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WEIERSTRASS POINTS ON RATIONAL NODAL CURVES OF GENUS 3

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ABSTRACT. We determine, except for one unsettled case, which combinations of Weierstrass weights can occur on irreducible rational nodal curves of arithmetic genus three. It is shown that the number of nonsingular Weierstrass points on such curves can be any integer between 0 and 6, except 1.

The classical notion of Weierstrass point on a smooth, projective curve has recently been extended to the case of integral, projective Gorenstein curves [8]. To begin to understand the phenomena which may occur in the singular case, we present here examples of Weierstrass points on rational nodal curves of arithmetic genus 3. We determine, except for one unsettled case, which combinations of Weierstrass weights can occur on such curves. Even in the classical case, few results are known concerning the possible combinations of Weierstrass weights which may coexist on a given smooth curve (cf. Eisenbud-Harris [2]). In the case of smooth curves of genus 3, this problem has been solved by Vermuelen [7] who showed that the number of weight two Weierstrass points on such a curve may be any integer between 0 and 12, except for 10 or 11. Including the hyperelliptic curves, there are thus twelve different possible combinations of weights on smooth curves of genus 3. We give examples of thirteen different combinations of weights on rational curves with three nodes. Another result of particular interest is that the number of nonsingular Weierstrass points on such curves can be any integer between 0 and 6, except 1. We will work over $\mathbb{C}$ and we will let $\mathbb{P}^1$ denote $\mathbb{P}^1_{\mathbb{C}}$. We refer the reader to [3] for details concerning the classical theory of Weierstrass points on smooth, projective curves (compact Riemann surfaces).

We begin by reviewing the definition of Weierstrass point in the singular case. Let X be an integral, projective Gorenstein curve of arithmetic genus $g > 0$ and let ω denote the sheaf of dualizing differentials on X. Let σ, $\sigma_1, \ldots, \sigma_g$ be nonzero rational differentials on X. We define the Wronskian of $\sigma_1, \ldots, \sigma_g$ with respect to σ, denoted $W_\sigma(\sigma_1, \ldots, \sigma_g)$, by

$$W_\sigma(\sigma_1, \ldots, \sigma_g) = \det[F_{ij}] \quad \text{for} \quad 1 \leq i, j \leq g,$$

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where the F_{ij} are rational functions defined by

$$\sigma_j = F_{1j}\sigma \quad \text{for} \quad j = 1, \ldots, g,$$

and for $i > 1$, the F_{ij} are defined recursively by the formula

$$dF_{i-1,j} = F_{ij}\sigma \quad \text{for} \quad i = 2, \ldots, g \quad \text{and} \quad j = 1, \ldots, g.$$

Now suppose that $\sigma_1, \ldots, \sigma_g$ are a basis for $H^0(X, \omega)$. Define a section $\alpha \in H^0(X, \omega^{\otimes N})$, where $N = 1 + \cdots + g$, as follows. Suppose that $\{U_i; i \in I\}$ is an open cover of X such that for each $i \in I$, $\Gamma(U_i, \omega)$ is a free rank one $\Gamma(U_i, \mathcal{O}_X)$-module with generator τ_i. For each $i \in I$, define $\alpha_i \in \Gamma(U_i, \omega^{\otimes N})$ by

$$\alpha_i = W_{\tau_i}(\sigma_1, \ldots, \sigma_g)\tau_i^{\otimes N}.$$

It is not hard to see that the α_i's patch to give a section $\alpha \in H^0(X, \omega^{\otimes N})$.

For $P \in X$, let σ generate ω_P and write $\alpha_P = f\sigma^{\otimes N}$, where $f \in \mathcal{O}_P$.

DEFINITION. *The Weierstrass weight of P, denoted $W(P)$, is defined by $W(P) = \operatorname{ord}_P f = \dim_{\mathbb{C}} \mathcal{O}_P/(f)$. The point P is called a Weierstrass point if $W(P) > 0$.*

It is easy to see that $W(P)$ is independent of the choice of σ and of the choice of the basis of $H^0(X, \omega)$. A computation of the degree of $\omega^{\otimes N}$ shows that the total of the Weierstrass weights of all points on X is $g(g-1)(g+1)$.

