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SELF θ-CONGRUENT MINIMAL SURFACES IN $\mathbb{R}^3$

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Abstract

A minimal surface is a surface with vanishing mean curvature. In this paper we study self θ-congruent minimal surfaces, that is, surfaces which are congruent to their θ-associates under rigid motions in $\mathbb{R}^3$ for $0 \leq \theta < 2\pi$. We give necessary and sufficient conditions in terms of its Weierstrass pair for a surface to be self θ-congruent. We also construct some examples and give an application.

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1. Introduction

In [1] the first author studies a special class of minimal surfaces in $\mathbb{R}^3$, those congruent with their conjugates. In this paper we study the more general case of minimal surfaces in $\mathbb{R}^3$ congruent to one of their associated surfaces.

Let M be a Riemann surface, $g : M \to \mathbb{C} \cup \{\infty\}$ a meromorphic function, and η a holomorphic 1-form such that $\eta(p) = 0$ if and only if $g(p) = \infty$, with the order of η at p being twice the order of g at p. If the three meromorphic forms

$$\omega_1 = \frac{1}{2}(1 - g^2)\eta, \quad \omega_2 = \frac{\sqrt{-1}}{2}(1 + g^2)\eta, \quad \omega_3 = g\eta,$$

have purely imaginary periods then the *Weierstrass representation*

$$(1) \qquad X(p) = \mathrm{Re} \int_{p_0}^{p} (\omega_1, \omega_2, \omega_3)$$

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gives an immersed minimal surface $S = X(M) \subset \mathbb{R}^3$. The pair (g, η) is called the *Weierstrass pair* of S. When M is simply connected there are no period conditions to be satisfied, and thus any pair (g, η) on M gives a minimal surface.

From now on we will write $i = \sqrt{-1}$.

In terms of a local coordinate z on M, $\eta = f(z)dz$, where f is holomorphic. The local geometry of S via the Weierstrass pair (g, η) is then as follows:

$$I = \frac{1}{4}(1+|g|^2)^2|\eta|^2 = \frac{1}{4}(1+|g(z)|^2)^2|f(z)|^2|dz|^2 = \lambda^2|dz|^2, \tag{2}$$

$$II = -\operatorname{Re}(\eta dg) = -\operatorname{Re}[f(z)g'(z)(dz)^2], \tag{3}$$

$$K = -\left(\frac{4|dg|}{|\eta|(1+|g|^2)^2}\right)^2 = -\left(\frac{4|g'(z)|}{|f(z)|(1+|g(z)|^2)^2}\right)^2 = -\frac{\Delta \log \lambda}{\lambda^2}, \tag{4}$$

where I and II are the first and second fundamental forms induced by X and K is the Gaussian curvature. The last equation in (4) is true for any metric $\lambda^2|dz|^2$ on M.

One more fact about the meromorphic function g is that if N is the unit normal vector of S and $\pi : S^2 \to \mathbb{C} \cup \{\infty\}$ is the stereographic projection from the north pole $(0, 0, 1)$, then $g = \pi \circ N$, and S is minimal if and only if g is meromorphic.

From (2) we see that any local holomorphic coordinate z is an isothermal coordinate for the surface and X is a conformal harmonic immersion. The conformality is equivalent to the identity:

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = 0.$$

For more details about the Weierstrass representation see [4, page 63] or [2, Section 6].

2. Self θ-congruent minimal surfaces in $\mathbb{R}^3$

A minimal immersion $X : M \to \mathbb{R}^3$ is equivalent to a Weierstrass pair (g, η) with $(\omega_1, \omega_2, \omega_3)$, which have purely imaginary periods. For any $\theta \in \mathbb{R}$, we can define another pair $(g, e^{i\theta}\eta)$. If $e^{i\theta}(\omega_1, \omega_2, \omega_3)$ also have purely imaginary periods, then $(g, e^{i\theta}\eta)$ also defines a minimal surface via (1), that is,

$$X_\theta(p) = \operatorname{Re} e^{i\theta} \int_{p_0}^{p} (\omega_1, \omega_2, \omega_3).$$

These surfaces $S_\theta = X_\theta(M)$ are called the *associated* surfaces of the minimal surface $S = X(M) = X_0(M)$. In particular, $S_{\pi/2}$ is called the *conjugate* surface of S. For example if M is simply connected, then X_θ is well defined for all $\theta \in \mathbb{R}$. Hence when studying the local properties, we always know the associated surfaces exist.

By the formulas (2), (3) and (4), every S_θ is isometric to S. The question is then when is this isometry induced by a congruence in $\mathbb{R}^3$?

DEFINITION 1. Let $A(3) = SO(3) \oplus \mathbb{R}^3$ be the rigid motion group of $\mathbb{R}^3$ and $0 \leq \theta < 2\pi$. A minimal surface S is *self θ-congruent induced by F* if there is a $\sigma \in A(3)$ and a map $F : M \to M$ such that

$$\sigma \circ X_\theta = X \circ F.$$

In particular, if $\theta = 0$ or π, then S has a symmetry by a congruence σ. If $\theta = \pi/2$, then we call S a *self-conjugate minimal surface* (as in [1]).

