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DISTRIBUTIONAL PROPERTIES OF SOLUTIONS OF $\mathrm{d}V_T = V_{T-}\,\mathrm{d}U_T + \mathrm{d}L_T$ WITH LÉVY NOISE

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Abstract

For a given bivariate Lévy process $(U_t, L_t)_{t\geq 0}$, distributional properties of the stationary solutions of the stochastic differential equation $\mathrm{d}V_t = V_{t-}\,\mathrm{d}U_t + \mathrm{d}L_t$ are analysed. In particular, the expectation and autocorrelation function are obtained in terms of the process (U, L) and in several cases of interest the tail behavior is described. In the case where U has jumps of size -1, necessary and sufficient conditions for the law of the solutions to be (absolutely) continuous are given.

Keywords: Stochastic differential equation; generalized Ornstein–Uhlenbeck process; stationarity; moment conditions; tail behavior

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1. Introduction

For a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$, the generalized Ornstein–Uhlenbeck process driven by (ξ, η) with starting random variable V_0 is defined via the integral equation

$$V_t = \mathrm{e}^{-\xi_t}\biggl(V_0 + \int_{(0,t]} \mathrm{e}^{\xi_{s-}}\,\mathrm{d}\eta_s\biggr), \qquad t \geq 0.$$

The associated stochastic differential equation (SDE) is given by

$$\mathrm{d}V_t = V_{t-}\,\mathrm{d}U_t + \mathrm{d}L_t, \qquad t \geq 0, \tag{1.1}$$

where $(U_t, L_t)_{t\geq 0}$ is again a bivariate Lévy process which is completely determined by $(\xi_t, \eta_t)_{t\geq 0}$. In particular, it holds that $\mathrm{e}^{-\xi_t} = \mathcal{E}(U)_t,\ t \geq 0$, where $\mathcal{E}(U)_t$ denotes the Doléans-Dade exponential of U (see, e.g. [15, pp. 84–86]). This relation forces the process U to admit no jumps which are smaller than or equal to -1, i.e. it holds that $\Pi_U((-\infty, -1]) = 0$, where Π_U denotes the Lévy measure of U. Having many applications in physics, insurance, and risk theory, as well as in financial settings, generalized Ornstein–Uhlenbeck processes have been studied in various papers; see, e.g. Maller *et al.* [13] for general properties of the processes and Lindner and Maller [12], who stated necessary and sufficient conditions for the existence of stationary solutions.

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For a general bivariate Lévy process $(U_t, L_t)_{t\geq 0}$, the unique solution of the SDE (1.1) is given by (see [2, Proposition 3.2] or [15, Exercise V.27])

$$V_t = \mathcal{E}(U)_t\bigg(V_0 + \int_{(0,t]}\Big[\mathcal{E}(U)_{s-}\Big]^{-1}\,\mathrm{d}\eta_s\bigg)\mathbf{1}_{\{K(t)=0\}} + \mathcal{E}(U)_{(T(t),t]}\bigg(\Delta L_{T(t)} + \int_{(T(t),t]}\Big[\mathcal{E}(U)_{(T(t),s)}\Big]^{-1}\,\mathrm{d}\eta_s\bigg)\mathbf{1}_{\{K(t)\geq 1\}}, \qquad t \geq 0, \tag{1.2}$$

where

$$\eta_t = L_t - \sum_{\substack{0<s\leq t\\ \Delta U_s\neq -1}} \frac{\Delta U_s \Delta L_s}{1+\Delta U_s} - t\,\mathrm{cov}(B_{U_1}, B_{L_1}), \tag{1.3}$$

$\Delta M_t = M_t - M_{t-}$ denotes the jump height at time t of a càdlàg process $(M_t)_{t\geq 0}$, B_{U_1} and B_{L_1} denote the Brownian motion parts of U_1 and L_1, respectively, and, consistently with the notation introduced in [2],

$$K(t) := \text{number of jumps of size } -1 \text{ of } U \text{ in [0,t]}$$

and

$$T(t) := \sup\{s \leq t\colon \Delta U_s = -1\} \quad \text{for } t \geq 0.$$

The generalization of the Doléans-Dade exponential $\mathcal{E}(U)_t$ for $0 \leq s < t$ is given by

$$\mathcal{E}(U)_{(s,t]} := \exp\bigg((U_t - U_s) - \frac{\sigma_U^2(t-s)}{2}\bigg)\prod_{s<u\leq t}(1+\Delta U_u)\mathrm{e}^{-\Delta U_u}$$

and

$$\mathcal{E}(U)_{(s,t)} := \exp\bigg((U_{t-} - U_s) - \frac{\sigma_U^2(t-s)}{2}\bigg)\prod_{s<u<t}(1+\Delta U_u)\mathrm{e}^{-\Delta U_u},$$

while, for $s \geq t$, we set $\mathcal{E}(U)_{(s,t]} := 1$. Here σ_U^2 is the quadratic variation of the Brownian motion part of U. If the starting random variable V_0 is independent of $(U_t, L_t)_{t\geq 0}$, the process V_t in (1.2) is called causal, otherwise it is called non-causal.

Since every jump of U of size -1 restarts the process $(V_t)_{t\geq 0}$ defined in (1.2), as already remarked in [2], in applications these jumps can be interpreted as occurrence of default. Jumps of U of size less than -1 have an interpretation, e.g. in financial settings when positive values of a contract described by U turn into obligations that have to be paid.

Necessary and sufficient conditions for the existence of stationary solutions of (1.2) have been given in [2]. In this paper we study the distributional properties of these stationary solutions. In particular, in Section 3 we give the moment conditions and quote first and second moments as well as the autocorrelation function of the stationary solutions in terms of (U, L). In Section 4 we investigate the tail behavior of the stationary solutions by applying the results of [5], [6], and [11]. Our results show that, depending on the properties of U and L, the resulting solutions can have a different tail behavior, heavy tailed or exponentially decreasing.

As observed by Watanabe, one can conclude from Theorem 1.3 of [1] that the law of the stationary processes in the case $\Pi_U(\{-1\}) = 0$ is a pure-type measure, i.e. it is either absolutely continuous, continuous singular, or a Dirac measure. In the case of generalized Ornstein–Uhlenbeck processes conditions for continuity of the stationary solutions have already been

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established in [3]. In Section 5 of this paper we will study the case $\Pi_U(\{-1\}) > 0$. It turns out that the distributions of the stationary solutions do not fulfill a pure-type theorem in this case. We then give necessary and sufficient conditions for them to be (absolutely) continuous. Some examples are given for illustration. Note that the results given in Section 5 hold for any solution X of a random fixed-point equation $X \overset{\mathrm{D}}{=} AX' + B$ with $X \overset{\mathrm{D}}{=} X'$ and (A, B) independent of X' such that $\mathrm{P}(A = 0) > 0$. Here '$\overset{\mathrm{D}}{=}$' denotes equality in distribution.

Finally, in Section 6, we provide the proofs of our results.

2. Preliminaries

For any Lévy process $(X_t)_{t\geq 0}$, we write its Lévy–Khintchine triplet as $(\sigma_X^2, \gamma_X, \Pi_X)$ and, to avoid trivial cases, throughout this paper we will assume that the processes U and L are not equal to 0 constants. For a random variable X, its distribution will be denoted by $\mathcal{L}(X)$.

In this section we briefly recall some results about the solutions $(V_t)_{t\geq 0}$ of the SDE (1.1) which we will require throughout this paper. All of these results are proved in [2].

First note that it is easy to see from the formula of the general solutions (1.2) that the process $(V_t)_{t\geq 0}$ fulfills the random recurrence equation

$$V_t = A_{s,t}\,\mathbf{1}_{\{K(t)=K(s)\}}\,V_s + B_{s,t}\,\mathbf{1}_{\{K(t)=K(s)\}} + [A_{T(t),t}\,\Delta L_{T(t)} + B_{T(t),t}]\,\mathbf{1}_{\{K(t)>K(s)\}} \tag{2.1}$$

for $0 \leq s < t$, where

$$A_{s,t} := \mathcal{E}(\tilde{U})_{(s,t]} \quad \text{and} \quad B_{s,t} := \mathcal{E}(\tilde{U})_{(s,t]} \int_{(s,t]} [\mathcal{E}(\tilde{U})_{(s,u)}]^{-1}\,\mathrm{d}\tilde{\eta}_u, \tag{2.2}$$

with the processes $\tilde{U}$ and $\tilde{\eta}$ given by

$$\tilde{U}_t = U_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta U_s \quad \text{and} \quad \tilde{\eta}_t = \eta_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta \eta_s, \qquad t \geq 0. \tag{2.3}$$

Hence, under the assumption that it is causal, i.e. V_0 is independent of $(U_t, L_t)_{t\geq 0}$, $(V_t)_{t\geq 0}$ is a time-homogeneous Markov process.

The use of $\tilde{U}$ and $\tilde{\eta}$ instead of U and η is not necessary for the above formulae, but it allows the definition of $B_{s,t}$ for arbitrary $0 < s \leq t$. This will be of benefit later in Section 6. Also, observe that $(\tilde{U}_t, \tilde{\eta}_t)_{t\geq 0}$ is independent of $(K(t), T(t))_{t\geq 0}$.

Since in this paper we concentrate on the stationary solutions of (1.1), our research is based on the following theorem [2, Theorems 2.1 and 2.2].

Theorem 2.1. *Let (U, L) be a bivariate Lévy process, and let $V = (V_t)_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ be defined by (1.2) and (1.3). Then a finite random variable V_0 can be chosen such that V is strictly stationary if and only if one of the following conditions hold.*

(i) *There exists a $k \neq 0$ such that $U = -L/k$.*

(ii) *The integral $\int_0^t \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ converges almost surely (a.s.) to a finite random variable as $t \to \infty$.*

(iii) $\Pi_U(\{-1\}) = 0$ *and the integral $\int_0^t [\mathcal{E}(U)_{s-}]^{-1}\,\mathrm{d}\eta_s$ converges a.s. to a finite random variable as $t \to \infty$.*

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If one of these conditions is satisfied then the distributions of V_0 and the corresponding strictly stationary process V are uniquely determined. In particular, for the three cases above, the following statements respectively hold.

(i) *The strictly stationary solution is indistinguishable from the constant process $t \mapsto k$.*

(ii) *If $\lambda := \Pi_U(\{-1\}) = 0$, the distribution of V_0 is given by the distribution of the integral $\int_{(0,\infty)} \mathcal{E}(U)_{s-}\, \mathrm{d}L_s$. Otherwise, if $\lambda \neq 0$, it holds that $V_0 \stackrel{\mathrm{D}}{=} Z_\tau$ for the process*

$$Z_t = \mathcal{E}(\tilde{U})_t \left(Y + \int_{(0,t]} [\mathcal{E}(\tilde{U})_{s-}]^{-1}\, \mathrm{d}\tilde{\eta}_s \right), \qquad t \geq 0, \tag{2.4}$$

evaluated at an exponentially distributed time τ with parameter λ, independent of (U, L) and Y. Here Y is a random variable, independent of (U, L), with the same distribution as ΔL_{T_1}, where T_1 denotes the time of the first jump of U of size -1, i.e. $\mathrm{P}_Y(\mathrm{d}y) = \Pi_{U,L}(\{-1\}, \mathrm{d}y)/\Pi_U(\{-1\})$.

(iii) *The strictly stationary solution is given by $V_t = -\mathcal{E}(U)_t \int_{(t,\infty)} [\mathcal{E}(U)_{s-}]^{-1}\, \mathrm{d}\eta_s$ a.s., $t \geq 0$, and, hence, it is strictly non-causal in the sense that V_t is independent of $(U_s, L_s)_{0<s<t}$.*

Owing to the required convergence of the integral $\int_{(0,\infty)} \mathcal{E}(U)_{s-}\, \mathrm{d}L_s$, Theorem 2.1(ii) for $\lambda = 0$ can only occur if $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ a.s., as shown in [2]. In the same way $\lim_{t\to\infty} [\mathcal{E}(U)_t]^{-1} = 0$ is a necessary condition for the convergence of $\int_{(0,\infty)} [\mathcal{E}(U)_{s-}]^{-1}\, \mathrm{d}\eta_s$ and, hence, for the existence of a stationary solution as described in Theorem 2.1(iii).

3. Moment conditions and the autocorrelation function

Recall that, by [17, Theorem 25.17], for a Lévy process $(X_t)_{t\geq 0}$ and a constant $\kappa \geq 0$, we have $\mathrm{E}\, \mathrm{e}^{-\kappa X_1} < \infty$ if and only if $\mathrm{E}\, \mathrm{e}^{-\kappa X_t} < \infty$ for all $t \geq 0$. In particular, if we define

$$\psi_X(\kappa) := \log \mathrm{E}[\mathrm{e}^{-\kappa X_1}],$$

it holds that $\mathrm{E}\, \mathrm{e}^{-\kappa X_t} = \mathrm{e}^{t\psi_X(\kappa)}$ for all $t \geq 0$.

