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DUAL PROBLEMS OF QUASICONVEX MAXIMISATION

ALEXANDER M. RUBINOV AND BELGIN ŞIMŞEK

A conjugacy operation is introduced on the set $Q(X)$ of all quasiconvex lower semicontinuous nonnegative functions vanishing at zero. This operation is used in order to introduce and study a dual problem with respect to a maximisation problem where both constraint and objective functions belong to $Q(X)$.

1. Let X be a locally convex Hausdorff topological vector space and $\overline{R}_+ = R_+ \cup \{+\infty\}$ where R_+ is the set of all nonnegative real numbers. Let us consider the set $Q(X)$ of all quasiconvex lower semicontinuous functions q defined on X and mapping into $\overline{R}_+$ with the property $q(0) = 0$. Recall that a function q defined on X is called quasiconvex if the sets $S_c(q) = \{x \in X : q(x) \leqslant c\}$ are convex for all c. Clearly, $q \in Q(X)$ if and only if the set $S_c(q)$ is convex and closed and $0 \in S_c(q)$ for all $c \geqslant 0$.

The purpose of this paper is to present a new concept of the dual problem with respect to a maximisation problem where both constraint and objective functions belong to $Q(X)$. Duality for convex extremal problems is constructed as a rule by the following scheme: if the primal problem is a maximisation then the dual problem is a minimisation. As it turns out the scheme: maximisation in the primal problem and maximisation in the dual problem is more suitable for our nonconvex case. First we introduce a conjugacy operation on the set $Q(X)$.

2. Let us consider the level sets:

$$S_c(q) = \{x \in X : q(x) \leqslant c\} \text{ and } T_c(q) = \{x : q(x) < c\}$$

of the given function $q \in Q(X)$. Now we determine a conjugate function q^* which is defined on the space X', dual with respect to X and such that a level set $S_{1/c}(q^*)$ is equal to the polar of the level set $S_c(q)$ for all $0 \leqslant c \leqslant +\infty$. Recall that the polar with respect to a nonempty subset S of X is the set $S^\circ = \{\ell \in X' : \ell(x) \leqslant 1, \quad \forall x \in S\}$. By definition the polar of the empty set coincides with X'.

DEFINITION 2.1: Let $q \in Q(X)$. The function q^* defined on the space X' by the formula

$$q^*(\ell) = \sup\left\{\frac{1}{q(x)} : \ell(x) > 1\right\}$$

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is called the **conjugate function** with respect to q. Let us note that this definition is close to the definition which is given by Thach [**1, 2**].

PROPOSITION 2.2. *Let $q \in Q(X)$ and $0 \leqslant c \leqslant +\infty$. Then*

$$(i) \quad S_{1/c}(q^*) = T_c^\circ(q)$$
$$(ii) \quad T_{1/c}(q^*) = \bigcup_{c'>c} (S_{c'}(q))^\circ$$

PROOF: We consider only the case where $0 < c < +\infty$.

(i) By definition of the conjugate function we have $\ell \in S_{1/c}(q^*)$ if and only if the inequality $\ell(x) > 1$ implies $q(x) \geqslant c$. Let $\ell \in S_{1/c}(q^*)$ and $x \in T_c(q)$. Since $q(x) < c$ it follows that $\ell(x) \leqslant 1$ and $\ell \in T_c^\circ(q)$. We have $S_{1/c}(q^*) \subset T_c^\circ(q)$. Similar reasoning shows that $T_c^\circ(q) \subset S_{1/c}(q^*)$.

ii) If $\ell \in T_{1/c}(q^*)$ and $c' > c$ then the inequality $\ell(x) > 1$ implies $q(x) > c' > c$. Let $x \in S_{c'}(q)$. By definition, $q(x) \leqslant c'$ so $\ell(x) \leqslant 1$. Thus $\ell \in S_{c'}^\circ(q)$. Hence $\ell \in \bigcup_{c'>c} (S_{c'}(q))^\circ$ and $T_{1/c}(q^*) \subset \bigcup_{c'>c} (S_{c'}(q))^\circ$. It is easy to check that the reverse inclusion holds. ∎

