Heinzer, *J-noetherian integral domains with* 1 *in the stable range*, Proc. Amer. Math. Soc. *19* (1968), 1369–1372.

**7.** I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc. *66* (1949), 464–491.

**8.** M. Larsen, W. Lewis, and T. Shores, *Elementary divisor rings and finitely presented modules*, Trans. Amer. Math. Soc. *187* (1974), 231–248.

**9.** J. Ohm and R. L. Pendleton, *Rings with noetherian spectrum*, Duke Math. J. *35* (1968), 631–640.

**10.** ——— *Addendum to 'Rings with noetherian spectrum'*, Duke Math. J. *35* (1968), 875.

**11.** Y. Quentel, *Sur la compacité du spectre minimal d'un anneau*, Bull. Soc. Math. France *99* (1971), 265–272.

**12.** T. Shores, *The structure of Loewy modules*, J. Reine Angew. Math. *254* (1972), 204–220.


*University of Nebraska-Lincoln,*
*Lincoln, Nebraska*