As evidence that this definition is an appropriate generalization of the notion of Weierstrass point to the singular case, the following key result is shown in [8].

THEOREM 1. *Suppose X is an integral, projective Gorenstein curve of arithmetic genus $g > 1$ and suppose $P \in X$. Then the following statements are equivalent.*

(1) $W(P) > 0$
(2) There is a nonzero $\sigma \in H^0(X, \omega)$ satisfying $\operatorname{ord}_P \sigma \geq g$.
(3) There is a 1-special subscheme with support P and length equal to g.
(4) There is a 1-special subscheme with support P and length at most g.

The concept of an r-special subscheme was introduced by Kleiman [5]. In the case of a nonsingular curve, a multiple of a Weierstrass point is a special divisor. In the singular case, it becomes necessary to replace "special divisor" with "1-special subscheme", since there will now be non-principal subschemes supported at a singularity.

At a smooth point, one may define Weierstrass gaps and the semigroup of non-gaps and prove results completely similar to the classical case. In particular, if the gap sequence at a smooth point P is $1, \gamma_2, \ldots, \gamma_g$ then the Weierstrass weight at P is $\sum_{i=1}^{g}(\gamma_i - i)$ and an integer r is a non-gap at (a smooth point) P if and only if there exists a function on X which has a pole of order r at P and is regular everywhere else.

However, the notions of Weierstrass gaps and semigroup of non-gaps appear not to extend to singular points. If P is a singular point, then the objects of interest are not the Weil divisors nP, but rather all the subschemes supported at P.

Set $\delta_P = \dim(\tilde{\mathcal{O}}_P/\mathcal{O}_P)$, where $\tilde{\mathcal{O}}_P$ denotes the integral closure of $\mathcal{O}_P$. The following is also proved in [8].

THEOREM 2. *$W(P) \geq \delta_P g(g-1)$. In particular, if $g > 1$, then every singular point of X is a Weierstrass point.*

We note that this Theorem may be viewed as a generalization of a result of S. Diaz [1] who showed that every non-separating node on a stable curve is a limit of Weierstrass points on nearby smooth curves and that the generic non-separating node on a uninodal stable curve is a limit of exactly $g(g-1)$ Weierstrass points on nearby smooth curves.

The fact that singular points must be Weierstrass points becomes clear if one looks more closely at the above Wronskian. Suppose that $\sigma_1, \ldots, \sigma_g$ is a basis for $H^0(X, \omega)$. Let $\pi: \tilde{X} \to X$ denote the normalization of X and let $t \in K(\tilde{X})$ be a rational function such that $\operatorname{ord}_Q t = 1$ for all $Q \in \pi^{-1}(P)$. Let h be a generator (in $\tilde{\mathcal{O}}_P$) of the conductor of $\mathcal{O}_P$ in $\tilde{\mathcal{O}}_P$. Then $\sigma = dt/h$ generates ω_P. Write $\sigma_i = f_i\sigma$ for $i = 1, 2, \ldots, g$. Then it is not hard to see that

$$\begin{aligned} W_\sigma(\sigma_1, \ldots, \sigma_g) &= \det(h^{i-1}f_j^{(i-1)}(t)) \quad \text{for} \quad i,j = 1, \ldots, g \\ &= h^{g(g-1)/2}W_t(f_1, \ldots, f_g), \end{aligned} \tag{1.1}$$

where $W_t(f_1, \ldots f_g)$ is the usual Wronskian of rational functions. Since the function h vanishes at P if P is a singular point, it is obvious that singularities are Weierstrass points of high weight.

Another result from [8] which we will need is:

PROPOSITION 1. *Let P be a node of X with $\theta: Y \to X$ the partial normalization of X at P. Put $\theta^{-1}(P) = \{Q_1, Q_2\}$. Then*

$$W(P) = g(g-1) + W(Q_1) + W(Q_2).$$

EXAMPLES.

(1) Gorenstein curves of arithmetic genus 0 or 1 have no Weierstrass points.