In order to deal with negative moments of the stochastic exponential in the case $\Pi_U(\{-1\}) = 0$, we define the auxiliary Lévy process

$$W_t := -U_t + \sigma_U^2 t + \sum_{0<s\leq t} \frac{(\Delta U_s)^2}{1 + \Delta U_s}, \tag{3.1}$$

which fulfills $[\mathcal{E}(U)_t]^{-1} = \mathcal{E}(W)_t$, $t \geq 0$. See [2] or [10] for details.

The following result on moments of the Doléans-Dade exponential will be needed later. Although we expect it might be known, we were unable to find a ready reference and so we provide a proof in Section 6.1.

Proposition 3.1. *Let $(U_t)_{t\geq 0}$ be a Lévy process, and let $\kappa \geq 1$.*

(i) *$|\mathcal{E}(U)_t|^\kappa$ is integrable if and only if $\mathrm{E}\,|U_1|^\kappa < \infty$. In particular, for $\kappa = 1$ and $\kappa = 2$, it respectively holds that*

$$\mathrm{E}[\mathcal{E}(U)_t] = \mathrm{e}^{\mathrm{E}[U_1]t} \tag{3.2}$$

and

$$\operatorname{var}(\mathcal{E}(U)_t) = \mathrm{e}^{2t\,\mathrm{E}[U_1]}(\mathrm{e}^{t \operatorname{var}(U_1)} - 1). \tag{3.3}$$

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(ii) *Additionally, suppose that* $\Pi_U(\{-1\}) = 0$. *Then* $|\mathcal{E}(U)_t|^{-\kappa} = |\mathcal{E}(W)_t|^{\kappa}$ *is integrable if and only if*

$$\int_{(-1-\mathrm{e}^{-1},-1+\mathrm{e}^{-1})} |1+x|^{-\kappa}\,\Pi_U(\mathrm{d}x) < \infty. \tag{3.4}$$

In particular, for $\kappa = 1$ *and* $\kappa = 2$, *(3.2) and (3.3) respectively hold with* U *replaced by* W, *where* $\mathrm{E}[W_1]$ *and* $\mathrm{var}(W_1)$ *are given by*

$$\mathrm{E}[W_1] = -\gamma_U + \sigma_U^2 + \int_{[-1,1]} \frac{x^2}{1+x}\Pi_U(\mathrm{d}x) - \int_{|x|>1} \frac{x}{1+x}\Pi_U(\mathrm{d}x)$$

and

$$\mathrm{var}(W_1) = \sigma_U^2 + \int_{\mathbb{R}} \frac{x^2}{(1+x)^2}\Pi_U(\mathrm{d}x) = \mathrm{var}(U_1) - \int_{\mathbb{R}} \frac{x^3(2+x)}{(1+x)^2}\Pi_U(\mathrm{d}x).$$

In the following we will examine second-order properties of the stationary process $(V_t)_{t\geq 0}$. We start with a short lemma characterizing the constant solutions.

Lemma 3.1. *The process* $(V_t)_{t\geq 0}$ *as in (1.2) is a.s. constant, equal to* $k \in \mathbb{R}$ *if and only if* $kU_t = -L_t$ *a.s. and* $V_0 = k$ *a.s.*

The next theorem gives us the moment conditions, the expectation and variance, of the nonconstant stationary solutions of (1.2). For $\kappa \geq 1$, the moment conditions could have been deduced from [18, Theorem 5.1]. We extend to $\kappa > 0$ and give a proof in Section 6.1 which is based on the proof of Proposition 4.1 of [12]. Compared to the special case treated there, we obtain sharper conditions for the existence of the moments by omitting the use of Hölder's inequality. Indeed, a comparison with Theorems 4.1 and 4.2 below shows that the moment conditions in the following theorem are sharp.

Theorem 3.1. *Let* $(V_t)_{t\geq 0}$ *be a nonconstant strictly stationary solution of (1.2).*

(i) *Suppose that* $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$ *a.s. and that, for fixed* $\kappa > 0$,

$$\mathrm{E}\,|\tilde{U}_1|^{\max\{1,\kappa\}} < \infty, \qquad \mathrm{E}\,|L_1|^{\max\{1,\kappa\}} < \infty, \quad \textit{and} \quad \mathrm{E}\,|\mathcal{E}(\tilde{U})_1|^{\kappa} < \mathrm{e}^{\lambda}, \tag{3.5}$$

for $\lambda = \Pi_U(\{-1\}) \geq 0$. *Then* $\mathrm{E}\,|V_0|^{\kappa} < \infty$. *In particular, for* $\kappa = 1$ *and* $\kappa = 2$, *it respectively holds that*

$$\mathrm{E}[V_0] = -\frac{\mathrm{E}[L_1]}{\mathrm{E}[U_1]} \tag{3.6}$$

and

$$\mathrm{var}(V_0) = -\frac{\mathrm{E}[(U_1\,\mathrm{E}[L_1] - \mathrm{E}[U_1]L_1)^2]}{(\mathrm{E}[U_1])^2(2\,\mathrm{E}[U_1] + \mathrm{var}(U_1))}. \tag{3.7}$$

Note that, for $\kappa = 1$, $\mathrm{E}[U_1]$ *is negative by (3.2) and (3.5) while, for* $\kappa = 2$, *by (3.3) and (3.5), it holds that* $2\,\mathrm{E}[U_1] + \mathrm{var}(U_1) < 0$.

(ii) *Suppose that* $\Pi_U(\{-1\}) = 0$ *and* $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$ *a.s., and that, for fixed* $\kappa > 0$,

$$\mathrm{E}\,|W_1|^{\max\{1,\kappa\}} < \infty, \qquad \mathrm{E}\,|\eta_1|^{\max\{1,\kappa\}} < \infty, \quad \textit{and} \quad \mathrm{E}\,|\mathcal{E}(W)_1|^{\kappa} < 1. \tag{3.8}$$

Then it holds that $\mathrm{E}\,|V_0|^{\kappa} < \infty$. *In particular, for* $\kappa = 1$ *and* $\kappa = 2$, *(3.6) and (3.7) hold for* U *and* L *replaced by* W *and* η, *respectively.*

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Finally, we give the autocorrelation function of the stationary processes $(V_t)_{t\geq 0}$ in the following theorem.

Theorem 3.2. *Let* $(V_t)_{t\geq 0}$ *be a nonconstant strictly stationary solution of (1.2).*

(i) *Suppose that* $\lim_{t\to\infty}\mathcal{E}(U)_t = 0$ *a.s. and that (3.5) holds for* $\kappa = 2$. *Then*

$$\operatorname{cov}(V_s, V_t) = -\mathrm{e}^{\mathrm{E}[U_1](t-s)} \frac{\mathrm{E}[(U_1\,\mathrm{E}[L_1] - \mathrm{E}[U_1]L_1)^2]}{(\mathrm{E}[U_1])^2(2\,\mathrm{E}[U_1] + \operatorname{var}(U_1))}. \tag{3.9}$$

(ii) *Suppose that* $\Pi_U(\{-1\}) = 0$ *and* $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$. *Then if (3.8) holds for* $\kappa = 2$, *(3.9) is true with* U *and* L *replaced by* W *and* η, *respectively.*

It should be mentioned here that the proof of Theorem 3.2 does not make use of the stationarity assumption. In fact, every solution $(V_t)_{t\geq 0}$ of (1.1) such that V_u is independent of $(U_{u+v} - U_u, L_{u+v} - L_u)_{v\geq 0}$ fulfills $\operatorname{cov}(V_s, V_t) = \mathrm{e}^{\mathrm{E}[U_1](t-s)}\operatorname{var} V_s$ given that $\operatorname{var} V_s$ and $\mathrm{E}\,|U_1|$ are finite. In the same way, given $\Pi_U(\{-1\}) = 0$, every $(V_t)_{t\geq 0}$ with V_u independent of $(U_v, L_v)_{0\leq v<u}$ satisfies $\operatorname{cov}(V_s, V_t) = \mathrm{e}^{\mathrm{E}[W_1](t-s)}\operatorname{var} V_t$ if $\operatorname{var} V_t$ and $\mathrm{E}\,|W_1|$ are finite.

4. Tail behavior

In this section we study the tail behavior of the stationary solutions of (1.2) which were given in Theorem 2.1. To analyse the nonconstant stationary solutions, we start with a result corresponding to Theorem 2.1(ii) which is based on classical results of [5] and [11] on the tails of solutions of random recurrence equations. For the special case of generalized Ornstein–Uhlenbeck processes, this result is also given in [12, Theorem 4.5] with slightly stronger conditions.

Theorem 4.1. *Let* $(U_t, L_t)_{t\geq 0}$ *be a bivariate Lévy process, and suppose that there exists* $\kappa > 0$ *such that*

$$\mathrm{E}\,|\tilde{U}_1|^{\max\{1,\kappa+\varepsilon\}} < \infty, \qquad \mathrm{E}\,|L_1|^{\max\{1,\kappa\}} < \infty, \quad \textit{and} \quad \mathrm{E}\,|\mathcal{E}(\tilde{U})_1|^{\kappa} = \mathrm{e}^{\lambda}, \tag{4.1}$$

for some $\varepsilon > 0$ *and* $\lambda = \Pi_U(\{-1\}) \geq 0$. *If* U *is of finite variation, additionally assume that the drift of* U *is nonzero or that there is no* $r > 0$ *such that* $\operatorname{supp}(\Pi_U) \subset \{-1 \pm \mathrm{e}^{rz},\ z \in \mathbb{Z}\}$. *Then* $\lim_{t\to\infty}\mathcal{E}(U)_t = 0$ *a.s. and there exist a strictly stationary solution* $(V_t)_{t\geq 0}$ *of (1.2) and constants* $C_+, C_- \geq 0$ *such that*

$$\lim_{x\to\infty} x^{\kappa}\,\mathrm{P}(V_0 > x) = C_+ \quad \textit{and} \quad \lim_{x\to\infty} x^{\kappa}\,\mathrm{P}(V_0 < -x) = C_-. \tag{4.2}$$

If $(V_t)_{t\geq 0}$ *is not constant, it holds that* $C_+ + C_- > 0$, *and in the case in which* $\Pi_U((-\infty, -1)) > 0$ *we obtain* $C_+ = C_-$.

In the analogue statement for the non-causal stationary solution corresponding to Theorem 2.1(iii) we have to ensure that $\lambda = 0$ holds, since otherwise such a solution does not exist. Apart from that the result is similar to the one before and can be stated as follows.

Theorem 4.2. *Let* $(U_t, L_t)_{t\geq 0}$ *be a bivariate Lévy process with* $\Pi_U(\{-1\}) = 0$, *and suppose that there exists* $\kappa > 0$ *such that*

$$\mathrm{E}\,|W_1|^{\max\{1,\kappa\}} < \infty, \qquad \mathrm{E}\,|\eta_1|^{\max\{1,\kappa\}} < \infty, \quad \textit{and} \quad \mathrm{E}\,|\mathcal{E}(W)_1|^{\kappa} = 1.$$

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If U is of finite variation, additionally assume that the drift of U is nonzero or that there is no $r > 0$ such that $\operatorname{supp}(\Pi_U) \subset \{-1 \pm e^{rz}, z \in \mathbb{Z}\}$. *Then* $\lim_{t\to\infty} \mathcal{E}(U)_t^{-1} = 0$ *a.s. and there exist a strictly stationary solution $(V_t)_{t\geq 0}$ of (1.2) and constants $C_+, C_- \geq 0$ such that*

$$\lim_{x\to\infty} x^\kappa \, \mathrm{P}(V_0 > x) = C_+ \quad \textit{and} \quad \lim_{x\to\infty} x^\kappa \, \mathrm{P}(V_0 < -x) = C_-.$$

If $(V_t)_{t\geq 0}$ is not constant, it holds that $C_+ + C_- > 0$, and in the case in which $\Pi_U((-\infty, -1)) > 0$ we obtain $C_+ = C_-$.

Since in the following we want to apply the results on tails of perpetuities given by Goldie and Grübel [6], we first reveal that the process V as defined in (1.2) can be interpreted as a perpetuity. This formulation will then also be used in Section 5.

In fact, it is known that the fixed-point random equation

$$X \overset{\mathrm{D}}{=} AX' + B, \tag{4.3}$$

where X and X' are equally distributed random variables and X' is independent of the random vector (A, B), is related to the almost-sure absolute convergence of the perpetuity

$$X_\infty := \sum_{k=0}^{\infty} \bigg(\prod_{i=0}^{k-1} A_i \bigg) B_k, \tag{4.4}$$

where $(A_k, B_k)_{k\in\mathbb{N}_0}$ is an independent and identically distributed (i.i.d.) sequence with the same distribution as (A, B).

Proposition 4.1 below is shown in greater detail in [7, Theorems 2.1 and 3.1]. One direction of Proposition 4.1(ii) is already due to Vervaat [18].