REMARK 1. Let $D \subset M$ be an open set such that X is one-to-one on $F(D)$. Then $F = X^{-1} \circ \sigma \circ X_\theta$ induces an isometry between (D, I_{X_θ}) and $(F(D), I_X)$, where I_{X_θ} and I_X are metrics induced by X_θ and X respectively. Thus F must be a holomorphic or anti-holomorphic mapping.

In the following we also consider branched surfaces, that is, η may vanish at points where g does not have a pole.

Now we want to derive necessary and sufficient conditions of the Weierstrass pair (g, η) such that the induced surface is a self θ-congruent branched minimal surface. First, we define a Möbius transformation $\Phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ to be *orthonormal* if

$$\Phi(z) = \frac{az - \bar{c}}{cz + \bar{a}}, \quad a, c \in \mathbb{C} \quad \text{and} \quad |a|^2 + |c|^2 = 1.$$

THEOREM 1. *Let S be a minimal surface and (g, η) be its Weierstrass pair. Then S is self θ-congruent induced by F if and only if there is an orthonormal Möbius transformation Φ such that*

$$g \circ F = \Phi \circ g, \quad \text{or} \quad g \circ F = \overline{\Phi} \circ g = \overline{\left(\frac{ag(z) - \bar{c}}{cg(z) + \bar{a}}\right)}, \tag{5}$$

and

$$\operatorname{Re}\{F^*(\eta) F^*(dg)\} = \operatorname{Re}\{e^{i\theta} \eta dg\}, \tag{6}$$

where $F^*(\eta)$ is the pullback of η, etc.

PROOF. First suppose S is self θ-congruent, then there is a holomorphic or anti-holomorphic map $F : M \to M$ and a $\sigma = \tau + t \in A(3)$ such that $\sigma \circ X_\theta = X \circ F$, where $\tau \in SO(3)$ and $t \in \mathbb{R}^3$. Then by comparing the second fundamental forms $II_\theta = F^* II$ we have

$$-\operatorname{Re}\{e^{i\theta} \eta dg\} = -\operatorname{Re}\{F^*(\eta) F^*(dg)\},$$

and (6) follows.

Next let $N : M \to S^2$ be the unit normal map of S. Since S and S_θ have the same Gauss map g, N is also the unit normal of S_θ, and so $\tau \circ N = \pm N \circ F$, where the sign is determined by whether F is holomorphic or antiholomorphic. Since τ is a rotation, it induces an orientation-preserving isometry on the unit sphere S^2.

It is well known that a transformation which preserves orientation and circles on S^2 is a Möbius transformation, and therefore through the stereographic projection $\pi : S^2 - \{(0,0,1)\} \to \mathbb{C}$, τ can be expressed as $\pi \circ \tau = \Phi \circ \pi$, where

$$\Phi(z) = \frac{az+b}{cz+d}, \quad a,\ b,\ c,\ d \in \mathbb{C}, \quad \text{and} \quad ad - bc = 1.$$

Because τ is an orientation-preserving isometry on the unit sphere S^2, Φ preserves the metric $4|dz|^2/(1+|z|^2)^2$, which means

$$\frac{|dz|^2}{(1+|z|^2)^2} = \frac{|ad-bc|^2|dz|^2}{(|cz+d|^2+|az+b|^2)^2},$$

therefore $a = \overline{d}$, $c = -\overline{b}$, and $|a|^2 + |c|^2 = 1$.

Combining these facts we get

$$\Phi \circ g = \Phi \circ \pi \circ N = \pi \circ \tau \circ N = \pi \circ (\pm N \circ F) = g \circ F \ (\text{or } -\overline{g}^{-1} \circ F),$$

and (5) is true.

Now suppose there exist F and Φ such that (5) and (6) hold, then $II_\theta = F^* II$. If we can prove that $I_\theta = F^* I$, then by the fundamental theorem for surfaces, we know that S is self θ-congruent.

Supposing that F is holomorphic, it follows from (5) and (6) that

$$g'(F(z))F'(z) = \frac{g'(z)}{(cg(z)+\overline{a})^2},$$
$$\mathrm{Re}\{f(F(z))g'(F(z))(F'(z))^2(dz)^2\} = \mathrm{Re}\{e^{i\theta} f(z)g'(z)(dz)^2\}.$$

Taking directions such that $(dz)^2$ is real and purely imaginary respectively, we have

$$f(F(z))g'(F(z))(F'(z))^2 = e^{i\theta} f(z)g'(z) = e^{i\theta} f(z)g'(F(z))F'(z)(cg(z)+\overline{a})^2,$$

so

$$f(F(z))F'(z) = e^{i\theta} f(z)(cg(z)+\overline{a})^2.$$

Therefore, we get

$$(1+|g(z)|^2)^2|e^{i\theta} f(z)|^2 = (1+|g(z)|^2)^2 \frac{|f(F(z))F'(z)|^2}{|cg(z)+\overline{a}|^4}$$

$$\begin{aligned}
&= (|ag(z) - \bar{c}|^2 + |cg(z) + \bar{a}|^2)^2 \frac{|f(F(z))F'(z)|^2}{|cg(z) + \bar{a}|^4} \\
&= \left(1 + \frac{|ag(z) - \bar{c}|^2}{|cg(z) + \bar{a}|^2}\right)^2 |f(F(z))F'(z)|^2 \\
&= (1 + |\Phi \circ g(z)|^2)^2 |f(F(z))F'(z)|^2 \\
&= (1 + |g(F(z))|^2)^2 |f(F(z))F'(z)|^2,
\end{aligned}$$

which means $I_\theta = F^* I$.