(2) Suppose X is a rational nodal curve of arithmetic genus 2. Then by Proposition 1 and (1) above, each node has Weierstrass weight 2.

PROPOSITION 2. *([8]). Let X denote an integral, rational nodal curve of arithmetic genus 2.*

(i) If X is obtained from $\mathbb{P}^1$ by identifying 0 with ∞ and a with b, then the nonsingular Weierstrass points on X are $\pm\sqrt{ab}$.

(ii) If X is obtained from $\mathbb{P}^1$ by identifying a with b and c with d, where $\infty \notin \{a, b, c, d\}$, then ∞ is a Weierstrass point of X exactly when $a + b = c + d$.

PROOF. (i) For a basis of $H^0(X, \omega)$ we may take dt/t and $dt/(t-a)(t-b)$. The nonsingular Weierstrass points are the zeros of the (ordinary) Wronskian of the functions $1/t$ and $1/(t-a)(t-b)$. Direct computation shows that this Wronskian vanishes at $\pm\sqrt{ab}$.

(ii) For a basis of $H^0(X, \omega)$ we may take $dt/(t-a)(t-b)$ and $dt/(t-c)(t-d)$. However, to compute at ∞ we must replace t by $1/z$, obtaining $-dz/(az-1) \times (bz-1)$ and $-dz/(cz-1)(dz-1)$. Computing the (ordinary) Wronskian of the functions $-1/(az-1)(bz-1)$ and $-1/(cz-1)(dz-1)$, we find that the constant term of the numerator is $c+d-a-b$. ■

The situation becomes much more complicated with rational nodal curves of arithmetic genus 3. Note that, by Proposition 1 and Example (2) above, each of the three nodes on such a curve may have Weierstrass weight 6, 7, or 8. At a smooth point, where one can consider the gap sequence, we have, as in the classical case of smooth curves of genus 3, that the Weierstrass weight can be 0, 1, 2, or 3. The total of the Weierstrass weights of all points is 24. We now investigate which combinations of weights actually occur.

Several possible combinations of weights may be ruled out by the following proposition, which is similar to the fact that a smooth hyperelliptic curve of genus g has $2g+2$ (hyperelliptic) Weierstrass points.

PROPOSITION 3. *Suppose that X is a rational nodal curve of arithmetic genus g and suppose that there exists a morphism $\phi: X \to \mathbb{P}^1$ of degree two. Then each node on X has Weierstrass weight $g(g-1)$ and there are two nonsingular Weierstrass points of weight $g(g-1)/2$.*

PROOF. Let $\pi: \mathbb{P}^1 \to X$ denote the normalization of X. By the Riemann-Hurwitz formula, the degree two map $\pi \circ \phi: \mathbb{P}^1 \to \mathbb{P}^1$ will have two ramification points P_1 and P_2. The points $\pi(P_1)$ and $\pi(P_2)$ will then be smooth points of X and it is easy to see that 2 will be a non-gap at each of these points. As in the smooth case, since 2 is a non-gap, it follows that the semigroup of non-gaps will be $\{0, 2, 4, 6, \ldots\}$ and that the weight of each of these points will be $g(g-1)/2$. Since the total of the weights of all the Weierstrass points is $g(g-1)(g+1)$ and each node has weight at least $g(g-1)$, we see that each node must have weight exactly $g(g-1)$. ■

Proposition 3 shows that there cannot exist a rational nodal curve of arithmetic genus 3 which has only one (necessarily nonsingular) Weierstrass point of weight 3. There are 24 remaining possible combinations of weights. These cases are shown in Table 1.

For the remainder of this article X will denote an integral rational nodal curve of arithmetic genus 3. We will let

$$(a_1, b_1; a_2, b_2; \ldots; a_g, b_g)$$

denote the rational nodal curve of arithmetic genus g obtained by identifying the points a_j and b_j of $\mathbb{P}^1$ for $j = 1, 2, \ldots, g$. By the "standard basis" of dualizing differentials on X, we mean the basis $\sigma_j = dt/(t-a_j)(t-b_j)$ for $j = 1, \ldots, g$, where it is understood that if b_j, say, is ∞, then we take $dt/(t-a_j)$ for σ_j.