Proposition 4.1. (i) *Suppose that* $\mathrm{P}(A = 0) > 0$. *Then the sum in (4.4) converges a.s. to X_∞ and (4.3) has a unique solution which is given by $\mathcal{L}(X_\infty)$.*

(ii) *Suppose that* $\mathrm{P}(A = 0) = 0$ *and* $\mathrm{P}(Ac + B = c) < 1$ *for all $c \in \mathbb{R}$. Then (4.3) has a solution if and only if the sum in (4.4) converges a.s. absolutely in which case $\mathcal{L}(X_\infty)$ is the unique solution of the random fixed-point equation (4.3).*

From (2.1) we know that the stationary solutions $(V_t)_{t\geq 0}$ of the SDE (1.1) satisfy the distributional fixed-point equation

$$V_0 \overset{\mathrm{D}}{=} V_t = A_t V_0 + B_t \tag{4.5}$$

for any $t \geq 0$, where

$$A_t = \mathcal{E}(\tilde{U})_t \, \mathbf{1}_{\{K(t)=0\}}, \tag{4.6}$$

$$\begin{aligned} B_t = {} & \mathcal{E}(\tilde{U})_t \int_{(0,t]} [\mathcal{E}(\tilde{U})_{s-}]^{-1} \, \mathrm{d}\tilde{\eta}_s \, \mathbf{1}_{\{K(t)=0\}} \\ & + \mathcal{E}(\tilde{U})_{(T(t),t]} \bigg(\Delta L_{T(t)} + \int_{(T(t),t]} [\mathcal{E}(\tilde{U})_{(T(t),s)}]^{-1} \, \mathrm{d}\tilde{\eta}_s \bigg) \mathbf{1}_{\{K(t)>0\}}, \end{aligned} \tag{4.7}$$

which are independent of V_0, if the solution is causal as in Theorem 2.1(ii). In the case of strictly non-causal solutions, as in Theorem 2.1(iii), we may rearrange (4.5) to obtain

$$V_t \overset{\mathrm{D}}{=} V_0 = A_t^{-1} V_t - A_t^{-1} B_t \tag{4.8}$$

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with $(A_t^{-1}, -A_t^{-1}B_t) = (\mathcal{E}(W)_t, -\int_{(0,t]} \mathcal{E}(W)_{s-}\,\mathrm{d}\eta_s)$ independent of V_t. Hence, in our applications the case $\mathrm{P}(A=0)=0$ coincides with the case $\lambda = \Pi_U(\{-1\}) = 0$, while $\mathrm{P}(A=0)>0$ holds if and only if $\lambda > 0$ and, therefore, only occurs in the causal case. If there exists no $k \in \mathbb{R}$ such that $kU = -L$, the resulting process is nondegenerate by Lemma 3.1. Hence, convergence of the perpetuity is given in both cases under the conditions given in Theorem 2.1, since then a nondegenerate stationary solution exists, as has been shown in [2].

Now that we can interpret our stationary solutions as perpetuities, we can apply the results on the tail behavior of perpetuities in [6]. We start with the following proposition, which is a direct consequence of [6, Theorem 4.1]. Note that we do not need any assumptions on the process L here.

Proposition 4.2. *Let $(U_t, L_t)_{t\ge 0}$ be a bivariate Lévy process, and let $(V_t)_{t\ge0}$ be a nonconstant strictly stationary solution of (1.2).*

(i) *Assume that $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$. If U is of finite variation, suppose that it has strictly positive drift or that $\Pi_U(\mathbb{R}\setminus[-2,0]) > 0$. Then the law of V_0 has at least a power-law tail, i.e.*

$$\liminf_{x\to\infty} \frac{\log(\mathrm{P}(|V_0| \ge x))}{\log x} > -\infty. \tag{4.9}$$

(ii) *Assume that $\lambda = \Pi_U(\{-1\}) = 0$ and that $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$. If U is of finite variation, suppose that it has strictly negative drift or that $\Pi_U([-2,0]) > 0$. Then (4.9) holds.*

The conditions on U formulated in Proposition 4.2(i) ensure that we have $\mathrm{P}(|\mathcal{E}(\tilde{U})_t| > 1) > 0$ for all $t \ge 0$. If, by contrast, $|\mathcal{E}(\tilde{U})_t|$ is bounded by 1 and not constant, then the tails of V_0 decrease at least exponentially fast under some additional condition on L as formulated in the following theorem.

Theorem 4.3. *Let $(U_t, L_t)_{t\ge0}$ be a bivariate Lévy process, and let $(V_t)_{t\ge0}$ be a strictly stationary, nonconstant solution of (1.2).*

(i) *Assume that $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$. Suppose that U is of finite variation and has nonpositive drift, and that $\Pi_U(\mathbb{R}\setminus[-2,0]) = 0$. Assume that either the drift is nonzero or that $\Pi_U(\mathbb{R}\setminus\{-1\}) > 0$. Then, given that there exists $\kappa > 0$ such that $\mathrm{E}\,\mathrm{e}^{\kappa|L_1|} < \infty$, the tails of $\mathcal{L}(V_0)$ decrease at least exponentially fast, i.e.*

$$\limsup_{x\to\infty} x^{-1}\log(\mathrm{P}(|V_0| \ge x)) < 0. \tag{4.10}$$

(ii) *Assume that $\lambda = \Pi_U(\{-1\}) = 0$ and that $\lim_{t\to\infty}[\mathcal{E}(U)_t]^{-1} = 0$. Suppose that U is of finite variation and has nonnegative drift, and that $\Pi_U([-2,0]) = 0$. Assume that either the drift is nonzero or that $\Pi_U(\mathbb{R}\setminus\{-1\}) > 0$. Then, given that there exists $\kappa > 0$ such that $\mathrm{E}\,\mathrm{e}^{\kappa|\eta_1|} < \infty$, (4.10) holds.*

5. Absolute continuity

In this section we determine necessary and sufficient conditions for the stationary solutions of (1.2) in the case that $\lambda = \Pi_U(-1) > 0$ is (absolutely) continuous. By the above exposition, this corresponds to studying the law of the perpetuity (4.4). In the case that $\mathrm{P}(A=0)=0$ this problem was first treated by Grincevičius [8], [9] and later on by Alsmeyer *et al.* [1]. An application of Grincevičius' results to generalized Ornstein–Uhlenbeck processes was given

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in [3]. Here we concentrate on the case $\mathrm{P}(A = 0) > 0$, and give necessary and sufficient conditions for the law of a perpetuity to be (absolutely) continuous as follows.

Theorem 5.1. *Let (A, B) be a pair of real-valued random variables with $\mathrm{P}(A = 0) > 0$, and let X_∞ be the unique solution of the fixed-point random equation (4.3).*

(i) *The distribution of X_∞ is continuous if and only if the conditional distribution of B given $A = 0$ is continuous.*

(ii) *The distribution of X_∞ is absolutely continuous if and only if the conditional distribution of B given $A = 0$ is absolutely continuous.*

If we apply Theorem 5.1 to the stationary solutions of (1.2), using (4.6), and (4.7), we find that the distribution of V_0 is (absolutely) continuous if and only if the distribution of

$$R_t := \mathcal{E}(\tilde{U})_{(T(t),t]}\Delta L_{T(t)} + \mathcal{E}(\tilde{U})_t \int_{(T(t),t]} [\mathcal{E}(\tilde{U})_{s-}]^{-1}\,\mathrm{d}\tilde{\eta}_s$$

given $K(t) > 0$ is (absolutely) continuous, and, by the proof of Theorem 2.2 of [2], it holds that, for all $B \in \mathcal{B}_1$,

$$\mathrm{P}(R_t \in B \mid K(t) > 0) = \lambda \int_{(0,t]} \mathrm{P}(Z_y \in B)\mathrm{e}^{-\lambda y}\,\mathrm{d}y,$$

with Z_t as in (2.4). Hence, we can formulate the following corollary.

Corollary 5.1. *Suppose that $(V_t)_{t\geq 0}$ is a strictly stationary solution of (1.2) with $\lambda := \Pi_U(\{-1\}) > 0$.*

(i) *$\mathcal{L}(V_0)$ is continuous if and only if*

$$\int_{(0,1]} \mathrm{P}(Z_y = a)\mathrm{e}^{-\lambda y}\,\mathrm{d}y = 0 \quad \textit{for all } a \in \mathbb{R}.$$

(ii) *$\mathcal{L}(V_0)$ is absolutely continuous if and only if*

$$\int_{(0,1]} \mathrm{P}(Z_y \in B)\mathrm{e}^{-\lambda y}\,\mathrm{d}y = 0 \quad \textit{for all } B \in \mathcal{B}_1 \textit{ with Lebesgue measure } 0.$$

In particular, we can conclude that if $\mathcal{L}(Z_t)$ is (absolutely) continuous for Lebesgue-almost every $t > 0$, then so is $\mathcal{L}(V_0)$. In the following we will discuss some examples for the behavior of the distributions of Z_t and, hence, of $\mathcal{L}(V_0)$.

Example 5.1. Suppose that the processes U and L are independent. Then, by (2.4), it holds almost surely that $Z_t = \mathcal{E}(\tilde{U})_t \int_{(0,t]}[\mathcal{E}(\tilde{U})_{s-}]^{-1}\,\mathrm{d}\tilde{\eta}_s$, $t \geq 0$, so that, by Lemma 3.1 of [2],

$$Z_t \overset{\mathrm{D}}{=} \int_{(0,t]} \mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s,$$

with the process $(\tilde{L}_t)_{t\geq 0}$ defined by

$$\tilde{L}_t = L_t - \sum_{\substack{0<s\leq t\\ \Delta U_s=-1}} \Delta L_s, \qquad t \geq 0, \tag{5.1}$$

which in this setting is almost surely equal to $(L_t)_{t\geq 0}$.

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Assume additionally that L_t is a standard Brownian motion. Then, for all $t > 0$, the conditional distribution of Z_t given $(\mathcal{E}(\tilde{U})_s)_{0\le s<t}$ is normally distributed with mean 0 and variance $\int_{(0,t]} |\mathcal{E}(\tilde{U})_{s-}|^2 \,\mathrm{d}s > 0$ a.s. Hence, $\mathrm{P}(Z_t \in B \mid (\mathcal{E}(\tilde{U})_s)_{0\le s<t}) = 0$ for all $B \in \mathcal{B}_1$ with Lebesgue measure 0 and it follows that $\mathrm{P}(Z_t \in B) = \mathrm{E}[\mathrm{P}(Z_t \in B \mid (\mathcal{E}(\tilde{U})_s)_{0\le s<t})] = 0$ for all $B \in \mathcal{B}_1$ with Lebesgue measure 0. Hence, $\mathcal{L}(V_0)$ is absolutely continuous.

Example 5.2. Suppose that U and L are independent, and let L_t be a compound Poisson process. Then $\mathcal{L}(V_0)$ has an atom, since $\mathcal{L}(Z_t)$ has an atom at $a = 0$ for $t \ge 0$.

If additionally the jump distribution of L is continuous then the distribution of Z_t given $L_t \ne 0$ is also continuous, so that $\mathrm{P}(Z_t = a) = 0$ for all $t \ge 0$ and $a \ne 0$. Thus, $\mathcal{L}(V_0)$ has a continuous part and an atom at 0; hence, it is not a pure-type measure.

Example 5.3. Suppose that the distribution of Y is (absolutely) continuous. Then $\mathcal{L}(V_0)$ is (absolutely) continuous too.

Indeed, from (2.4) for $B \in \mathcal{B}_1$ with Lebesgue measure 0 in the absolutely continuous case or for a single point set $B = \{b\}$ in the continuous case, we obtain

$$\begin{aligned}\mathrm{P}(Z_t \in B) &= \mathrm{P}\biggl(Y \in [\mathcal{E}(\tilde{U})_t]^{-1}B - \int_{(0,t]} [\mathcal{E}(\tilde{U})_{s-}]^{-1}\,\mathrm{d}\tilde{\eta}_s\biggr)\\ &= \int_{\mathbb{R}^2} \mathrm{P}\biggl(Y \in xB - y \biggm| [\mathcal{E}(\tilde{U})_t]^{-1} = x,\ \int_{(0,t]} [\mathcal{E}(\tilde{U})_{s-}]^{-1}\,\mathrm{d}\tilde{\eta}_s = y\biggr)\nu(\mathrm{d}x,\,\mathrm{d}y)\\ &= \int_{\mathbb{R}^2} 0\nu(\mathrm{d}x,\,\mathrm{d}y)\\ &= 0,\end{aligned}$$

where ν is the distribution of $([\mathcal{E}(\tilde{U})_t]^{-1}, \int_{(0,t]}[\mathcal{E}(\tilde{U})_{s-}]^{-1}\,\mathrm{d}\tilde{\eta}_s)$. It follows that Z_t is (absolutely) continuous for all $t > 0$. Hence, $\mathcal{L}(V_0)$ is (absolutely) continuous by Corollary 5.1.

6. Proofs

For the proofs of the preceding results, we need to define some auxiliary Lévy processes. In the case that $\lambda = \Pi_U(\{-1\}) = 0$ we will often make use of the formulation

$$\mathcal{E}(U)_t = (-1)^{N_t}\mathrm{e}^{-\hat{U}_t}, \tag{6.1}$$

where the processes $N = (N_t)_{t\ge0}$ and $\hat{U} = (\hat{U}_t)_{t\ge0}$ are defined by

$$\begin{aligned}N_t &:= \text{number of jumps of size less than } -1 \text{ of } U \text{ in } [0,t],\\ \hat{U}_t &:= -U_t + \frac{\sigma_U^2 t}{2} + \sum_{0<s\le t} [\Delta U_s - \log|1 + \Delta U_s|].\end{aligned}$$

See [2] for details on $N = (N_t)_{t\ge0}$ and $\hat{U} = (\hat{U}_t)_{t\ge0}$.