COROLLARY 2.3. *$q^* \in Q(X')$ for all $q \in Q(X)$.*

3. Let $f, g \in Q(X')$. We consider an extremal problem (P_c):

$$f(x) \to \sup \quad \textit{under condition} \quad g(x) < c,$$

where $c \in (0, +\infty)$. Clearly, this problem is not convex even if f and g are convex functions. Let us remark that the problem

$$f(x) \to \sup \quad \textit{under condition} \quad g_i(x) < c_i \ (i = 1, \cdots, m)$$

can be rewritten as the following problem which is of type (P_c) :

$$f(x) \to \sup \quad \textit{under condition} \quad g(x) < 1$$

where $g = \sup_i (1/c_i) g_i$. A point $\overline{x}$ is called a solution of the problem (P_c) if $g(\overline{x}) = c$ and $f(\overline{x}) = \sup\{f(x) : g(x) < c\}$. Therefore the solution is not an admissible element. If f is continuous and $S_c(g) = c\ell\, T_c(g)$ then

$$\sup_{g(x)<c} f(x) = \sup_{g(x)\leqslant c} f(x)$$

and the vector $\overline{x}$ is a solution of the problem

$$f(x) \to \max \quad \textit{under condition} \quad g(x) \leqslant c$$

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and $\bar{x}$ is an admissible element for this problem.

Let $\sup\limits_{g(x)<c} f(x) = d < +\infty$ and consider the problem

$$g^*(\ell) \to \sup \quad \textit{under condition} \quad f^*(\ell) < \frac{1}{d}.$$

This problem is called the dual with respect to the problem (P_c). We denote this problem by $(D_{1/d})$. It is not usual for the value of the primal problem to be used in the formulation of a dual problem but we believe this approach is suitable for the theoretical investigation of the problem (P_c). Now we consider a function $\varphi(c)$ which coincides with value of problem (P_c),

$$\varphi(c) = \sup\{f(x) : g(x) < c\} \qquad c \in (0, +\infty) \tag{1}$$

Theorem 2.4. *If φ is a strictly increasing function then the value of the dual problem $(D_{1/d})$ coincides with $1/c$, that is, if*

$$\sup_{g(x)<c} f(x) = d \quad \textit{then} \quad \sup_{f^*(\ell)<\frac{1}{d}} g^*(\ell) = \frac{1}{c}.$$

Proof: Let $d' > d$. Since $\sup\limits_{g(x)<c} f(x) < d'$ we have $T_c(g) \subset T_{d'}(f)$ and therefore by Proposition 2.2:

$$(T_c(g))^\circ = S_{1/c}(g^*) \supset S_{1/d'}(f^*) = (T_{d'}(f))^\circ.$$

If $\ell \in T_{1/d}(f^*)$ then there exists $d' > d$ such that $\ell \in S_{1/d'}(f^*)$ and thus $g^*(\ell) \leqslant 1/c$. Hence

$$\sup_{f^*(\ell)<1/d} g^*(\ell) \leqslant \frac{1}{c}.$$

Let $\sup\limits_{f^*(\ell)<1/d} g^*(\ell) < 1/c$ and number $c' > c$ such that

$$\sup_{f^*(\ell)<1/d} g^*(\ell) < \frac{1}{c'} < \frac{1}{c}.$$

So, we have $T_{1/d}(f^*) \subset T_{1/c'}(g^*)$ and $S_d(f) \supset S_{c'}(g)$. This inclusion shows that $\sup\limits_{g(x)\leqslant c'} f(x) \leqslant d$. Thus

$$\varphi(c') = \sup_{g(x)<c'} f(x) \leqslant \sup_{g(x)\leqslant c'} f(x) \leqslant d = \varphi(c).$$