The case that F is antiholomorphic is similar. □

Now taking $\theta = 0$ or π, we get a criterion for minimal surfaces with a symmetry.

COROLLARY 1. *Let $X : M \to \mathbb{R}^3$ be a minimal surface with Weierstrass pair (g, η). Then $S = X(M)$ has a symmetry if and only if there is an orthonormal Möbius transformation Φ and a holomorphic or anti-holomorphic map $F : M \to M$ such that*

$$g \circ F = \Phi \circ g, \quad \text{or} \quad g \circ F = \overline{\Phi} \circ g, \tag{7}$$

and

$$\mathrm{Re}\{F^*(\eta) F^*(dg)\} = \pm \mathrm{Re}\{\eta dg\} = \mathrm{Re}\{e^{i\theta} \eta dg\}, \quad \theta = 0, \pi. \tag{8}$$

REMARK 2. If, more generally, we define the self θ-congruence by

$$\sigma \circ X_\theta = X \circ F,$$

where $\sigma \in O(3) \oplus \mathbb{R}^3$, then (6) becomes

$$\mathrm{Re}\{F^*(\eta) F^*(dg)\} = \pm \mathrm{Re}\{e^{i\theta} \eta dg\}.$$

Theorem 1 continues to hold, the proof being similar. The point is, the fundamental theorem for surfaces is also true when two surfaces have the same first fundamental form and their second fundamental forms differ by a sign, in which case, the linear transformation is orientation reversing.

REMARK 3. For the case of maximal surfaces in L^3, with the help of a similar Weierstrass representation (see [3]) we can also study the theory of self θ-congruent maximal surfaces and get similar results.

3. Special coordinates

Although Theorem 1 gives necessary and sufficient conditions for S to be self θ-congruent, the condition (6) is hard to verify in general. To rectify this, we study some special isothermal coordinates for minimal surfaces.

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3.1. Normal isothermal coordinate Let $p \in M$ be a non-flat point of S, that is, $K(p) < 0$, and equivalently, $dg \neq 0$ at p. Assume that z is an isothermal coordinate such that $z(p) = 0$ and $\eta = f(z)dz$, where f is a holomorphic function, $f(0) \neq 0$. Then on an open disk $D \subset M$ we can define a holomorphic function which is also a local isothermal coordinate,

$$w(z) = \int_0^z \sqrt{f(u)g'(u)}du. \tag{9}$$

We call such w defined above a *normal isothermal coordinate*. Under a normal isothermal coordinate w, the Weierstrass pair (g, η) can be expressed as

$$\begin{aligned} &g(w) = g(z(w)),\\ &\eta = w^*(f(z)\,dz) = f(z(w))\frac{dz}{dw}\,dw = \sqrt{\frac{f(z(w))}{g_z'(z(w))}}\,dw = \frac{dw}{g_w'(w)}. \end{aligned} \tag{10}$$

Thus by (2), (3), and (4), we have

$$\lambda^4 = -\frac{1}{K} = \frac{(1+|g|^2)^4}{16|g'|^4}, \qquad II = -\operatorname{Re}[(dw)^2]. \tag{11}$$

Let S be self θ-congruent induced by F. If F has a fixed point p with $K(p) < 0$, then we say that S is *self θ-congruent induced by F with a fixed point*. Note that we always assume that the fixed point is a non-flat point.

For self θ-congruent minimal surfaces induced by F with a fixed point, we can use the normal isothermal coordinate to simplify the statement of Theorem 1.

THEOREM 2. *Let $X : M \to \mathbb{R}^3$ be an immersed minimal surface with Weierstrass pair (g, η). Then $S = X(M)$ is self θ-congruent induced by F with a fixed point p if and only if there is an orthonormal Möbius transformation Φ such that*

$$g(\pm e^{i\theta/2}w) = \Phi(g(w)), \quad \text{or} \quad g(\pm e^{-i\theta/2}\overline{w}) = \overline{\Phi}(g(w)), \tag{12}$$

where w is the normal isothermal coordinate such that $w(p) = 0$.

If in addition $g(p) = 0$, then (12) *becomes*

$$g(\pm e^{i\theta/2}w) = e^{i\phi}g(w), \quad \text{or} \quad g(\pm e^{-i\theta/2}\overline{w}) = e^{-i\phi}\overline{g}(w), \tag{13}$$

for some $\phi \in \mathbb{R}$.

REMARK 4. By a rotation in $\mathbb{R}^3$ if necessary, we can always make $g(p) = 0$.

As long as the fixed point p is not a branch point, the theorem is true even S has branch points.

PROOF. Since $dg \neq 0$ at p and p is not a branch point, a normal isothermal coordinate as defined in (9) exists.