On the other hand, if $\lambda = \Pi_U(\{-1\}) > 0$, we will use the processes $\tilde{U}$, $\tilde{\eta}$, $\tilde{L}$, and $\tilde{W}$ defined by (2.3), (5.1), and

$$\tilde{W}_t = W_t - \sum_{\substack{0<s\le t\\ \Delta U_s=-1}} \Delta W_s, \qquad t \ge 0. \tag{6.2}$$

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6.1. Proofs for Section 3

For the following calculations, we need a short lemma on stochastic integrals with respect to Lévy processes which can be deduced from [4, Proposition 4.6.16]. Since the proof in that reference is not carried out completely, we present an alternative proof here. Note that the following lemma allows us to circumvent Hölder's inequality in the proof of Theorem 3.1 so that we get slightly sharper results than the corresponding results obtained in [12] for generalized Ornstein–Uhlenbeck processes.

Lemma 6.1. *Let $(L_s)_{s\geq 0}$ be a Lévy process, and let $(H_s)_{s\geq 0}$ be an adapted, càdlàg process. Suppose that there exists $\kappa \geq 1$ such that $\mathrm{E}\,|L_1|^\kappa < \infty$ and $\mathrm{E}\sup_{0<s\leq 1}|H_s|^\kappa < \infty$. Then*

$$\mathrm{E}\sup_{0<t\leq 1}\left|\int_{(0,t]} H_{s-}\,\mathrm{d}L_s\right|^\kappa < \infty. \tag{6.3}$$

In particular, if $\mathrm{E}\,|L_1| < \infty$ and $\mathrm{E}\sup_{0<s\leq t}|H_s| < \infty$ for $t > 0$, it holds that

$$\mathrm{E}\int_{(0,t]} H_{s-}\,\mathrm{d}L_s = \mathrm{E}[L_1]\int_{(0,t]} \mathrm{E}\,H_{s-}\,\mathrm{d}s. \tag{6.4}$$

Proof. Define the processes L^+ and L^- such that $L = L^+ + L^-$ and $\mathrm{E}\,L_1 = \mathrm{E}\,L_1^+$ with L^- having only jumps of size in $(-\frac{1}{2},\frac{1}{2})$ and $L_s^+ = \sum_{i=1}^{N_s} Y_i + \gamma s$ being a compound Poisson process with parameter a, jump times T_i, $i = 1, 2, \ldots$, and jump heights Y_i, $i = 1, 2, \ldots$, such that $|Y_i| \geq \frac{1}{2}$ for all $i \in \mathbb{N}$. Then we can derive, by a standard calculation using Minkowski's inequality,

$$\begin{aligned}
&\left(\mathrm{E}\sup_{0<t\leq 1}\left|\int_{(0,t]} H_{s-}\,\mathrm{d}L_s^+\right|^\kappa\right)^{1/\kappa} \\
&\quad\leq \left(\mathrm{E}\left(\sum_{i=1}^{N_1}|H_{T_i-}||Y_i|\right)^\kappa\right)^{1/\kappa} + \left(\mathrm{E}\left(|\gamma|\sup_{0<t\leq 1}\left|\int_{(0,t]} H_{s-}\,\mathrm{d}s\right|\right)^\kappa\right)^{1/\kappa} \\
&\quad\leq \left(\sum_{j=0}^{\infty}\mathrm{P}(N_1 = j)\,j\,\mathrm{E}\,|Y_1|^\kappa\,\mathrm{E}\sup_{0<s\leq 1}|H_s|^\kappa\right)^{1/\kappa} + |\gamma|\left(\mathrm{E}\sup_{0<s\leq 1}|H_s|^\kappa\right)^{1/\kappa} \\
&\quad= \left(\mathrm{E}[N_1]\,\mathrm{E}\,|Y_1|^\kappa\,\mathrm{E}\sup_{0<s\leq 1}|H_s|^\kappa\right)^{1/\kappa} + |\gamma|\left(\mathrm{E}\sup_{0<s\leq 1}|H_s|^\kappa\right)^{1/\kappa} \\
&\quad< \infty.
\end{aligned}$$

On the other hand, using the notation of [15, Theorem V.2] for some constant c_1,

$$\mathrm{E}\sup_{0<t\leq 1}\left|\int_{(0,t]} H_{s-}\,\mathrm{d}L_s^-\right|^\kappa = \left\|\int_{(0,\cdot]}\mathbf{1}_{(0,1]}(s)H_{s-}\,\mathrm{d}L_s^-\right\|_{\underline{\underline{S}}^\kappa}^\kappa \leq c_1\left\|\int_{(0,\cdot]}\mathbf{1}_{(0,1]}(s)H_{s-}\,\mathrm{d}L_s^-\right\|_{\underline{\underline{H}}^\kappa}^\kappa$$

and by Equation (14) of [14] we know that, for some constant c_2,

$$\left\|\int_{(0,\cdot]}\mathbf{1}_{(0,1]}(s)H_{s-}\,\mathrm{d}L_s^-\right\|_{\underline{\underline{H}}^\kappa}^\kappa \leq c_2\,\mathrm{E}\sup_{0<s\leq 1}|H_s|^\kappa\,\|(L_{s\wedge 1}^-)_{s\geq 0}\|_{\mathrm{BMO}}^\kappa,$$

where $s\wedge 1 := \min\{s, 1\}$ and $\|\cdot\|_{\mathrm{BMO}}$ denotes the BMO norm as defined, e.g. in [15, p. 197]. Since L^- is a zero-mean Lévy process with bounded jumps, $(L_{s\wedge 1}^-)_{s\geq 0}$ is a BMO process.

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Hence, in the above inequality, the right-hand side is finite and so is $\mathrm{E}\sup_{0<t\le 1}|\int_{(0,t]} H_{s-}\,\mathrm{d}L_s^-|^\kappa$, which completes the proof of (6.3).

For the second assertion, note that since $L_s - s\,\mathrm{E}\,L_1$ is a local martingale, the same holds for $M_t := \int_{(0,t]} H_{s-}\,\mathrm{d}(L_s - s\,\mathrm{E}[L_1])$. By calculations similar to the above, it holds that $\mathrm{E}\sup_{0<s\le t}|M_t| < \infty$ and, hence, by [15, Theorem I.51], M_t is a martingale. Thus, we have $\mathrm{E}\int_{(0,t]} H_{s-}\,\mathrm{d}L_s = \mathrm{E}[L_1]\,\mathrm{E}\int_{(0,t]} H_{s-}\,\mathrm{d}s$, and using Fubini's theorem, the second assertion follows.

Proof of Proposition 3.1. (i) First note that $|\mathcal{E}(U)_t|^\kappa = |\mathcal{E}(\tilde{U})_t|^\kappa\,\mathbf{1}_{\{K(t)=0\}}$ is integrable if and only if $|\mathcal{E}(\tilde{U})_t|^\kappa$ is. Thus, it is sufficient to show the integrability condition under the assumption that $\Pi_U(\{-1\}) = 0$.

Because of (6.1) we have $\mathrm{E}[|\mathcal{E}(U)_t|^\kappa] = \mathrm{E}[\mathrm{e}^{-\kappa\hat{U}_t}]$ and, hence, by [17, Theorem 25.17], it holds that $\mathrm{E}[|\mathcal{E}(U)_t|^\kappa] < \infty$ if and only if $\int_{|x|>1}\mathrm{e}^{-\kappa x}\,\Pi_{\hat{U}}(\mathrm{d}x) < \infty$. Using $\Pi_{\hat{U}} = X(\Pi_U)$ for the transformation

$$X\colon \mathbb{R}\setminus\{-1\}\to\mathbb{R}, \qquad x\mapsto X(x) = -\log|1+x|, \tag{6.5}$$

as introduced in [2, Lemma 3.4], this is equivalent to

$$\int_{\mathbb{R}\setminus([-1-\mathrm{e},-1-\mathrm{e}^{-1}]\cup[-1+\mathrm{e}^{-1},-1+\mathrm{e}])} |1+x|^\kappa\,\Pi_U(\mathrm{d}x) < \infty,$$

which is fulfilled if and only if $\int_{|x|>1}|x|^\kappa\,\Pi_U(\mathrm{d}x) < \infty$, viz. $|U_1|^\kappa$ is integrable.

To compute $\mathrm{E}[\mathcal{E}(U)_t]$ for U with $\Pi_U(\{-1\}) \ge 0$, recall that the Doléans-Dade exponential fulfills the integral equation

$$\mathcal{E}(U)_t = 1 + \int_0^t \mathcal{E}(U)_{s-}\,\mathrm{d}U_s. \tag{6.6}$$

Under the given assumptions, we have $\mathrm{E}\sup_{0<s\le t}|\mathcal{E}(U)_s| < \infty$ by [17, Theorem 25.18] and, hence, using Lemma 6.1, it holds that

$$\mathrm{E}[\mathcal{E}(U)_t] = 1 + \mathrm{E}[U_1]\int_0^t \mathrm{E}[\mathcal{E}(U)_s]\,\mathrm{d}s.$$

Thus, by differentiation, $\mathrm{d}\mathrm{E}[\mathcal{E}(U)_t]/\mathrm{d}t = \mathrm{E}[U_1]\,\mathrm{E}[\mathcal{E}(U)_t]$ so that $\mathrm{E}[\mathcal{E}(U)_t] = c\mathrm{e}^{\mathrm{E}[U_1]t}$ for some constant $c \ne 0$. But since $\mathcal{E}(U)_0 = 1$ a.s., we easily see that $c = 1$, which gives (3.2).

To show (3.3) using integration by parts and [15, Theorems II.19 and II.29], we obtain, from (6.6),

$$\begin{aligned}
\mathrm{E}[(\mathcal{E}(U)_t)^2] &= 1 + 2\,\mathrm{E}\bigg[\int_0^t \mathcal{E}(U)_{s-}\,\mathrm{d}U_s\bigg] + \mathrm{E}\bigg[\bigg[\int_0^\bullet \mathcal{E}(U)_{s-}\,\mathrm{d}U_s, \int_0^\bullet \mathcal{E}(U)_{s-}\,\mathrm{d}U_s\bigg]_t\bigg] \\
&\quad + 2\,\mathrm{E}\bigg[\int_0^t\bigg(\int_0^s \mathcal{E}(U)_{u-}\,\mathrm{d}U_u\bigg)\mathrm{d}\bigg(\int_0^s \mathcal{E}(U)_{u-}\,\mathrm{d}U_u\bigg)\bigg] \\
&= 1 + 2(\mathrm{E}[\mathcal{E}(U)_t] - 1) + \mathrm{E}\bigg[\int_0^t (\mathcal{E}(U)_{s-})^2\,\mathrm{d}[U,U]_s\bigg] \\
&\quad + 2\,\mathrm{E}\bigg[\int_0^t (\mathcal{E}(U)_{s-} - 1)\mathcal{E}(U)_{s-}\,\mathrm{d}U_s\bigg],
\end{aligned}$$

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which, by a standard calculation using (3.2), (6.3), and (6.4), leads to

$$\frac{\mathrm{d}(\mathrm{E}[(\mathcal{E}(U)_t)^2])}{\mathrm{d}t} = (\mathrm{E}[[U, U]_1] + 2\,\mathrm{E}[U_1])\,\mathrm{E}[(\mathcal{E}(U)_t)^2],$$

so that

$$\mathrm{E}[(\mathcal{E}(U)_t)^2] = \mathrm{e}^{t(\mathrm{E}[[U,U]_1]+2\,\mathrm{E}[U_1])}$$

since $\mathcal{E}(U)_0 = 1$ a.s. Finally, note that $\mathrm{E}[[U, U]_1] = \mathrm{E}[U_1^2] - 2\,\mathrm{E}[U_1]\int_0^1 s\,\mathrm{E}[U_1]\,\mathrm{d}s = \mathrm{var}(U_1)$, which gives

$$\mathrm{E}[(\mathcal{E}(U)_t)^2] = \mathrm{e}^{t(2\,\mathrm{E}[U_1]+\mathrm{var}(U_1))}, \tag{6.7}$$

and, hence, (3.3).