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But we assumed $c' > c$ therefore $\varphi(c') > \varphi(c)$ and we have a contradiction. $\square$

Corollary 2.5. *Let the function φ be defined by formula (1) if $c > 0$ and $\varphi(c) = 0$ if $c \leqslant 0$. Suppose that φ is strictly increasing and lower semicontinuous (that is, continuous from the left) on $(0, +\infty)$. Let*

$$\psi(d) = \begin{cases} 0 & \text{if } d \leqslant 0 \\ \sup\{g^*(\ell) : f^*(\ell) < d\} & \text{if } d > 0 \end{cases}$$

Then $\psi = (\varphi^*)^{-1}$ on $(0, +\infty)$. (Let us note that φ belongs to $Q(R)$ and therefore the conjugate to φ exists. At the same time $\psi(d)$ is the value of the dual problem (D_d) for positive d.)

Proof: Since φ is strictly increasing and continuous from the left we have

$$\varphi^*(y) = \sup_{yx>1} \frac{1}{\varphi(x)} = \frac{1}{\inf\limits_{x>1/y} \varphi(x)} = \frac{1}{\varphi(1/y)}.$$

Let $\psi(c) = d$. Then by Theorem 2.4 we have $\psi(1/d) = 1/c$ and $\varphi^*(1/c) = 1/(\varphi(c)) = 1/d$. Therefore $\varphi^*(\psi(1/d)) = \varphi^*(1/c) = 1/d$ for all $0 < d < +\infty$ and $\psi = (\varphi^*)^{-1}$. $\square$

Let $q \in Q(X)$ and q^* be its conjugate function. Let $q(x) > 0$ and $q^*(\ell) > 0$. Then, using the definition, we have that the inequality $\ell(x) > 1$ implies the inequality $q^*(\ell)q(x) \leqslant 1$. A linear functional ℓ is called a *subgradient of the function q at the point* x° if $q^*(\ell)q(x^\circ) = 1$ and $\ell(x^\circ) = 1$. Let ℓ be a subgradient at the point x°. Then $q^*(\ell)q(x^\circ) = 1$ shows that

$$\sup_{\ell(x)>1} \frac{1}{q(x)} = \frac{1}{q(x^\circ)}.$$

Let $c < q(x^\circ)$. It is easy to check that $q(x) > c$ if $\ell(x) > 1$. Hence the inequality $q(x) \leqslant c$ implies $\ell(x) \leqslant 1$. Since c is an arbitrary number with the property $c < q(x^\circ)$ we have that the inequality $q(x) < q(x^\circ)$ implies $\ell(x) \leqslant 1$ and $\sup\limits_{q(x)<d} \ell(x) \leqslant 1$ for $\ell(x^\circ) = 1$. So

$$\sup_{q(x)<d} \ell(x) = 1 \quad \text{where} \quad d = q(x^\circ). \tag{2}$$

We see that x° is a solution in our sense of the extremal problem

$$\ell(x) \to \sup \quad \textit{under condition} \quad q(x) < d$$

because $q(x^\circ) = d$ and $\ell(x^\circ) = 1$. It is easy to check that the reverse assertion is true, that is, if functional $\ell \in X'$ and x° is a solution of the problem (2) then ℓ is a subgradient at the point x°.

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Let us give a geometrical interpretation of the subgradient. We consider the level set $T_{q(x^\circ)}(q)$ of function q, the hyperplane $H = \{x : \ell(x) = 1\}$ and closed half space $H^- = S_1(\ell) = \{x : \ell(x) \leqslant 1\}$. The vector x° is a solution of problem (2) if and only if $T_{q(x^\circ)}(q)$ is a subset of H^- and $x^\circ \in H$. Let us assume that q is a continuous function and $q(x^\circ) > 0$. Then the set $T_{q(x^\circ)}(q)$ is open and convex. If $x^\circ \in c\ell\, T_{q(x^\circ)}(q)$ then there exists a support hyperplane H with respect to $T_{q(x^\circ)}(q)$ at the point x°. If $H = \{x : \ell(x) = 1\}$ then ℓ is a subgradient of q at the point x°. So if q is continuous and $c\ell\, T_{q(x^\circ)}(q) = S_{q(x^\circ)}(q)$ then a subgradient exists at every point x° with the property $q(x^\circ) > 0$.