First suppose S is self θ-congruent. Let w be the normal isothermal coordinate w such that $w(p) = 0$ and let $D_r := \{|w| < r\}$. On D_r we have $\eta dg = (dw)^2$, thus by (3),

$$II_{X_\theta} = -\operatorname{Re}\{e^{i\theta}\eta dg\} = -\operatorname{Re}\{(e^{i\theta/2}dw)^2\}.$$

On the other hand, since $F(0) = 0$ there exists $0 < r_1 < r$ such that $F(D_{r_1}) \subset D_r$, so that on D_r,

$$F^* II_X = -\operatorname{Re}\{(dF(w))^2\}.$$

Since S is self θ-congruent, comparison of the two formulas gives

$$F(w) = \pm e^{i\theta/2}w, \quad \text{or} \quad F(w) = \pm e^{-i\theta/2}\overline{w}. \tag{14}$$

Thus (12) follows immediately from (5) and (14).

If $g(0) = 0$, then (12) implies $\Phi(0) = 0$ as well. Thus $\Phi(w) = e^{i\phi} \cdot w$, and so (13) is true.

Conversely, if there is an orthonormal Möbius transformation Φ for an immersed minimal surface with Weierstrass data $(g, dw/g')$ under the normalized isothermal coordinate w such that (12) is true, we can take

$$F(w) = \pm e^{i\theta/2}w, \qquad \text{or} \quad F(w) = \pm e^{-i\theta/2}\overline{w}. \tag{15}$$

Then (12) and (15) are just the local versions of (5) and (6). So by Theorem 1, S is self θ-congruent. □

REMARK 5. Under a normal isothermal coordinate, we can relax the definition of self θ-congruent with a fixed point by dropping the fixed point condition and only requiring that $F(D_{r_1}) \subset F(D_r)$, for some $0 < r_1 < r$. Then (14) can be modified to

$$w(F(p)) = \pm e^{i\theta/2}w(p) + c, \quad \text{or} \quad w(F(p)) = \pm e^{-i\theta/2}\overline{w}(p) + c, \tag{16}$$

for $p \in D_{r_1}$, $c = w(q)$, $q \in D_r$. And then (12) can be stated as

$$g(\pm e^{i\theta/2}w + c) = \Phi_\pm(g(w)), \quad \text{or} \quad g(\pm e^{-i\theta/2}\overline{w} + c) = \overline{\Phi}_\pm(g(w)), \tag{17}$$

But (13) has no generalizations.

By (11) we have the following corollary.

COROLLARY 2. *Let S be a minimal surface in $\mathbb{R}^3$, and let w be the normal isothermal coordinate around a point $p \in S$. If the Gaussian curvature K is invariant under the transformation in* (16), *then S is self θ-congruent.*

PROOF. X and $X \circ F$ induce the same first and second fundamental forms. □

Taking $\theta = 0$ or π again, we have another corollary.

COROLLARY 3. *Let $X : M \to \mathbb{R}^3$ be an immersed minimal surface with Weierstrass pair (g, η). If $F : M \to M$ induces a non-trivial symmetry of X and if there is a non-flat point p such that $F(p) = p$, then under coordinates of $\mathbb{R}^3$ such that $X(p) = (X_1(p), X_2(p), X_3(p)) = (0, 0, 0)$ and $N(p) = (0, 0, -1)$, we have*

$$X' \circ F = \pm \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X', \quad \text{or} \quad X' \circ F = \pm \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} X',$$

where X' means the transpose of X.

PROOF. We again use a normal isothermal coordinate w such that $w(p) = 0$. Since $g(0) = 0$, by (13) there are ϕ and $\phi_\pm \in \mathbb{R}$ such that

$$g(-w) = e^{i\phi} g(w), \quad \text{or} \quad g(\pm\overline{w}) = e^{-i\phi_\pm}\overline{g}(w).$$

Hence we have either $F(w) = -w$, $\Phi(z) = e^{i\phi}z$; or $F(w) = \pm\overline{w}$, $\Phi_\pm(z) = e^{i\phi_\pm}z$.

Let $\Psi = \Phi$ or $\Phi_\pm$, then there is a $\tau \in SO(3)$ such that $\pi \circ \tau = \Psi \circ \pi$. Because $X(p) = (0, 0, 0)$ and $F(p) = p$, $\pm\tau \circ X = X \circ F$.

Under our coordinate system, τ has the form

$$\tau = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\psi = \phi$ or $\phi_\pm$.

Since $g(0) = 0$, the tangent plane of S at $X(p)$, $T_{X(p)}S$, is the X_1X_2-plane. By (10) we have locally

$$\omega_3 = \frac{g\,dw}{g'}, \quad g(0) = 0, \quad g'(0) \neq 0.$$

Let

$$G(w) = \int_0^w \frac{g(w)\,dw}{g'(w)}, \quad \text{then} \quad G(0) = 0, \quad G'(0) = 0, \quad G''(0) \neq 0.$$

Hence G is a two-to-one covering branched at $w = 0$. Then $X_3(w) = \operatorname{Re} G(w)$, and $X_3^{-1}(0)$ is a one-dimensional variety with a singularity at $w = 0$. Moreover, X_3^{-1} contains exactly 4 curves emitting from $w = 0$ with adjacent angle $\pi/2$. Since X is conformal, S and $T_{X(p)}S$ intersect ($X_3 = 0$) locally in 4 curves emitting from $X(p) = 0$ with adjacent angle $\pi/2$.