(ii) Recall that $\mathrm{E}\,|\mathcal{E}(U)_t|^{-\kappa} < \infty$ if and only if $\int_{|x|>1} \mathrm{e}^{-\kappa x}\,\Pi_{\hat{W}}(\mathrm{d}x) < \infty$, where $\hat{W}$ is the process corresponding to W via (6.1). Since $\Pi_{\hat{W}} = X(\Pi_W) = X(Y(\Pi_U))$ with

$$Y\colon \mathbb{R}\setminus\{-1\} \to \mathbb{R}\setminus\{-1\}, \qquad x \mapsto Y(x) = \frac{-x}{1+x}, \tag{6.8}$$

as defined in [2, Lemma 3.4], this is equivalent to

$$\int_{\mathbb{R}\setminus([-1-\mathrm{e},-1-\mathrm{e}^{-1}]\cup[-1+\mathrm{e}^{-1},-1+\mathrm{e}])} |1+x|^{-\kappa}\,\Pi_U(\mathrm{d}x) < \infty,$$

and, hence, to (3.4). Equations (3.2) and (3.3) can then be shown by similar calculations as above while the formula for $\mathrm{E}[W_1]$ is given in [2, Lemma 3.4]. The variance of W_1 is given by $\mathrm{var}(W_1) = \sigma_W^2 + \int_{\mathbb{R}} x^2\,\Pi_W(\mathrm{d}x)$ (see [17, Example 25.12]). Using the transformation Y in (6.8), this directly leads to the given formula.

Proof of Lemma 3.1. Suppose that $V_t = k$ a.s. By (1.1) we know that $V_t = V_0 + \int_{(0,t]} V_{s-}\,\mathrm{d}U_s + L_t$, which gives $k = k + kU_t + L_t$ and, hence, the desired result.

For the converse, note that $kU_t = -L_t$, $t \geq 0$, implies that $\tilde{\eta}_t = k\tilde{W}_t$ by (1.3), (2.3), (3.1), and (6.2), and also $\Delta L_{T(t)} = k$ for all $t \geq 0$ so that we obtain, from (1.2),

$$\begin{aligned} V_t = {} & \mathcal{E}(U)_t\left(k + k\int_{(0,t]} \mathcal{E}(\tilde{W})_{s-}\,\mathrm{d}\tilde{W}_s\right)\mathbf{1}_{\{K(t)=0\}} \\ & + \mathcal{E}(U)_{(T(t),t]}\left(k + k\int_{(T(t),t]} \mathcal{E}(\tilde{W})_{(T(t),s)}\,\mathrm{d}\tilde{W}_s\right)\mathbf{1}_{\{K(t)\geq 1\}}. \end{aligned}$$

From (6.6), it follows that the Doléans-Dade exponential fulfills the integral equation $\mathcal{E}(\tilde{W})_{(T(t),t]} = 1 + \int_{(T(t),t]} \mathcal{E}(\tilde{W})_{(T(t),s)}\,\mathrm{d}\tilde{W}_s$ for all $t > 0$ with $K(t) > 0$, and this together with (6.6) directly gives $V_t = k$ a.s.

Proof of Theorem 3.1. (i) Using Proposition 3.1(i), it follows from (3.5) for $k := \max\{1, \kappa\}$ that we have $\mathrm{E}\,|\mathcal{E}(\tilde{U})_s|^k < \infty$. By [17, Theorem 25.18], this is equivalent to $\mathrm{E}\sup_{0<s\leq 1}|\mathcal{E}(\tilde{U})_s|^k = \mathrm{E}\sup_{0<s\leq 1} \mathrm{e}^{-k\hat{\tilde{U}}_s} < \infty$ and, hence, from Lemma 6.1, it follows that

$$\mathrm{E}\sup_{0<t\leq 1}\left|\int_{(0,t]} \mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\right|^k < \infty. \tag{6.9}$$

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Set $\alpha := \lfloor \kappa \rfloor$, the integer part of κ. Then it can be shown exactly as in the proof of [12, Proposition 4.1] that, for any $m, n \in \mathbb{N}_0$, $m < n$,

$$\begin{aligned} \mathrm{E}\bigg|\int_{(m,n]} \mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\bigg|^{\kappa} \le \mathrm{E}\bigg|\int_{(0,1]} \mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\bigg|^{\kappa} \bigg(\sum_{j=m}^{n-1} \exp\bigg(\frac{j}{\kappa}\psi_{\hat{\hat{U}}}(\kappa)\bigg)\bigg)^{\alpha} \\ \times \bigg(\sum_{j=m}^{n-1} \exp\bigg(\frac{j(\kappa-\alpha)}{\kappa}\psi_{\hat{\hat{U}}}(\kappa)\bigg)\bigg), \end{aligned} \tag{6.10}$$

which is Equation (8.4) of [12], where the last factor can be omitted if $\kappa = \alpha$.

(a) Assume first that $\lambda = \Pi_U(\{-1\}) = 0$, i.e. by Theorem 2.1(ii), it holds that $V_0 \stackrel{\mathrm{D}}{=} \int_{(0,\infty)} \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$. Since we have $\mathrm{E}\,|\mathcal{E}(U)_1|^{\kappa} < 1$, which is equivalent to $\psi_{\hat{U}}(\kappa) < 0$, the sums in (6.10) converge absolutely when $n \to \infty$ so that, with (6.9), it follows that $\int_{(0,t]} \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ is a Cauchy sequence in L^{κ} and, thus, converges in L^{κ} to $\int_{(0,\infty)} \mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ as $t \to \infty$ so that we have $\mathrm{E}\,|V_0|^{\kappa} < \infty$.

For the expectation, we obtain, using (6.4) and (3.2),

$$\mathrm{E}[V_0] = \mathrm{E}\bigg[\int_{(0,\infty)} \mathcal{E}(U)_{s-}\,\mathrm{d}L_s\bigg] = \mathrm{E}[L_1]\int_{(0,\infty)} \mathrm{e}^{\mathrm{E}[U_1]s}\,\mathrm{d}s = -\frac{\mathrm{E}[L_1]}{\mathrm{E}[U_1]}.$$

To compute the variance, using integration by parts and Lemma 6.1, we obtain

$$\begin{aligned} \mathrm{E}[V_0^2] &= \mathrm{E}\bigg[\bigg[\int_0^{\bullet} \mathcal{E}(U)_{s-}\,\mathrm{d}L_s, \int_0^{\bullet} \mathcal{E}(U)_{s-}\,\mathrm{d}L_s\bigg]_{\infty}\bigg] \\ &\quad + 2\,\mathrm{E}\bigg[\int_0^{\infty}\bigg(\int_0^t \mathcal{E}(U)_{s-}\,\mathrm{d}L_s\bigg)\,\mathrm{d}\bigg(\int_0^t \mathcal{E}(U)_{s-}\,\mathrm{d}L_s\bigg)\bigg] \\ &= \mathrm{E}[[L,L]_1]\int_0^{\infty} \mathrm{E}[(\mathcal{E}(U)_{s-})^2]\,\mathrm{d}s \\ &\quad + 2\,\mathrm{E}[L_1]\int_0^{\infty} \mathrm{E}\bigg[\mathcal{E}(U)_{t-}\bigg(\int_0^t \mathcal{E}(U)_{s-}\,\mathrm{d}L_s\bigg)\bigg]\,\mathrm{d}t. \end{aligned}$$

By (6.7), it holds that

$$\int_0^{\infty} \mathrm{E}[(\mathcal{E}(U)_{s-})^2]\,\mathrm{d}s = -(2\,\mathrm{E}[U_1] + \mathrm{var}(U_1))^{-1},$$

which is strictly positive and finite since $\mathrm{E}[\mathcal{E}(U)_1^2] < 1$ holds by assumption. For the calculation of

$$\int_0^{\infty} \mathrm{E}\bigg[\mathcal{E}(U)_{t-}\bigg(\int_0^t \mathcal{E}(U)_{s-}\,\mathrm{d}L_s\bigg)\bigg]\,\mathrm{d}t =: \int_0^{\infty} X_t\,\mathrm{d}t,$$

again, by integration by parts, the use of Lemma 6.1, (3.2), (6.6), and (6.7), we obtain

$$X_t = \frac{\mathrm{E}[[U,L]_1] + \mathrm{E}[L_1]}{2\,\mathrm{E}[U_1] + \mathrm{var}(U_1)}(\mathrm{e}^{t(2\,\mathrm{E}[U_1]+\mathrm{var}(U_1))} - 1) + \mathrm{E}[U_1]\int_0^t X_s\,\mathrm{d}s.$$

Solving this integral equation leads to

$$\mathrm{E}[V_0^2] = \frac{1}{2\,\mathrm{E}[U_1] + \mathrm{var}(U_1)}\bigg(\frac{2\,\mathrm{E}[L_1](\mathrm{cov}(U_1, L_1) + \mathrm{E}[L_1])}{\mathrm{E}[U_1]} - \mathrm{var}(L_1)\bigg), \tag{6.11}$$

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where we replaced

$$\mathrm{E}[[U,L]_1]=\mathrm{E}[U_1L_1]-\mathrm{E}[L_1]\int_0^1 s\,\mathrm{E}[U_1]\,\mathrm{d}s-\mathrm{E}[U_1]\int_0^1 s\,\mathrm{E}[L_1]\,\mathrm{d}s=\operatorname{cov}(U_1,L_1).$$

Using the results above, the variance can now be derived by standard algebra.

(b) Suppose that $\lambda=\Pi_U(\{-1\})>0$, and deduce from Theorem 2.1(ii) that

$$\mathrm{E}\,|V_0|^\kappa=\mathrm{E}\,|Z_\tau|^\kappa=\int_{(0,\infty)}\lambda\mathrm{e}^{-\lambda t}\,\mathrm{E}\,|Z_t|^\kappa\,\mathrm{d}t,$$

where $(Z_t)_{t\geq 0}$ is defined in (2.4).

Let $\lambda':=\psi_{\hat{\tilde{U}}}(\kappa)=\log\mathrm{E}\,|\mathcal{E}(\tilde{U})_1|^\kappa$. Then we have $\lambda'<\lambda$ by assumption. Choose λ'' such that $\lambda'\leq\lambda''<\lambda$. First observe that we have

$$\begin{aligned}\mathrm{E}\,|Z_t|^\kappa&=\mathrm{E}\left|\mathcal{E}(\tilde{U})_tY+\mathcal{E}(\tilde{U})_t\int_{(0,t]}[\mathcal{E}(\tilde{U})_{s-}]^{-1}\,\mathrm{d}\tilde{\eta}_s\right|^\kappa\\&\leq 2^\kappa\left(\mathrm{E}\,|\mathcal{E}(\tilde{U})_tY|^\kappa+\mathrm{E}\left|\int_{(0,t]}\mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\right|^\kappa\right),\end{aligned}$$

where we have used [2, Lemma 3.1].

Since $\mathrm{E}\,|L_1|^\kappa<\infty$ implies that $\mathrm{E}\,|Y|^\kappa<\infty$, and since $\tilde{U}$ and Y are independent, we conclude, using (3.2), that

$$\int_0^\infty \mathrm{e}^{-t\lambda}2^\kappa\,\mathrm{E}\,|\mathcal{E}(\tilde{U})_tY|^\kappa\,\mathrm{d}t\leq 2^\kappa\int_0^\infty \mathrm{e}^{-t\lambda}\mathrm{e}^{t\lambda''}\,\mathrm{E}\,|Y|^\kappa\,\mathrm{d}t<\infty.$$

For the second term, observe that, from (6.9) and (6.10), it follows that

$$\begin{aligned}\mathrm{E}\left|\int_{(0,n]}\mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\right|^\kappa&\leq\mathrm{E}\left|\int_{(0,1]}\mathcal{E}(\tilde{U})\,\mathrm{d}\tilde{L}_s\right|^\kappa\left(\sum_{j=0}^{n-1}\exp\left(\frac{j}{\kappa}\lambda'\right)\right)^\alpha\\&\quad\times\left(\sum_{j=0}^{n-1}\exp\left(\frac{j(\kappa-\alpha)}{\kappa}\lambda'\right)\right)\\&\leq C\exp\left(\frac{n\alpha}{\kappa}\lambda''\right)\exp\left(\frac{n(\kappa-\alpha)}{\kappa}\lambda''\right)\\&\leq C\mathrm{e}^{n\lambda''}\end{aligned}$$

for any $n\in\mathbb{N}$ and a suitable constant C. Finally, for arbitrary $t>0$, we obtain

$$\begin{aligned}\mathrm{E}\left|\int_{(0,t]}\mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\right|^\kappa&=\mathrm{E}\left|\int_{(0,\lfloor t\rfloor]}\mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s+\mathcal{E}(\tilde{U})_{\lfloor t\rfloor}\int_{(\lfloor t\rfloor,t]}\mathcal{E}(\tilde{U})_{(\lfloor t\rfloor,s)}\,\mathrm{d}\tilde{L}_s\right|^\kappa\\&\leq 2^\kappa\left(C\mathrm{e}^{\lfloor t\rfloor\lambda''}+\mathrm{e}^{\lfloor t\rfloor\lambda''}\,\mathrm{E}\sup_{0<t\leq 1}\left|\int_{(0,t]}\mathcal{E}(\tilde{U})\,\mathrm{d}\tilde{L}_s\right|^\kappa\right)\\&\leq C'\mathrm{e}^{t\lambda''},\end{aligned}$$

by (6.9) for some constant C'.