TABLE 1

Case	Number of Weierstrass Points of Weight 1	2	3	6	7	8
1	6	0	0	3	0	0
2	4	1	0	3	0	0
3	2	2	0	3	0	0
4	0	3	0	3	0	0
5	0	0	2	3	0	0
6	5	0	0	2	1	0
7	3	1	0	2	1	0
8	1	2	0	2	1	0
9	4	0	0	2	0	1
10	2	1	0	2	0	1
11	0	2	0	2	0	1
12	4	0	0	1	2	0
13	2	1	0	1	2	0
14	0	2	0	1	2	0
15	3	0	0	1	1	1
16	1	1	0	1	1	1
17	2	0	0	1	0	2
18	0	1	0	1	0	2
19	3	0	0	0	3	0
20	1	1	0	0	3	0
21	2	0	0	0	2	1
22	0	1	0	0	2	1
23	1	0	0	0	1	2
24	0	0	0	0	0	3

PROPOSITION 4. *If one node of X has weight* 8, *then no node of X can have weight* 7.

PROOF. Suppose that X has one node of weight 8 and one node of weight 7. We may assume that $X = (0, \infty;\ 1, a;\ b, c)$, that the node obtained by identifying b with c has weight 8, and that the node P obtained by identifying 1 with a has weight 7. This implies that b and c are Weierstrass points on the rational nodal curve of genus two $X_1 = (0, \infty;\ 1, a)$. Hence, by Proposition 2, we may assume tht $b = \sqrt{a}$ and $c = -\sqrt{a}$.

Now, by Proposition 2, the Weierstrass points of the rational nodal curve of genus two $X_2 = (0, \infty;\ \sqrt{a}, -\sqrt{a})$ are $\pm\sqrt{-a}$. Since a cannot be 0, the assumption that either 1 or a is a Weierstrass point of X_2 implies that $a = -1$. But then both a and 1 are Weierstrass points of X_2 so P must have weight 8. ■

COROLLARY 1. *Cases 15, 16, 21, 22, and 23 of Table 1 do not occur.*

PROPOSITION 5. *If two nodes of X have weight* 8, *then the third node also has weight* 8.

PROOF. We may assume that $X = (0, \infty;\ 1, a;\ b, c)$ and that the nodes P_1 and P_2 obtained by identifying 1 with a and b with c, respectively, each have weight 8. But then, as in the proof of Proposition 4, we must have $a = -1$ and $\{b, c\} = \{i, -i\}$. If we let $Y = (1, -1;\ i, -i)$, then ∞ is a Weierstrass point of Y by Proposition 2. Hence the node P of X obtained by identifying 0 with ∞ has weight at least 7. Proposition 4 now shows that P must have weight 8. Note that X is an example of Case 24 in Table 1. ■

COROLLARY 2. *Cases 17 and 18 of Table 1 do not occur.*

COROLLARY 3. *The curve $X = (0, \infty;\ 1, -1;\ i, -i)$ is, up to isomorphism, the only rational nodal curve of arithmetic genus 3 with no nonsingular Weierstrass points.*

The curve X of Corollary 3 is indeed a curious example. It can be shown ([8]) that there does not exist a locally *principal* 1-special subscheme of length at most three supported at any of the nodes. The presence of many automorphisms of $\mathbb{P}^1$ which leave invariant the set $\{0, \infty, 1, -1, i, -i\}$ is apparently what makes this example so unusual.

PROPOSITION 6. *There is no rational nodal curve of arithmetic genus 3 with exactly one nonsingular Weierstrass point.*

PROOF. This follows from Table 1, since we have now eliminated Cases 18, 22, and 23. ■

PROPOSITION 7. *There is no rational nodal curve of arithmetic genus 3 such that each node has weight* 7.

PROOF. We may suppose that $X = (0, \infty;\ 1, a;\ b, c)$ and that

(1) b, but not c, is a Weierstrass point of $X_1 = (0, \infty;\ 1, a)$
(2) 1, but not a, is a Weierstrass point of $X_2 = (0, \infty;\ b, c)$
(3) ∞, but not 0, is a Weierstrass point of $X_3 = (1, a;\ b, c)$.