But the tangent plane, X_1X_2-plane, is also invariant under $\pm\tau$. If $\sin\psi \neq 0$ or ± 1, then S meets its tangent plane at $(0,0,0)$ in more than 4 curves emitting from $(0,0,0)$, a contradiction to the above observation.

Thus ψ must be either $0, \pi, \pi/2$, or $3\pi/2$. Then according to $\tau \circ N = N \circ F$ or $\tau \circ N = -N \circ F$, we have $X \circ F = \pm\tau \circ X$. The proof is complete. □

3.2. The Gaussian coordinate At a non-flat point $p \in M$, $dg \neq 0$, so we can use the Gauss map g as an isothermal coordinate as well. In this situation, the *local Weierstrass pair* is $(z, f(z)dz)$, and all local properties of S are determined by f alone.

We call the coordinate $z(p) = g(p)$ the *Gaussian coordinate*.

We restate this important special case of Theorem 1.

THEOREM 3. *Let $X : M \to \mathbb{R}^3$ be a minimal surface and $p \in M$ a non-flat point. Let (D', z) be a Gaussian coordinate around p. If $S = X(M)$ is self θ-congruent induced by F for which there exists an open set $p \in D \subset D'$ such that $F(D) \subset D'$, then F is an orthonormal Möbius transformation*

$$F(z) = \frac{az - \bar{c}}{cz + \bar{a}}, \qquad \text{and} \qquad e^{i\theta} f(z) = \frac{f(F(z))}{(cz + \overline{a})^4}, \tag{18}$$

or

$$F(z) = \overline{\left(\frac{az - \bar{c}}{cz + \bar{a}}\right)}, \qquad \text{and} \qquad e^{i\theta} f(z) = \frac{\overline{f}(F(z))}{(cz + \overline{a})^4}, \tag{19}$$

where $a, c \in \mathbb{C}$ and $|a|^2 + |c|^2 = 1$.

And if $g(p) = 0$ and $F(p) = p$, then

$$F(z) = e^{i\phi} z, \qquad \text{and} \qquad e^{i\theta} f(z) = e^{i\cdot 2\phi} f(e^{i\phi} z), \tag{20}$$

or

$$F(z) = e^{-i\phi} \bar{z}, \qquad \text{and} \qquad e^{i\theta} f(z) = e^{i\cdot 2\phi}\overline{f(e^{-i\phi}\bar{z})} \tag{21}$$

for some $\phi \in \mathbb{R}$.

PROOF. Note that $g(z) = z$ and $F(D) \subset D'$ imply $F(z) = g \circ F(z) = \Phi \circ g(z) = \Phi(z)$. □

REMARK 6. Suppose S has a finite total curvature annular end corresponding to a punctured disk $D - \{p\}$. If S is self θ-congruent induced by F, then F can be holomorphically extended to p. So our theory can be applied to complete minimal surfaces which have finite total curvature annular ends. For such surfaces, see for example, [4, Section 9] or [2, Section 11].

4. Some examples

Let us look at some examples.

EXAMPLE 1. Let $M = \mathbb{C}$ or $\mathbb{C} - \{0\}$ and $(g, \eta) = (z, f(z)dz)$, where

$$f(z) = az^k, \quad 0 \neq a \in \mathbb{C},\ k \in \mathbb{Z},\ k \neq -1, -2, -3.$$

Then (g, η) defines a self θ-congruent (branched if $k > 0$) minimal surface for any $0 < \theta < 2\pi$, induced by $F(z) = e^{i\phi}z$ such that

$$\theta - (k+2)\phi = 2l\pi, \quad \text{for some} \quad l \in \mathbb{Z},$$

or by $F(z) = e^{-i\psi}\overline{z}$ with

$$\theta + 2\arg a - (k+2)\psi = 2l\pi, \quad \text{for some} \quad l \in \mathbb{Z},$$

where $\arg a$ is the angle of a, that is, $a = |a|e^{i\arg a}$.

For $k = -2$, the surface is a catenoid or its associated surfaces, these surfaces are not well defined in $\mathbb{C} - \{0\}$, except the catenoid, that is, when $a \in \mathbb{R}$. It is well known that the catenoid is a rotation surface, and from our criterion, self 0-congruent.

PROOF. By Theorem 3 and Remark 6, notice that $g(0) = 0$ so $\Phi(z) = e^{i\phi}z$. When F is holomorphic, then by (20)

$$ae^{i\theta}z^k = e^{2i\phi}a(e^{i\phi}z)^k = ae^{i(k+2)\phi}z^k.$$

Similarly, when F is anti-holomorphic, then by (21)

$$ae^{i\theta}z^k = e^{2i\phi}\overline{a(e^{-i\psi}\overline{z})^k} = \overline{a}e^{i(k+2)\phi}z^k,$$

that is,

$$e^{i2\arg a}e^{i\theta}z^k = e^{i(k+2)\phi}z^k. \qquad \square$$

EXAMPLE 2. Let $q > 1$ be a positive integer,

$$f(z) = z^{k_0}\sum_{k=0}^{\infty} a_k z^{2q(k_0+2)k}, \quad a_{k_0} \neq 0, \quad k_0 \neq -1, -2, -3,$$

and $M \subset \mathbb{C}$ be the convergent disk of f. Then $(z, f(z)dz)$ generates a self θ-congruent minimal surface for all $\theta = (p/q)\pi$ but may induced by different F, where p is an integer and $0 < p < 2q$.