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We conclude that

$$\int_0^\infty \mathrm{e}^{-t\lambda} 2^\kappa \, \mathrm{E} \left| \int_{(0,t]} \mathcal{E}(\tilde{U})_{s-} \, \mathrm{d}\tilde{L}_s \right|^\kappa \mathrm{d}t < \infty,$$

and altogether $\int_0^\infty \mathrm{e}^{-t\lambda} |Z_t|^\kappa \, \mathrm{d}t < \infty$, giving the finiteness of $\mathrm{E}\,|V_0|^\kappa$.

To compute the expectation, we use result (3.2) and carry out calculations as in part (a) to derive, under the assumption that $\mathrm{E}[\tilde{U}_1] \neq 0$,

$$\mathrm{E}[Z_t] = \mathrm{e}^{\mathrm{E}[\tilde{U}_1]t} \, \mathrm{E}\, Y + \frac{\mathrm{E}[\tilde{L}_1]}{\mathrm{E}[\tilde{U}_1]} (\mathrm{e}^{\mathrm{E}[\tilde{U}_1]t} - 1),$$

so that, by integration and using $\mathrm{E}[U_1] = \mathrm{E}[\tilde{U}_1] - \lambda$,

$$\mathrm{E}[V_0] = -\frac{\mathrm{E}[\tilde{L}_1] + \lambda \, \mathrm{E}\, Y}{\mathrm{E}[U_1]}.$$

If, on the other hand, $\mathrm{E}[\tilde{U}_1] = 0$, it follows that $\mathrm{E}[Z_t] = \mathrm{E}\, Y + t\, \mathrm{E}[\tilde{L}_1]$ and then $\mathrm{E}[V_0] = \mathrm{E}\, Y + \lambda^{-1} \, \mathrm{E}[\tilde{L}_1]$, which is a special case of the formula derived above. Hence, since $\mathrm{E}[L_1] = \mathrm{E}[\tilde{L}_1] + \lambda \, \mathrm{E}\, Y$, we have shown the formula for $\mathrm{E}[V_0]$, provided that $\mathrm{E}\,|V_0| < \infty$.

Because

$$\mathrm{E}[V_0^2] = \mathrm{E}[Z_\tau^2] = \int_{(0,\infty)} \lambda \mathrm{e}^{-\lambda t} \, \mathrm{E}[Z_t^2] \, \mathrm{d}t,$$

to prove the formula for $\operatorname{var}(V_0)$, we first have to derive $\mathrm{E}[Z_t^2]$, for which, by a long calculation starting from (2.4), we obtain

$$\begin{aligned}
\mathrm{E}[Z_t^2] = \mathrm{e}^{t(2\,\mathrm{E}[\tilde{U}_1] + \operatorname{var}(\tilde{U}_1))} \Bigg(\mathrm{E}[Y^2] &+ \frac{\operatorname{cov}(\tilde{U}_1, \tilde{L}_1) + \mathrm{E}[\tilde{L}_1]}{\mathrm{E}[\tilde{U}_1] + \operatorname{var}(\tilde{U}_1)} \bigg(2\, \mathrm{E}\, Y + \frac{2\, \mathrm{E}[\tilde{L}_1]}{2\, \mathrm{E}[\tilde{U}_1] + \operatorname{var}(\tilde{U}_1)} \bigg) \\
&+ \frac{\operatorname{var}(\tilde{L}_1)}{2\, \mathrm{E}[\tilde{U}_1] + \operatorname{var}(\tilde{U}_1)} \Bigg) \\
&- \mathrm{e}^{t\, \mathrm{E}[\tilde{U}_1]} \bigg(\frac{\operatorname{cov}(\tilde{U}_1, \tilde{L}_1) + \mathrm{E}[\tilde{L}_1]}{\mathrm{E}[\tilde{U}_1] + \operatorname{var}(\tilde{U}_1)} \bigg(2\, \mathrm{E}\, Y + \frac{2\, \mathrm{E}[\tilde{L}_1]}{\mathrm{E}[\tilde{U}]} \bigg) \bigg) \\
&+ \frac{2\, \mathrm{E}[\tilde{L}_1](\operatorname{cov}(\tilde{U}_1, \tilde{L}_1) + \mathrm{E}[\tilde{L}_1]) - \mathrm{E}[\tilde{U}_1] \operatorname{var}(\tilde{L}_1)}{\mathrm{E}[\tilde{U}_1](2\, \mathrm{E}[\tilde{U}_1] + \operatorname{var}(\tilde{U}_1))}.
\end{aligned}$$

By integration and standard algebra, this leads to (6.11) and, hence, to the given formula for $\operatorname{var}(V_0)$ where we used the following relationships (all sums are meant over the jumps of U of size -1 during the time interval $[0, 1]$):

$$\mathrm{E}[U_1^2] = \mathrm{E}[\tilde{U}_1^2] + 2\, \mathrm{E}[\tilde{U}_1] \, \mathrm{E}[\Sigma \Delta U] + \mathrm{E}[(\Sigma \Delta U)^2] = \mathrm{E}[\tilde{U}_1^2] - 2\lambda \, \mathrm{E}[\tilde{U}_1] + \lambda + \lambda^2,$$

since $\Sigma \Delta U$ is Poisson distributed, so that

$$\operatorname{var}(\tilde{U}_1) = \operatorname{var}(U_1) - \lambda.$$

On the other hand, for L and $\tilde{L}$, we have

$$\mathrm{E}[L_1^2] = \mathrm{E}[\tilde{L}_1^2] + 2\lambda \, \mathrm{E}\, Y \, \mathrm{E}[\tilde{L}_1] + \operatorname{var}(\Sigma \Delta L) + (\mathrm{E}[\Sigma \Delta L])^2,$$

and, since $\Sigma \Delta L$ is compound Poisson distributed, this gives

$$\mathrm{E}[L_1^2] = \mathrm{E}[\tilde{L}_1^2] + 2\lambda \, \mathrm{E}\, Y \, \mathrm{E}[\tilde{L}_1] + \lambda \, \mathrm{E}[Y^2] + \lambda^2 (\mathrm{E}\, Y)^2$$

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and, hence,

$$\operatorname{var}(\tilde{L}_1) = \operatorname{var}(L_1) - \lambda \operatorname{E}[Y^2],$$

while, for the covariance, we deduce that $\operatorname{cov}(\tilde{U}_1, \tilde{L}_1) = \operatorname{cov}(U_1, L_1) + \lambda \operatorname{E} Y$.

(ii) The proof of this part is the same as that of part (i) in the $\lambda = 0$ case. We leave the details to the reader.

Proof of Theorem 3.2. (i) For $s < t$, take $A_{s,t}$ and $B_{s,t}$ as defined in (2.2), and recall (2.1). Since, by Theorem 2.1, the stationary solution $(V_t)_{t\geq 0}$ is unique in law, we may and will assume that V_0 is independent of $(U_t, L_t)_{t\geq 0}$. Observe that, due to the independence, it follows from Proposition 3.1 and Theorem 3.1 that $A_{s,t}\,\mathbf{1}_{\{K(s)=K(t)\}}\,V_s$ has finite expectation. Hence, if $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration induced by $(U_t, L_t)_{t\geq 0}$, we obtain

$$\begin{aligned}\operatorname{E}[V_t \mid \mathcal{F}_s] &= \operatorname{E}[(A_{s,t}V_s + B_{s,t})\,\mathbf{1}_{\{K(s)=K(t)\}} + (A_{T(t),t}\Delta\eta_{T(t)} + B_{T(t),t})\,\mathbf{1}_{\{K(s)<K(t)\}} \mid \mathcal{F}_s] \\ &= V_s \operatorname{E}[A_{s,t}\,\mathbf{1}_{\{K(s)=K(t)\}}] \\ &\quad + \operatorname{E}[A_{T(t),t}\Delta\eta_{T(t)}\,\mathbf{1}_{\{K(s)<K(t)\}} + B_{s,t}\,\mathbf{1}_{\{K(s)=K(t)\}} + B_{T(t),t}\,\mathbf{1}_{\{K(s)<K(t)\}}].\end{aligned}$$

On the other hand, since V_s is independent of $A_{s,t}\,\mathbf{1}_{\{K(s)=K(t)\}}$, it holds that

$$\begin{aligned}\operatorname{E}[V_t] &= \operatorname{E}[V_s]\operatorname{E}[A_{s,t}\,\mathbf{1}_{\{K(s)=K(t)\}}] \\ &\quad + \operatorname{E}[A_{T(t),t}\Delta\eta_{T(t)}\,\mathbf{1}_{\{K(s)<K(t)\}} + B_{s,t}\,\mathbf{1}_{\{K(s)=K(t)\}} + B_{T(t),t}\,\mathbf{1}_{\{K(s)<K(t)\}}],\end{aligned}$$

so that

$$\operatorname{E}[V_t \mid \mathcal{F}_s] = \operatorname{E}[V_t] + \operatorname{E}[A_{s,t}\,\mathbf{1}_{\{K(s)=K(t)\}}](V_s - \operatorname{E}[V_s]).$$

Finally, since

$$\operatorname{E}[A_{s,t}\,\mathbf{1}_{\{K(s)=K(t)\}}] = \operatorname{E}[\mathcal{E}(\tilde{U})_{(s,t]}]\operatorname{E}[\mathbf{1}_{\{K(s)=K(t)\}}] = \mathrm{e}^{\operatorname{E}[\tilde{U}_1](t-s)}\mathrm{e}^{-\lambda(t-s)} = \mathrm{e}^{\operatorname{E}[U_1](t-s)},$$

we obtain

$$\operatorname{cov}(V_s, V_t) = \operatorname{E}[V_s \operatorname{E}[V_t \mid \mathcal{F}_s]] - \operatorname{E}[V_s]\operatorname{E}[V_t] = \mathrm{e}^{\operatorname{E}[U_1](t-s)} \operatorname{var} V_s,$$

as had to be shown.

(ii) By Theorem 2.1(iii), the stationary solution is non-causal and from (2.1) we obtain $V_s = A_{s,t}^{-1}V_t - A_{s,t}^{-1}B_{s,t}$ for $s < t$. Defining $\mathcal{G}_t = \sigma((U_{u+t} - U_t, L_{u+t} - L_t)_{u\geq 0})$ we can then compute $\operatorname{E}[V_s \mid \mathcal{G}_t]$ for $s < t$ and, finally, $\operatorname{cov}(V_s, V_t)$ similarly as in (i).

6.2. Proofs for Section 4

Proof of Theorem 4.1. In the case that $\lambda = 0$ we first conclude from $\operatorname{E}|\mathcal{E}(U)_1|^\kappa = 1$, which is then equivalent to $\psi_{\hat{U}}(\kappa) = 0$, that $\lim_{t\to\infty}\mathcal{E}(U)_t = 0$, as in the proof of [12, Proposition 4.1]. Note here that $\operatorname{E}|U_1|^{\max\{1,\kappa\}} < \infty$ implies that $\operatorname{E}|\mathcal{E}(U)_1|^{\max\{1,\kappa\}} < \infty$ by Proposition 3.1(i). Furthermore, since $\operatorname{E}\log^+|L_1| < \infty$ by (4.1), it follows from Theorem 3.6 of [2] that $\int_{(0,\infty)}\mathcal{E}(U)_{s-}\,\mathrm{d}L_s$ converges a.s. Hence, a strictly stationary solution V_t exists by Theorem 2.1(ii).

If $\lambda > 0$, it is clear that $\lim_{t\to\infty}\mathcal{E}(U)_t = 0$ holds and the existence of a stationary solution is again guaranteed by Theorem 2.1(ii).

We know from (4.5) that, for all $t \geq 0$, the stationary solution fulfills $V_0 \overset{\mathrm{D}}{=} A_tV_0 + B_t$ for A_t and B_t defined by (4.6) and (4.7), respectively. Thus, we have, for any fixed $t > 0$,

$$\operatorname{E}|A_t|^\kappa = \operatorname{E}|\mathcal{E}(\tilde{U})_t\,\mathbf{1}_{\{K(t)=0\}}|^\kappa = \operatorname{E}|\mathcal{E}(\tilde{U})_t|^\kappa\,\mathrm{P}(K(t)=0) = \mathrm{e}^{t\lambda}\mathrm{e}^{-t\lambda} = 1$$

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by assumption, and with $\mathcal{E}(\tilde{U})_t = (-1)^{N_t} \mathrm{e}^{-\hat{\tilde{U}}_t}$, it holds that

$$\mathrm{E}\,|A_t|^\kappa \log^+ |A_t| = \mathrm{E}[\mathbf{1}_{\{K(t)=0\}}\, \mathrm{e}^{-\kappa \hat{\tilde{U}}_t} \log^+(\mathrm{e}^{-\kappa \hat{\tilde{U}}_t})] = \mathrm{e}^{-t\lambda}\, \mathrm{E}[\mathrm{e}^{-\kappa \hat{\tilde{U}}_t} \log^+(\mathrm{e}^{-\kappa \hat{\tilde{U}}_t})] < \infty,$$

since $\mathrm{E}[\mathrm{e}^{-(\kappa+\varepsilon)\hat{\tilde{U}}_t}] < \infty$ by (4.1) and Proposition 3.1. Additionally, for $k := \max\{1, \kappa\}$, by Minkowski's inequality, it holds that

$$\begin{aligned} 0 &< (\mathrm{E}\,|B_t|^k)^{1/k} \\ &\le (\mathrm{E}\,|\mathcal{E}(\tilde{U})_{(T(t),t]}\, \mathbf{1}_{\{K(t)>0\}}\, \Delta L_{T(t)}|^k)^{1/k} \\ &\quad + \left(\mathrm{E}\left|\mathcal{E}(\tilde{U})_{(T(t),t]} \int_{(T(t),t]} [\mathcal{E}(\tilde{U})_{(T(t),s)}]^{-1}\, \mathrm{d}\tilde{\eta}_s\right|^k\right)^{1/k}, \end{aligned} \tag{6.12}$$

where we set $T(t) := 0$ if $K(t) = 0$. The first term in the above sum vanishes if $\lambda = 0$ and in any case is less than or equal to

$$\begin{aligned} (\mathrm{E}[|\mathcal{E}(\tilde{U})_{(T(t),t]}|^k |\Delta L_{T(t)}|^k])^{1/k} &= (\mathrm{E}\,|\mathcal{E}(\tilde{U})_{(T(t),t]}|^k\ \mathrm{E}\,|\Delta L_{T(t)}|^k)^{1/k} \\ &= (\mathrm{E}\,\mathrm{e}^{-k(\hat{\tilde{U}}_t - \hat{\tilde{U}}_{T(t)})}\, \mathrm{E}\,|\Delta L_{T(t)}|^k)^{1/k}, \end{aligned}$$

which is finite by assumption.

