defined as

$$\exp\{A_n(s_n - s_{n-1})\}\exp\{A_{n-1}(s_{n-1} - s_{n-2})\}\ldots\exp\{A_1(s_1 - s_0)\},$$

a matrix product which is ordered since the matrices $A_r$ may not commute. The product integral of a continuous $A$ is defined as the limit of the integral of a step-function $A_P$ which converges to $A$ in an appropriate sense as the norm of $P$ tends to zero. Such an integral is to the product what an ordinary integral is to the sum. For example, a product integral over $[a, b]$ is the *product* of integral over $[c, b]$ and $[a, c]$. One of the first results is that if

$$F(x, a) = \prod_a^x e^{A(s)ds}$$

where $A$ is continuous, then $F$ is a solution of the initial value problem

$$\frac{dF(x, a)}{dx} = A(x)F(x, a), \quad F(a, a) = I.$$

This has an immediate application to the solution of a system $Y' = AY$ of $n$ linear ordinary differential equations in $n$ unknowns.

The first chapter of the present book gives an extensive treatment of this simple case; it is quite elementary, and the required matrix algebra is covered in an appendix. Later chapters, which extend the notion of product integration to contour integrals, to measures and to the case when $A$ is a more general operator-valued function, required some ideas from complex and functional analysis, and these are explained briefly. There are many examples throughout, with applications mainly to differential and integral equations but including the sufficiency part of the Hille-Yosida condition for an operator to be the infinitesimal generator of a contraction semigroup. The authors in a short final chapter, and P. R. Masani in an appendix, indicate several other areas of mathematics where product integration has been used, and some two hundred references are given.

Product integration, then, can be applied in many situations, and some results are obtained more easily than by conventional methods. Professors Dollard and Friedman have given the first modern survey, and have done so with admirable style and clarity. Their book can certainly be recommended to everyone whose research uses differential equations, and it may be of interest to other mathematicians also.

PHILIP HEYWOOD

SERRE, J.-P., *Trees* (Translated from the French by J. Stillwell) (Springer-Verlag, 1980), 142 pp., DM 48.

This is a translation of Serre's "Arbes, amalgames, $SL_2$", Astérisque no. 46, Soc. Math. France, 1977. It contains an exposition of the theory of groups acting on trees due to Serre and Bass. This important theory has had many applications and some of these are dealt with in this book. There are two chapters.

Chapter I consists mainly of an account of the basic theory. Given a group $G$ acting on an (oriented) tree $X$ the author describes (using methods from combinatorial group theory) how to obtain a presentation for $G$ from a spanning tree in the quotient graph $G\backslash X$. The fundamental theorem gives this presentation for $G$ in terms of the edges of $G\backslash X$ and the stabilizers (in $G$) of the vertices and edges of $X$. It is shown that free groups, tree products, HNN groups etc. can be realised in this way and the fundamental theorem is used to reprove the subgroup theorems of Schreier and Kurosh. Chapter I ends with the application of the basic theory to the case of groups which fix at least one point of every tree on which they act (the so-called "property ($FA$)"). In this way it is shown, for example, that $SL_3(\mathbb{Z})$ is not a proper free product with amalgamation.

In Chapter II the fundamental theorem is applied to subgroups of $GL(V)$, where $V = K^2$ and $K$ is a local field. A *lattice* of $V$ is an $\mathcal{O}$-submodule of $V$ which generates the $K$-vector space $V$, where $\mathcal{O}$ is the valuation ring of $K$. $GL(V)$ acts on the set of lattices in a natural way and two

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lattices are said to be *equivalent* if they are stabilized by the same elements of $GL(V)$. The author proves that $GL(V)$ acts on a tree $X$ in which each vertex is a lattice class and adjacent vertices are represented by nested lattices whose quotient is the residue field of $\mathcal{O}$. He shows that, provided a subgroup $G$ of $GL(V)$ satisfies certain hypotheses (some of which are topological), the quotient graph $G\backslash X$ can be determined. Applying the fundamental theorem he reproves in this way a theorem of Ihara on torsion-free subgroups of $SL_2(\mathbb{Q}_p)$ and a theorem of Nagao on $GL_2(k[t])$, where $k$ is a field.

In the last part of Chapter II the author considers the case where $K$ is the function field of a smooth projective curve $C$ over a field $k_0$ and he studies the subgroup $\Gamma = GL_2(A)$ of $GL(V)$, where $A$ is the affine algebra of the curve $C - \{P\}$ with a single point $P$ at infinity. (Nagao's theorem is concerned with the case $A = k_0[t]$.) By showing that there is a correspondence between the quotient $\Gamma\backslash X$ and vector bundles or rank 2 over $C$, he uses known results on such bundles to determine $\Gamma\backslash X$. The fundamental theorem is then applied. One important consequence of this, for example, is that, when $k_0$ is finite, $GL_2(A)$ has countably many congruence subgroups and uncountably many subgroups of finite index. (This is one of the main results in Serre's famous paper "Le problème des groupes de congruence pour $SL_2$", Ann. of Math., **92** (1970), 489–527.) Chapter II ends with sections on the homology of $\Gamma$ and its Euler-Poincaré characteristic.

Serre's notes on groups acting on trees have appeared in various forms (all in French) over the past ten years and they have had a profound influence on the development of many areas, for example, the theory of ends of discrete groups. This fine translation is very welcome and I strongly recommend it as an introduction to an important subject. In Chapter I, which is self-contained, the pace is fairly gentle. The author proves the fundamental theorem for the special cases of free groups and tree products before dealing with the (rather difficult) proof of the general case. One word of warning however—although Chapter II is well presented, considerable background reading is required for a full understanding of its contents.

A. W. MASON

BELLMAN, RICHARD, *Analytic Number Theory: An Introduction* (The Benjamin/Cummings Publishing Company Inc., 1980), 195 pp., U.S. $19.50.

Dr Bellman is well known for his distinguished work in number theory and in a very wide variety of branches of pure and applied mathematics. His objective in the volume under review has been to provide an introduction to certain parts of analytic number theory. The reader's interest is stimulated by studying the list of contents. In addition to covering a number of useful analytic techniques (methods of estimation, transforms, Poisson summation formula, etc.) this list includes the gamma and zeta function and all the most important arithmetical functions. There is particular emphasis on mean-value and Tauberian theorems.

It is when he progresses further that the reader begins almost immediately to realize the difficulties in store for him. There can be very few authors of books who do not suffer disappointment at the number of undetected errors and misprints in their published work, but the number to be found scattered through this book is excessive by any standard. The expert may have little trouble in making the necessary corrections, but the novice's difficulties with a new subject will be aggravated and his confidence shaken. Thus, on p. 3, formula (3) is false, in (4) $n$ should be replaced by $N$, and the formula at the bottom of the page is meaningless unless expressed more precisely. If the reader has courage to proceed he will find an average of one error per page in each of the first eleven pages. Mistakes such as *principle* for principal (p. 46), *formerly* for *formally* (p. 60) and $k^2$ for $k$ *squares* (p. 68) strongly suggest that the text was dictated and not checked thereafter. There are few grammatical solecisms, but the great bulk of the errors are typographical and mathematical. The latter range from dangerous half-truths, such as that 'Congruences may be manipulated like ordinary equations' (p. 3), to false statements such as (p. 65)

$$|\zeta(\sigma + it)| = 0(\log t^{3/2})$$

for $\sigma > \frac{1}{2}$. (Is it too pedantic to object to the universal use of the symbol for a capital italic $O$?)