In fact, let

$$\phi_l = \frac{\theta + 2l\pi}{k_0 + 2}, \quad l \in \mathbb{Z}, \qquad F(z) = e^{i\phi_l}z,$$

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then it is easy to check that F and f satisfy (20).

If $a_k/a_{k_0} \in \mathbb{R}$ for $k \geq 1$, let

$$\phi_l = \frac{\theta + 2\arg a_0 + 2l\pi}{k_0 + 2}, \qquad F(z) = e^{-i\phi_l}\bar{z}.$$

Then clearly F and f satisfy (21).

In particular, for $M = \mathbb{C}$, $f(z) = e^{z^{4q}}$, $(z, f(z)dz)$ gives a self $(p/q)\pi$-congruent immersed minimal surface induced by $F(z) = e^{i\phi}z$ or $F(z) = e^{-i\phi}\bar{z}$, for any integer $0 < p < 2q$, and any $\phi \in \mathbb{R}$ such that

$$\theta - 2\phi = 2l\pi, \quad \text{for some} \quad l \in \mathbb{Z}.$$

EXAMPLE 3. Let $M = \mathbb{D} := \{z \in \mathbb{C};\ |z| < 1\}$ and

$$f(z) = \frac{4}{z^4 + i}.$$

Then taking $\theta = \pi/2$ and substituting

$$F(z) = \pm e^{-i3\pi/4}\bar{z} \quad \text{and} \quad F(z) = \pm e^{i\pi/4}\bar{z}$$

into (21), we have

$$e^{i\theta} f(z) = if(z) = \frac{4i}{z^4 + i} = e^{i2\phi}\overline{f(F(z))}.$$

Thus $(z, f(z)\,dz)$ gives a self conjugate minimal surface, as shown in [1, page 566].

More generally,

$$f(z) = \sum_{-\infty<k<\infty} a_{2k}(z^{2q} + i)^{2k}, \quad a_{2k} \in \mathbb{R},$$
$$F(z) = \pm e^{-i\pi/(2q)}\bar{z} \quad \text{and} \quad F(z) = \pm e^{i(2q-1)\pi/(2q)}\bar{z},$$

gives a self π/q-congruent surface for any integer $q > 1$ *via* the Weierstrass pair $(z, f(z)\,dz)$;

$$f(z) = \sum_{-\infty<k<\infty} a_{2k+1}(z^{2q} + i)^{2k+1}, \quad a_{2k+1} \in \mathbb{R},$$
$$F(z) = \pm e^{-i\pi/(2q)}\bar{z} \quad \text{and} \quad F(z) = \pm e^{i(2q-1)\pi/(2q)}\bar{z},$$

gives a self $\pi + \pi/q$-congruent surface for any integer q via the Weierstrass pair $(z, f(z)\,dz)$. Similarly,

$$f(z) = \sum_{-\infty<k<\infty} a_k(z^{4q} + 1)^k, \quad a_k \in \mathbb{R},$$
$$F(z) = \pm e^{i\pi/(2q)}z \quad \text{and} \quad F(z) = \pm e^{-i\pi/(2q)}\bar{z}$$

give self π/q-congruent surfaces, and so on.

Now suppose $M = \Sigma := \mathbb{C} \cup \{\infty\}$ is the Riemannian sphere. For a meromorphic function $f : \Sigma \to \Sigma$ such that for any loop $L \subset \Sigma$,

$$\int_L z^k f(z)\, dz = 0, \quad k = 0, 1, 2, \tag{22}$$

the Weierstrass pair $(g, \eta) = (z, f(z)dz)$ defines a (branched) minimal surface on $\Sigma - f^{-1}(\infty)$. We say that such a surface is generated by f.

In Σ, any orthonormal Möbius transformation can be written as $F(z) = e^{i\phi}z$ under a suitable coordinate z. In fact, if $a \neq 0$, then

$$\frac{az(w) - \bar{c}}{cz(w) + \bar{a}} = \frac{a}{\bar{a}} \frac{|a|^2 z(w) - \overline{ac}}{(ac)z(w) + |a|^2} = e^{2i \arg a} w(z(w)) = e^{2i \arg a} w,$$

$$w(z) = \frac{|a|^2 z - \overline{ac}}{(ac)z + |a|^2};$$

if $a = 0$, then

$$F(z(w)) = -\frac{\bar{c}}{c} \frac{1}{z(w)} = e^{i(\pi - 2\arg c)} w, \quad w(z) = \frac{1}{z}.$$

If a surface S generated by $f : \Sigma \to \Sigma$ is self θ-congruent induced by F, then by Theorem 3 we know that F is an orthonormal Möbius transformation or the conjugate of one. Note that we can change coordinate by any Möbius transformation $w = \Phi(z)$, then up to a rotation, $(w, f \circ \Phi^{-1}(w)(dz/dw)\, dw)$ is the Weierstrass pair of S. Hence by the above observation we can assume that

$$F(z) = e^{i\phi}z, \quad \text{or} \quad F(z) = e^{-i\phi}\bar{z} \tag{23}$$

under a suitable coordinate z.

So when we study self θ-congruent minimal surfaces generated by f, we can always assume that it is induced by such F as in (23).