On the other hand, observe that, by conditioning on $T(t)$, which is independent of $(\tilde{U}_t, \tilde{L}_t)_{t\ge 0}$, it follows from [2, Lemma 3.1] that

$$\mathcal{E}(\tilde{U})_{(T(t),t]} \int_{(T(t),t]} [\mathcal{E}(\tilde{U})_{(T(t),s)}]^{-1}\, \mathrm{d}\tilde{\eta}_s \overset{\mathrm{D}}{=} \int_{(T(t),t]} \mathcal{E}(\tilde{U})_{(T(t),s)}\, \mathrm{d}\tilde{L}_t.$$

Hence, the second term of (6.12) is finite by Lemma 6.1 since $\mathrm{E}\sup_{0<s\le 1} |\mathcal{E}(\tilde{U})_s|^k < \infty$ by [17, Theorem 25.18] is equivalent to $\mathrm{E}\,|\mathcal{E}(\tilde{U})_1|^k < \infty$, which is given by assumption. Altogether, we obtain $0 < \mathrm{E}\,|B_t|^\kappa \le \mathrm{E}\,|B_t|^k < \infty$.

From (6.1), it is clear that $\hat{\tilde{U}}$ has infinite variation if and only if $\tilde{U}$ has and, thus, if and only if U has infinite variation. Hence, by [17, Corollary 24.6], in this case $\hat{\tilde{U}}_t$ has a nonarithmetic law for each $t > 0$. Otherwise, if we assume that $\hat{\tilde{U}}$ is deterministic then, from (4.1), it follows that $\kappa \hat{\tilde{U}}_1 = \lambda > 0$, which contradicts $\lim_{t\to\infty} \mathcal{E}(U)_t = 0$, so this case cannot occur. Thus, if U (and, hence, $\hat{\tilde{U}}$) is of finite variation, by [17, Corollary 24.6], it suffices to ensure that either the drift $\gamma^0_{\hat{\tilde{U}}}$ of $\hat{\tilde{U}}$ is nonzero or that there is no $r > 0$ with $\mathrm{supp}(\Pi_{\hat{\tilde{U}}}) \subset \mathrm{supp}(\Pi_{\hat{U}}) \subset r\mathbb{Z}$, to guarantee that $\hat{\tilde{U}}_t$ has a nonarithmetic law for t from a dense subset of $(0, \infty)$. Via the relations between U and $\hat{U}$ known from [2, Lemma 3.4], we have $\gamma^0_{\hat{\tilde{U}}} = -\gamma^0_U$ and $\mathrm{supp}(\Pi_{\hat{U}}) = X(\mathrm{supp}(\Pi_U))$ with X as defined in (6.5), so that this holds by assumption. Hence, $\mathcal{L}(\log|A_t| \mid A_t \ne 0) = \mathcal{L}(\hat{\tilde{U}}_t)$ is nonarithmetic for t from a dense subset of $(0, \infty)$.

Now, by [5, Theorem 4.1], it follows that there exists a unique law of V_0 fulfilling $V_0 \overset{\mathrm{D}}{=} A_t V_0 + B_t$ and, by uniqueness in law of the stationary solution, this law is equal to $\mathcal{L}(V_t)$ for all $t \ge 0$. Hence, [5, Theorem 4.1] shows the existence of $C_+, C_- \ge 0$, so (4.2) holds as well as the fact that $C_+ = C_-$ if $\Pi_U((-\infty, -1)) > 0$.

Finally, fix a sequence t_n tending to ∞ so that $\mathcal{L}(\hat{\tilde{U}}_{t_n})$ is nonarithmetic for all $n \in \mathbb{N}$. Now, from [5, Theorem 4.1], it follows additionally that if $C_+ + C_- = 0$, it holds that $B_{t_n} = (1 - A_{t_n})c_n$ for some real constants c_n. But letting n tend to ∞ we observe that $A_{t_n} \to 0$ a.s. as $n \to \infty$, and, hence, $B_{t_n} \overset{\mathrm{D}}{\to} V_0$ by (4.5). This implies that $c_n \overset{\mathrm{D}}{\to} V_0$ as $n \to \infty$, so V_0 and, hence, V_t, $t \ge 0$, is constant.

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The proof of Theorem 4.2 can be carried out analogously to that of Theorem 4.1 simplified to the case $\lambda = 0$. Observe that by defining a process $\hat{W}$ similar to $\hat{U}$, it holds that $\hat{W} = -\hat{U}$, so $\hat{W}$ has a nonarithmetic law if and only if $\hat{U}$ has.

Proof of Proposition 4.2. Part (i) follows from [6, Theorem 4.1] once it is shown that $\mathrm{P}(|A_t| > 1) > 0$ holds for A_t, $t > 0$, defined in (4.6). This is equivalent to $\mathrm{P}(\hat{\hat{U}}_t < 0) > 0$, so we have to ensure that $\hat{\hat{U}}$ is not a subordinator, which is equivalent to the assumptions stated in Proposition 4.2(i) using the relations between the processes U and $\hat{\hat{U}}$.

Owing to (4.8) and the arguments above, we have to ensure that $\hat{W}$ is not a subordinator to prove (ii). Since $\hat{W} = -\hat{U}$ and $\Pi_{\hat{U}} = X(\Pi_U)$, this is equivalent to the given assumptions.

Proof of Theorem 4.3. (i) We know that V_t fulfills (4.5) for all $t \geq 0$. Note that, owing to the relations between the processes U and $\hat{\hat{U}}$, the assumptions on U given in the theorem are equivalent to stating that $\hat{\hat{U}}$ is a subordinator with $\mathrm{P}(\hat{\hat{U}}_t > 0) = 1$ for all $t > 0$. By (4.6), this implies that $\mathrm{P}(|A_t| \leq 1) = \mathrm{P}(|\mathcal{E}(\tilde{U})_t| \, \mathbf{1}_{\{K(t)=0\}} \leq 1) = 1$ and $\mathrm{P}(|A_t| < 1) > 0$ for all $t > 0$. It remains to show that the moment generating function (MGF) of $|B_t|$ for $t > 0$ fixed is finite in some neighborhood $(-\varepsilon, \varepsilon)$ of 0; then the result follows directly from Theorem 2.1 of [6].

By (4.7), for $t > 0$ fixed and $\varepsilon > 0$, it holds that

$$\begin{aligned}
\exp(\varepsilon|B_t|) &= \exp\bigg(\varepsilon\bigg|\mathcal{E}(\tilde{U})_{(T(t),t]}\int_{(T(t),t]}[\mathcal{E}(\tilde{U})_{(T(t),s)}]^{-1}\,\mathrm{d}\tilde{\eta}_s \\
&\qquad\qquad + \mathcal{E}(\tilde{U})_{(T(t),t]}\Delta L_{T(t)}\,\mathbf{1}_{\{K(t)>0\}}\bigg|\bigg) \\
&\leq \exp\bigg(\varepsilon\bigg|\mathcal{E}(\tilde{U})_{(T(t),t]}\int_{(T(t),t]}[\mathcal{E}(\tilde{U})_{(T(t),s)}]^{-1}\,\mathrm{d}\tilde{\eta}_s\bigg|\bigg)\exp(\varepsilon|\Delta L_{T(t)}|\,\mathbf{1}_{\{K(t)>0\}}),
\end{aligned}$$

where we set $T(t) = 0$ if $K(t) = 0$ and, hence, by Hölder's inequality, it is enough to show that both factors have finite expectation in some neighborhood of 0. Owing to our assumption on L, this holds for the second factor, while for the first factor we obtain, with [2, Lemma 3.1],

$$\begin{aligned}
&\mathrm{E}\exp\bigg(\varepsilon\bigg|\mathcal{E}(\tilde{U})_{(T(t),t]}\int_{(T(t),t]}[\mathcal{E}(\tilde{U})_{(T(t),s)}]^{-1}\,\mathrm{d}\tilde{\eta}_s\bigg|\bigg) \\
&\quad = \mathrm{E}\bigg[\mathrm{E}\bigg[\exp\bigg(\varepsilon\bigg|\mathcal{E}(\tilde{U})_{t-T(t)}\int_{(0,t-T(t)]}[\mathcal{E}(\tilde{U})_{s-}]^{-1}\,\mathrm{d}\tilde{\eta}_s\bigg|\bigg)\,\bigg|\,T(t)\bigg]\bigg] \\
&\quad = \int_{[0,t]}\mathrm{E}\bigg[\exp\bigg(\varepsilon\bigg|\int_{(0,t-w]}\mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\bigg|\bigg)\bigg]\,\mathrm{dP}_{T(t)}(w).
\end{aligned}$$

Recall the process $\tilde{L}$, and define $\tilde{L}^+$ and $\tilde{L}^-$ similarly to L^+ and L^- in the proof of Lemma 6.1. Then

$$\begin{aligned}
&\mathrm{E}\bigg[\exp\bigg(\varepsilon\bigg|\int_{(0,t-w]}\mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\bigg|\bigg)\bigg] \\
&\quad \leq \bigg(\mathrm{E}\bigg[\exp\bigg(2\varepsilon\bigg|\int_{(0,t-w]}\mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s^+\bigg|\bigg)\bigg]\bigg)^{1/2} \\
&\quad\qquad \times \bigg(\mathrm{E}\bigg[\exp\bigg(2\varepsilon\bigg|\int_{(0,t-w]}\mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s^-\bigg|\bigg)\bigg]\bigg)^{1/2} \qquad (6.13)
\end{aligned}$$

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by Hölders inequality. Denoting the total variation of $\tilde{L}^+$ on $(0,t]$ by $||\tilde{L}^+||_t$ we obtain

$$\left|\int_{(0,t-w]} \mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s^+\right| \leq \int_{(0,t-w]} \sup_{0<u\leq t-w} |\mathcal{E}(\tilde{U})_{u-}||\mathrm{d}\tilde{L}_s^+| \leq \int_{(0,t-w]} |\mathrm{d}\tilde{L}_s^+| \leq ||\tilde{L}^+||_{t-w}.$$

Since $\tilde{L}^+$ is by definition a finite variation process and L has finite MGF in some neighborhood of 0, so has $||\tilde{L}^+||$. Thus, the first term on the right-hand side of (6.13) is finite for small enough ε.

For fixed $t > 0$, set $M_t := \int_{(0,t]} \mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s^-$. Then $(M_s)_{0<s\leq t}$ is a square integrable martingale by the lemma to Theorem IV.27 of [15], since $\mathrm{E}[\int_{(0,t)} \mathcal{E}(\tilde{U})_{s-}^2\,\mathrm{d}[\tilde{L}^-,\tilde{L}^-]_s] \leq \mathrm{E}[\sigma_L^2 t + \sum_{0<s\leq t}(\Delta\tilde{L}_s^-)^2] < \infty$. Additionally, it holds that $|\Delta M_t| = |\mathcal{E}(\tilde{U})_{t-}\Delta\tilde{L}_t^-| \leq \frac{1}{2}$ and $[M,M]_t \leq \sigma_L^2 t + \sum_{0<s\leq t}(\Delta\tilde{L}_s^-)^2$, where the latter is a Lévy process with bounded jumps having finite exponential moments by [17, Theorem 25.17]. Hence, by [16, Theorem 6.1], $(\mathcal{E}(M)_s)_{0<s\leq t}$ is a martingale.