Theorem 2.6. *Let $f, g \in Q(X)$. Assume that the function φ which is defined by formula (1) is strictly increasing on $(0, +\infty)$ and the function f is continuous. Let c be a positive number and $d \in (0, +\infty)$ be the value of the problem (P_c) and $c\ell T_d(f) = S_d(f)$. Then the vector $\overline{x}$ is a solution of problem (P_c) if and only if there is a common subgradient ℓ of the functions f and g at the point $\overline{x}$ such that ℓ is a solution of the dual problem $(D_{1/d})$.*

Proof: 1. Let $\overline{x}$ be a solution of the problem (P_c). Then $g(\overline{x}) = c$, $f(\overline{x}) = d$. Since $\sup_{g(x)<g(\overline{x})} f(x) = d$ we have $T_c(g) \subset S_d(f) = c\ell T_d(f)$. Since the function f is continuous, the convex set $T_d(f)$ is open and $\overline{x}$ is a boundary point of this set. Therefore there exists a support hyperplane $H = \{x : \ell(x) = 1\}$ with respect to the set $T_d(f)$ at the point $\overline{x}$. Since $T_c(g) \subset c\ell T_d(f)$ and $\ell(\overline{x}) = 1$ we have that H is the support hyperplane with respect to $T_c(g)$ at the same point.

So ℓ is a subgradient of f at the point $\overline{x}$ and a subgradient of g at this point. By definition,

$$f^*(\ell)f(\overline{x}) = 1, \quad \text{that is} \quad f^*(\ell) = \frac{1}{f(\overline{x})} = \frac{1}{d},$$
$$g^*(\ell)g(\overline{x}) = 1, \quad \text{that is} \quad g^*(\ell) = \frac{1}{g(\overline{x})} = \frac{1}{c}.$$

Let us consider the dual problem

$$g^*(\ell) \to \sup \quad \textit{under condition} \quad f^*(\ell) \leqslant \frac{1}{d}.$$

If function $\varphi(c)$ is strictly increasing then by Theorem 2.4 the value of this problem coincides with $1/c$. Since $f^*(\ell) = 1/d$ and $g^*(\ell) = 1/c$ the vector ℓ is a solution of this problem. So we have a necessary condition for a maximum.

2. Let a point x° be such that there is a common subgradient ℓ at the point x° of functions f and g which is a solution of the dual problem $(D_{1/d})$. Equalities hold as follows:

$$f^*(\ell)f(x^\circ) = 1, \quad g^*(\ell)g(x^\circ) = 1, \quad g^*(\ell) = \frac{1}{c}, \quad f^*(\ell) = \frac{1}{d}.$$

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Therefore $f(x^\circ) = d$ and $g(x^\circ) = c$, that is, x° is a solution of the problem (P_c). So we have a sufficient condition for a maximum.

A geometrical interpretation of duality for the minimisation of a convex function under convex constraints is the existence of a separating hyperplane for two convex sets. Theorem 2.6 shows that duality for the maximisation of a quasiconvex function can be interpreted geometrically through the existence of a common supporting hyperplane for two convex sets, one contained within the other. Note that Thach [**1**, **2**] considers one of these sets and the complement of the other. □

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Department of Mathematics and
Computer Science
Ben-Gurion University of the Negev
Beer-Sheva 84105
Israel

Mathematics Department
Hacettepe University
06532 Beytepe-Ankara
Turkey

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