Using Theorem 3 we can give a criterion for meromorphic functions f such that $(z, f(z)dz)$ defines a self θ-congruent minimal surfaces induced by F such that $F(0) = 0$ via the leading coefficients and order of pole (or zero) of f at 0.

Using (23) and the analytic continuation, we only need to study the behaviour of f at $z = 0$.

THEOREM 4. *Let f be a meromorphic function defined in a neighbourhood of $z = 0$ with the Laurent expansion,*

$$f(z) = \sum_{k=k_0}^{\infty} a_k z^k, \quad a_{k_0} \neq 0, \quad a_{-1} = a_{-2} = a_{-3} = 0.$$

Let S be the minimal surface generated by $(z, f(z)dz)$ and let $\theta = r\pi$, $0 < r < 2$. Then S is self $r\pi$-congruent induced by a holomorphic F such that $F(0) = 0$ if and only if

- *if r is irrational, then $f(z) = a_{k_0} z^{k_0}$, $F(z) = e^{i\phi_l} z$, and $\phi_l = (\theta + 2l\pi)/(k_0+2)$, $l \in \mathbb{Z}$. In particular, if $k_0 = 0$, then f is a constant and S is an Enneper's surface;*
- *if r is rational, then $f(z) = z^{k_0} P_l(z^{K_l})$, where $l \in \mathbb{Z}$, P_l is holomorphic, $P_l(0) \neq 0$, and K_l is the least positive integer such that*

$$\frac{K_l(r+2l)}{2k_0+4} \in \mathbb{Z}.$$

Furthermore, $F(z) = e^{i\phi_l} z$, $\phi_l = (\theta + 2l\pi)/(k_0+2)$.

Now suppose that $a_k/a_{k_0} \in \mathbb{R}$ and let $0 \le s := \arg a_{k_0}/\pi < 2$, then S is self $r\pi$-congruent induced by an anti-holomorphic F such that $F(0) = 0$ if and only if

- *if $r + 2s$ is irrational, then $f(z) = a_{k_0} z^{k_0}$, $F(z) = e^{i\phi_l} z$, and $\phi_l = (\theta + 2\arg a_{k_0} + 2l\pi)/(k_0+2)$, $l \in \mathbb{Z}$. In particular, if $k_0 = 0$, then f is a constant and S is an Enneper's surface;*
- *if $r + 2s$ is rational, then $f(z) = z^{k_0} P_l(z^{K_l})$, where $l \in \mathbb{Z}$, P_l is holomorphic, $P_l(0) \neq 0$, and K_l is the least positive integer such that*

$$\frac{K_l(r+2s+2l)}{2k_0+4} \in \mathbb{Z}.$$

Furthermore, $F(z) = e^{i\phi_l} z$, $\phi_l = (\theta + 2\arg a_{k_0} + 2l\pi)/(k_0+2)$.

PROOF. Since $a_{-1} = a_{-2} = a_{-3} = 0$, for any loop L in the definition domain of f, (22) is satisfied hence all associated surfaces exist.

First suppose S is self θ-congruent. If F is holomorphic, then Theorem 3 gives that $F(z) = e^{i\phi} z$. By (20) we have

$$e^{i\theta} \sum_{k=k_0}^{\infty} a_k z^k = e^{i2\phi} \sum_{k=k_0}^{\infty} a_k (e^{i\phi} z)^k. \tag{24}$$

Thus by comparing the coefficients we have

$$e^{i\theta} a_k = e^{i(k+2)\phi} a_k, \quad k \ge k_0. \tag{25}$$

Thus

$$\frac{\theta - (k+2)\phi}{2\pi} \in \mathbb{Z}, \quad \text{or} \quad a_k = 0. \tag{26}$$

In particular, since $k_0 \neq 0$, if

$$\frac{\theta - (k_0+2)\phi}{2\pi} = -l \in \mathbb{Z},$$

then let

$$\phi = \phi_l := \frac{2\pi l + \theta}{k_0 + 2}.$$

For any $k = k_1 + k_0$, $k_1 > 0$, we have

$$A_{kl} := \frac{\theta - (k+2)\phi_l}{2\pi} = \frac{\theta - (k_0+2)\phi_l}{2\pi} - \frac{k_1\phi_l}{2\pi} = -l - \frac{k_1\phi_l}{2\pi}.$$

So $A_{kl} \in \mathbb{Z}$ if and only if

$$\frac{k_1\phi_l}{2\pi} = \frac{k_1(2l+r)}{2k_0+4} \in \mathbb{Z}.$$

If r is rational, then the set of m such that

$$\frac{m(2l+r)}{2k_0+4} \in \mathbb{Z}$$

is an Abelian group $G(l) \subset \mathbb{Z}$ with infinitely many elements. Thus $G(l) = \{pK_l;\ p \in \mathbb{Z}\}$. This proves that if r is rational, then k for which $a_k \neq 0$ is of the form $k = k_0 + qK_l$, and therefore

$$f(z) = z^{k_0} \sum_{q=0}^{\infty} a_{k_0+qK_l} z^{qK_l}.$$

Setting

$$P_l = \sum_{q=0}^{\infty} a_{k_0+qK_l} z^q,$$

we have that $P_l(0) = a_{k_0} \neq 0$ and $f(z) = z^{k_0} P_l(z^{K_l})$.