Then, by Proposition 2, (1) implies that $b^2 = a$, (2) implies that $bc = 1$ and (3) implies that $b + c - a - 1 = 0$. Hence b satisfies the equation $b + \dfrac{1}{b} - b^2 - 1 = 0$, so $b = 1$ or i or $-i$. Clearly $b \neq 1$, and if $b = \pm i$, then $c = \mp i$ and $a = -1$, so c would also be a Weierstrass point of X_1. ■

COROLLARY 4. *Cases 19 and 20 of Table 1 do not occur.*

We are now reduced to investigating Cases 1 through 14 of Table 1. Many of the following results depend on very complicated algebraic computations. The MACSYMA software package was used to perform most of these, including computations of Wronskians and resultants. In some cases, approximate values of roots of complex polynomials were needed and these were obtained by using the algorithm built into MACSYMA. This algorithm is due to Jenkins and Traub [4].

When we refer below to a Wronskian, we mean the Wronskian of the "standard basis" of dualizing differentials.

CASE 1. As one would expect, this is the generic case. Indeed, the first author has shown in [6] that a generic integral rational nodal curve of arithmetic genus g has $g(g - 1)$ nonsingular Weierstrass points. It follows that each nonsingular Weierstrass point on such a curve has weight 1 and each node has weight $g(g - 1)$.

For a specific example of Case 1, one may take $X = (0, \infty; 1, 2; 3, 4)$. The nonsingular Weierstrass points are the zeros of the Wronskian of the functions $1/t$, $1/(t - 1)(t - 2)$ and $1/(t - 3)(t - 4)$. A computation shows that these zeros are the roots of

$$2t^6 - 15t^5 + 33t^4 - 35t^3 + 144t^2 - 360t + 264$$

and a calculation of the resultant of this polynomial and its derivative shows that all six roots are distinct.

CASE 2. Here we considered the curve $X = (0, \infty; 1, 2; 3, c)$. We computed the Wronskian as above and then computed the resultant of the numerator of the Wronskian and its derivative. This resultant is

$$6144c^5 - 28224c^4 + 50352c^3 - 49833c^2 + 30540c - 9604.$$

An approximation to the real root of this polynomial is

$$\frac{67131395}{33554432}.$$

We substituted this value for c in the above Wronskian and found that the resulting curve has one weight 2 Weierstrass point at approximately 1.86146 and four weight 1 Weierstrass points at approximately

$$0.90866 \pm 1.57376i \quad \textit{and} \quad 2.04780 \pm 0.06409i.$$

CASE 3. Put $X = (i, -i; 1 + i, -1 + i; (1 - i)/2, (-1 - i)/2)$. The numerator of the Wronskian on the finite smooth points of X is $-240t^2(3it^2 + 2t + 3i)$. Hence, 0 is a Weierstrass point of weight two and $(1 \pm \sqrt{10})i/3$ are Weierstrass points of weight one. A computation of the Wronskian at infinity shows that it is a Weierstrass point of weight two; or, one may see this by noting that the functions t^4 and $t^3 - (i/2)t^2 + t$ on X have poles only at infinity of orders 4 and 3, respectively.

CASE 4. Put $X = (i, -i; 1 + i, 1 - i; (1 - i)/2, (1 + i)/2)$. The numerator of the Wronskian on the finite smooth points of X is then $-240t^2(t - 1)^2$, showing that 0 and 1 are Weierstrass points of weight two. A computation of the Wronskian at infinity shows that it is also a Weierstrass point of weight two; or, one may see this by considering the functions t^4 and $t^3 - (3/2)t^2 + t$ on X.

CASE 5. This is the case of rational nodal curves which admit a degree two morphism onto $\mathbb{P}^1$ ("quasi-hyperelliptic" curves). The obvious examples are the curves $X = (a, -a; b, -b; c, -c)$, where a, b, and c are distinct, nonzero complex numbers. Then X has two weight 3 Weierstrass points at 0 and ∞. We note that another form for these quasi-hyperelliptic curves is $X = (0, \infty;\ 1, a;\ b, a/b)$. This curve has two weight 3 Weierstrass points at $\pm\sqrt{a}$. It follows that these quasi-hyperelliptic curves depend on two "moduli."