If r is irrational, then $G(l) = \{0\}$ and f must be the monomial $a_{k_0} z^{k_0}$. In particular, if $k_0 = 0$, then f is a constant, and S must be a piece of an Enneper's surface.

Clearly the converse is also true, thus the proof of the case when F is holomorphic is complete.

Now consider the case when F is anti-holomorphic and $F(z) = e^{-i\phi}\bar{z}$. By the assumption $a_k/a_{k_0} \in \mathbb{R}$, we have

$$e^{i\theta} a_{k_0} \sum_{k=k_0}^{\infty} \frac{a_k}{a_{k_0}} z^k = e^{i2\phi} \bar{a}_{k_0} \sum_{k=k_0}^{\infty} \frac{a_k}{a_{k_0}} (e^{i\phi} z)^k. \tag{27}$$

Thus by comparing the coefficients we have

$$e^{i(\theta + 2\arg a_{k_0})} \frac{a_k}{a_{k_0}} = e^{i(k+2)\phi} \frac{a_k}{a_{k_0}}, \quad k \geq k_0. \tag{28}$$

Thus

$$\frac{(r+2s)\pi - (k+2)\phi}{2\pi} \in \mathbb{Z}, \quad \text{or} \quad a_k = 0. \tag{29}$$

One then argues exactly as in the case when F is holomorphic. $\square$

5. An application

In [1], the first author used the normal isothermal coordinate to reduce the problem of finding self conjugate ($\theta = \pi/2$) minimal surfaces to the case of finding a $\pi/4$ rotationally invariant solution (that is, $\lambda(w) = \lambda(e^{i\pi/4}w)$) of the following equation:

$$(30) \qquad \lambda^2(u, v)\,\Delta \log \lambda(u, v) = 1, \quad (u, v) \in \Omega \subset \mathbb{R}^2, \quad w = u + iv,$$

where Δ is the usual Laplacian on $\mathbb{R}^2$.

It is also interesting to find θ rotationally invariant solutions (that is, $\lambda(w) = \lambda(e^{i\theta/2}w)$) to (30) in $\mathbb{R}^2$.

If we think of $\lambda^2|dz|^2$ as an intrinsic metric on $\mathbb{R}^2$, then by (2) and (4) (note that (4) is true for any metric $\lambda^2|dz|^2$) we have: λ is a solution to (30) is equivalent to (at $\lambda \neq 0$)

$$\lambda^4 = -1/K.$$

Recall that for a minimal surface S, by (10) and (11), under a normal isothermal coordinate w the Weierstrass pair of S has the form $(g, \eta) = (g, dw/g')$ and

$$\lambda^4 = \frac{-1}{K} = \frac{(1 + |g|^2)^4}{16|g'|^4}.$$

Thus λ is a solution to (30). Suppose the surface is self 2θ-congruent induced by $F(w) = e^{i\phi}w$, where w is a normal isothermal coordinate. Then by (13) of Theorem 2, if $g(0) = 0$ and $g'(0) \neq 0$, the function

$$(31) \qquad \lambda(w) = \frac{(1 + |g(w)|^2)}{2|g'(w)|}$$

gives a θ rotationally invariant solution to (30).

We have constructed many self θ-congruent minimal surfaces, their first fundamental forms (under a normal isothermal coordinate) give examples of θ invariant solution to (30).

Let us construct some more examples defined on the whole $\mathbb{C} = \mathbb{R}^2$. Let $g : \mathbb{C} \to \mathbb{C}$,

$$g(w) = \sum_{n=1}^{\infty} a_n w^n, \quad a_1 \neq 0, \quad \text{and} \quad \theta = \frac{p}{q}\pi, \quad p, q \in \mathbb{N},\ 0 < p < 2q.$$

We want the surface with Weierstrass pair $(g, dw/g')$ to be self 2θ-congruent induced by $F(w) = e^{i\theta}w$. Since $g(0) = 0$ and $g'(0) \neq 0$, (13) of Theorem 2 implies that this is the case if

$$(32) \qquad g(e^{i\theta}w) = e^{i\phi}g(w),$$

for some $\phi \in \mathbb{R}$. Let us take $\phi = \theta$, then comparing the coefficients we find that

$$e^{in\theta} a_n = e^{i\theta} a_n,$$

thus

$$\frac{\theta - n\theta}{2\pi} \in \mathbb{Z}, \quad \text{or} \quad a_n = 0.$$

Hence, when $a_n \neq 0$,

$$\frac{(n-1)p}{2q} \in \mathbb{Z}.$$

If we take $n = 2kq + 1$, then (32) is satisfied. Thus

$$g(w) = w \sum_{k=0}^{\infty} a_k w^{2kq}, \quad a_0 \neq 0,$$

gives a $\theta = p\pi/q$ rotationally invariant solution to (30) on $\mathbb{C} - g'^{-1}(0)$ via (31), for all positive integers p such that $0 < p < 2q$.

Of course we can also discuss other kinds of invariant solutions to (30), such as $\lambda(w) = \lambda(e^{-i\theta}\overline{w})$, corresponding to the self 2θ-congruent minimal surfaces induced by anti-holomorphic maps with a fixed point.

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