CASE 6. Consider $X = (0, \infty;\ 1, 2;\ i, -i)$. Then the node obtained by identifying 1 with 2 has weight 7, since 1 is a Weierstrass point of $X_1 = (0, \infty;\ i, -i)$. A computation of the Wronskian on the smooth locus shows that X has five nonsingular Weierstrass points which are the roots of the polynomial $t^5 - 3t^3 - 2t^2 + 4t + 2$.

CASE 7. Take $X = (0, \infty;\ 1, e^{2\pi i/3};\ b, e^{\pi i/3})$. Note that $e^{\pi i/3}$ is a Weierstrass point on $X_1 = (0, \infty;\ 1, e^{2\pi i/3})$. A factor of the resultant of the numerator of the Wronskian on the smooth locus of X and the derivative of this numerator is then

$$8(-1 + \sqrt{3}i)b^2 + 17b - 8(1 + \sqrt{3}i).$$

We note that the analogous factor of the similar resultant for the curve $(0, \infty;\ 1, c^2;\ b, c)$ is, in general, a quartic in b, but it is the above quadratic for $c = e^{\pi i/3}$. Take b to be one of the roots of this quadratic. Then numerical approximation of the zeros of the Wronskian of X shows that this Wronskian only has one repeated root (of multiplicity two). Thus X is an example of Case 7.

CASE 8. This case is the only one which is unsettled; i.e. we cannot give either an example of this case or an argument to show why it cannot occur.

CASES 9, 10, AND 11.

We may assume that $X = (0, \infty;\ 1, a;\ \sqrt{a}, -\sqrt{a})$, where $a \notin \{0, \infty, 1, -1\}$. The numerator of the Wronskian on the smooth locus of X is then

$$2[(a + 1)t^4 - 6at^3 + 4a(a + 1)t^2 - 6a^2t + a^2(a + 1)].$$

A computation of the resultant of this polynomial and its derivative shows that it has multiple roots only for $a = 0, -1, 1, \frac{1}{2}$, and 2. So for $a \notin \{\frac{1}{2}, 2\}$, the curve X is an example of Case 9. A check shows that for $a = \frac{1}{2}$ or $a = 2$, the curve X has two nonsingular Weierstrass points of weight 2 and is therefore an example of Case 11. As a consequence of this, we have

COROLLARY 5. Case 10 of Table 1 does not occur.

CASES 12, 13, AND 14. Take $X = (0, \infty;\ 1, b^2;\ b, b^3)$, where $b \notin \{0, \infty, 1, -1, i, -i, e^{2\pi i/3}, e^{4\pi i/3}\}$. Then b^2, but not 1, is a Weierstrass point on $X_1 = (0, \infty;\ b, b^3)$ and b, but not b^3, is a Weierstrass point on $X_2 = (0, \infty;\ 1, b^2)$. The resultant of the numerator of the Wronskian on the smooth locus of X and its derivative is

$$9b^{14}(b - 1)^2(b + 1)^4(b - i)(b + i)(b^2 - 3b + 1)^2(b^2 + b + 1)^3$$
$$\times\ (b^4 + 3b^3 + 20b^2 + 3b + 1).$$

If b is not a root of this resultant, then X is an example of Case 12. If $b = \frac{1}{2}(3 \pm \sqrt{5})$, then a check shows that X has two nonsingular Weierstrass points of weight 2 (at the two square roots of $-9 \mp 4\sqrt{5}$), so X is an example of Case 14. Finally, if b satisfies $b^4 + 3b^3 + 20b^2 + 3b + 1 = 0$, then a check shows that the Wronskian of X has only one repeated root (of multiplicity two) on the smooth locus. Thus X is an example of Case 13.

We note that we have established the following result.

THEOREM 3. *For every integer n between* 0 *and* 6, *except for* $n = 1$, *there exists an integral rational nodal curve of arithmetic genus 3 with n nonsingular Weierstrass points.*

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