By the definition of the Doléans-Dade exponential we have

$$\exp(\varepsilon M_s) = \mathcal{E}(M)_s^{\varepsilon} \exp\left(\frac{1}{2}\varepsilon\sigma_M^2 s\right)\left(\prod_{0<u\leq s}(1+\Delta M_u)^{-1}\mathrm{e}^{\Delta M_u}\right)^{\varepsilon},$$

where the first two factors on the right-hand side have bounded expectation uniformly in $s \in [0,t]$ and sufficiently small $\varepsilon > 0$. For the last factor, observe that

$$\sum_{0<u\leq s}(\Delta M_u - \log(1+\Delta M_u)) \leq \sum_{0<u\leq s}(\Delta M_u)^2 \leq \sum_{0<u\leq s}(\Delta\tilde{L}_u^-)^2,$$

since $|\Delta M_u| < \frac{1}{2}$, so $\sup_{0\leq s\leq t} \mathrm{E}(\prod_{0<u\leq s}(1+\Delta M_u)^{-1}\mathrm{e}^{\Delta M_u})^{\varepsilon}$ is finite for sufficiently small ε. An application of Hölder's inequality therefore gives $\mathrm{E}[\exp(\varepsilon M_s)] \leq C_1$ for all $s \leq t$, some constant $C_1 = C_1(t)$, and sufficiently small $\varepsilon > 0$.

Note that

$$\mathrm{E}[\mathrm{e}^{\varepsilon|M_t|}] = \mathrm{E}[\mathrm{e}^{\varepsilon M_t}\,\mathbf{1}_{\{M_t>0\}}] + \mathrm{E}[\mathrm{e}^{-\varepsilon M_t}\,\mathbf{1}_{\{M_t<0\}}] \leq \mathrm{E}[\mathrm{e}^{\varepsilon M_t}] + \mathrm{E}[\mathrm{e}^{-\varepsilon M_t}],$$

so $\mathrm{E}[\exp(\varepsilon|M_t|)] \leq 2C_1$ since the above calculations also hold for

$$-M_t = \int_{(0,t]\mathcal{E}(\tilde{U})_{s-s}} (-\tilde{L}^-)_s.$$

Hence, the second term on the right-hand side of (6.13) is bounded and we conclude that

$$\mathrm{E}\left[\exp\left(\varepsilon\left|\int_{(0,t-w]} \mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\right|\right)\right] \leq C_2$$

for some constant $C_2 = C_2(t)$ and sufficiently small $\varepsilon > 0$ uniformly in $w \in [0,t]$. Thus,

$$\int_{[0,t]} \mathrm{E}\left[\exp\left(\varepsilon\left|\int_{(0,t-w]} \mathcal{E}(\tilde{U})_{s-}\,\mathrm{d}\tilde{L}_s\right|\right)\right]\mathrm{dP}_{T(t)}(w) < \infty,$$

so $|B_t|$ is shown to have finite MGF in some neighborhood of 0.

To prove (ii), owing to (4.8), we have to show for $t > 0$ that $\mathrm{P}(|A_t^{-1}| \leq 1) = \mathrm{P}(|\mathcal{E}(W)_t| \leq 1) = 1$, $\mathrm{P}(|A_t| < 1) > 0$, and that the MGF of $|A_t^{-1}B_t| = |\int_{[0,t)} \mathcal{E}(W)_{s-}\,\mathrm{d}\eta_s|$, $t > 0$, is finite in some neighborhood $(-\varepsilon,\varepsilon)$ of 0. This can be done as in (i) and the result follows again from Theorem 2.1 of [6].

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6.3. Proofs for Section 5

The following lemma will be needed to prove our main theorem on absolute continuity. An analogous result for a.s. nonzero M_t has been shown in [3, Lemma 2.1].

Lemma 6.2. *For $t \in \mathbb{N}_0$, let ψ_t, Q_t, and M_t be random variables such that it holds that $\mathrm{P}(M_t = 0) > 0$, ψ_t is independent of (Q_t, M_t), and, for large enough $t \in \mathbb{N}$, the conditional distribution of Q_t given $M_t = 0$ is continuous. Suppose that ψ is a random variable satisfying $\psi = Q_t + M_t\psi_t$ and $\psi \overset{\mathrm{D}}{=} \psi_t$ for all $t \geq 0$ and such that $Q_t \overset{\mathrm{P}}{\to} \psi$ as $t \to \infty$. Then ψ has an atom if and only if it is a constant random variable.*

Proof. Suppose that ψ has an atom at $a \in \mathbb{R}$, i.e. $\mathrm{P}(\psi = a) =: \beta > 0$. Then, for all $\varepsilon \in (0, \beta)$, there exists $\delta > 0$ such that $\mathrm{P}(|\psi - a| < 2\delta) < \beta + \varepsilon$. Additionally, $Q_t \overset{\mathrm{P}}{\to} \psi$ implies the existence of some $t' = t'(\varepsilon)$ such that $\mathrm{P}(|\psi - Q_t| \geq \delta) = \mathrm{P}(|M_t\psi_t| \geq \delta) < \varepsilon$ for all $t \geq t'$.

Following the lines of the proof of [3, Lemma 2.1], we can now show that, for all $t \geq t'$, there exists $s_t \in \mathbb{R}$ such that $\beta_t := \mathrm{P}(Q_t + M_t s_t = a,\ |M_t s_t| < \delta) \geq \beta - \varepsilon$, and it holds that

$$\begin{aligned}\mathrm{P}(|\psi - Q_t| \geq \delta) + \mathrm{P}(|Q_t - a| < \delta) \geq \mathrm{P}(\psi = a) + \mathrm{P}(Q_t + M_t s_t = a,\ |M_t s_t| < \delta)\\ - \mathrm{P}(Q_t + M_t s_t = a,\ |M_t s_t| < \delta,\ \psi = a).\end{aligned} \tag{6.14}$$

Since $\mathrm{P}(M_t = 0) > 0$, we have

$$\begin{aligned}&\{Q_t + M_t s_t = a,\ |M_t s_t| < \delta,\ \psi = a\}\\ &\quad = \{Q_t + M_t s_t = a,\ |M_t s_t| < \delta,\ \psi = a,\ M_t = 0\}\\ &\qquad \cup \{Q_t + M_t s_t = a,\ |M_t s_t| < \delta,\ \psi = a,\ M_t \neq 0\}\\ &\qquad \subset \{Q_t = a,\ M_t = 0\} \cup \{Q_t + M_t s_t = a,\ |M_t s_t| < \delta,\ M_t \neq 0,\ \psi_t = s_t\}\\ &\qquad \subset \{Q_t = a,\ M_t = 0\} \cup (\{Q_t + M_t s_t = a,\ |M_t s_t| < \delta\} \cap \{\psi_t = s_t\}),\end{aligned}$$

and, by the continuity assumption on Q_t given $M_t = 0$, we obtain

$$\mathrm{P}(Q_t + M_t s_t = a,\ |M_t s_t| < \delta,\ \psi = a) \leq 0 + \beta_t\,\mathrm{P}(\psi_t = s_t),$$

so we can conclude from (6.14) that

$$\mathrm{P}(|\psi - Q_t| \geq \delta) + \mathrm{P}(|Q_t - a| < \delta) \geq \beta + \beta_t - \beta_t\,\mathrm{P}(\psi_t = s_t).$$

From here, again following directly the proof of [3, Lemma 2.1], we obtain the assumption that $\mathrm{P}(\psi = a) = 1$.

Now we can prove the conditions for the distribution of the perpetuity X_∞ to be (absolutely) continuous in the case $\mathrm{P}(A = 0) > 0$ as stated in Theorem 5.1.

Proof of Theorem 5.1. To show (i), first suppose that the conditional distribution of B given $A = 0$ is continuous. Let $(A_k, B_k)_{k\in\mathbb{N}_0}$ be an i.i.d. sequence of random variables such that (A_0, B_0) has the same distribution as (A, B). Define

$$\psi := X_\infty = \sum_{k=0}^{\infty}\Bigl(\prod_{i=0}^{k-1} A_i\Bigr)B_k, \qquad M_t = \prod_{i=0}^{t-1} A_i,$$

$$\psi_t = \sum_{k=t}^{\infty}\Bigl(\prod_{i=t}^{k-1} A_i\Bigr)B_k, \qquad Q_t = \sum_{k=0}^{t-1}\Bigl(\prod_{i=0}^{k-1} A_i\Bigr)B_k.$$

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Then it follows from Lemma 6.2 that ψ is continuous or a Dirac measure if we can show that the conditional distribution of Q_t given $M_t = 0$ is continuous for all $t \in \mathbb{N}$. We do this by induction.

For $t = 1$, the claim is true by assumption. Now suppose that the conditional distribution of Q_t given $M_t = 0$ is continuous, so $\mathrm{P}(Q_t = a,\ M_t = 0) = 0$ for each $a \in \mathbb{R}$. We then have, since $Q_{t+1} = Q_t + M_t B_t$, for each $a \in \mathbb{R}$,

$$\begin{aligned}\mathrm{P}(Q_{t+1} = a, M_{t+1} = 0)& \\ &= \mathrm{P}(Q_t + M_t B_t = a,\ M_t A_t = 0) \\ &= \mathrm{P}(Q_t = a,\ M_t = 0) + \mathrm{P}(Q_t + M_t B_t = a,\ M_t \neq 0,\ A_t = 0) \\ &= \mathrm{P}(B_t = M_t^{-1}(a - Q_t),\ M_t \neq 0,\ A_t = 0)\end{aligned}$$

by the induction hypothesis. Now, using regular conditional probabilities, we further obtain

$$\begin{aligned}\mathrm{P}(B_t = M_t^{-1}(a - Q_t),\ M_t \neq 0,\ A_t = 0)& \\ &= \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} \mathrm{P}(B_t = u^{-1}(a - v),\ A_t = 0 \mid M_t = u,\ Q_t = v)\, \mathrm{dP}_{(M_t, Q_t)}(u, v) \\ &= \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} \mathrm{P}(B_t = u^{-1}(a - v),\ A_t = 0)\, \mathrm{dP}_{(M_t, Q_t)}(u, v) \\ &\qquad\qquad \text{(since } (A_t, B_t) \text{ is independent of } (M_t, Q_t)\text{)} \\ &= \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} 0\, \mathrm{dP}_{(M_t, Q_t)}(u, v) \\ &= 0,\end{aligned}$$

the latter following from the fact that the conditional distribution of B_t given $A_t = 0$ is continuous. Hence, we see that the conditional distribution of Q_{t+1} given $M_{t+1} = 0$ is continuous too, completing the induction step. Lemma 6.2 hence shows that X_∞ is continuous or degenerate to a Dirac measure. But X_∞ cannot be degenerate to a Dirac measure, since $X_\infty \stackrel{\mathrm{D}}{=} B + AX' = B$ on $A = 0$, where $\mathrm{P}(A = 0) > 0$ and $\mathcal{L}(B \mid A = 0)$ is continuous.

To see the converse, suppose that the conditional distribution of B given $A = 0$ is not continuous. Then there exists an $a \in \mathbb{R}$ such that $\mathrm{P}(B = a \mid A = 0) = \beta > 0$. Since $\mathcal{L}(X_\infty)$ satisfies the fixed-point equation (4.3), we have

$$\mathrm{P}(X_\infty = a) = \mathrm{P}(AX' + B = a) \geq \mathrm{P}(B = a,\ A = 0) > 0.$$

Hence, $\mathcal{L}(X_\infty)$ has an atom.

For (ii), we will first show that $\mathcal{L}(X_\infty)$ is either absolutely continuous or a Dirac measure, given that the conditional distribution of B given $A = 0$ is absolutely continuous. Then it follows from part (i) that $\mathcal{L}(X_\infty)$ is absolutely continuous. In doing so we follow the arguments of Grincevičius [8], who considered the case $\mathrm{P}(A = 0) = 0$.

Assume that $\mathcal{L}(X_\infty)$ is not singular and denote its characteristic function by

$$f(x) = \mathrm{E}\,\mathrm{e}^{ixX_\infty} = \mathrm{E}[\mathrm{E}[\mathrm{e}^{iBx}\mathrm{e}^{iAxX'} \mid A, B]] = \mathrm{E}[\mathrm{e}^{iBx} f(Ax)].$$

Then by the Lebesgue decomposition theorem we may write $f(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$, where $\alpha_1 > 0, \alpha_2 \geq 0, \alpha_1 + \alpha_2 = 1$, and $f_1(x)$ and $f_2(x)$ are the characteristic functions of an absolutely continuous and a singular probability distribution, respectively. Hence,

$$\alpha_1 f_1(x) + \alpha_2 f_2(x) = \alpha_1\, \mathrm{E}[\mathrm{e}^{iBx} f_1(Ax)] + \alpha_2\, \mathrm{E}[\mathrm{e}^{iBx} f_2(Ax)].$$

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Let Y be a random variable independent of (A, B), having characteristic function f_1, and set $Z := AY + B$. Then, for $C \in \mathcal{B}_1$ with Lebesgue measure 0, it holds that

$$\begin{aligned}\mathrm{P}(Z \in C) &= \mathrm{P}(AY + B \in C)\\ &= \mathrm{P}(B \in C,\ A = 0) + \mathrm{P}(AY + B \in C,\ A \neq 0)\\ &= 0 + \int_{(\mathbb{R}\setminus\{0\})\times\mathbb{R}} \mathrm{P}(Y \in u^{-1}(C - v))\, \mathrm{dP}_{A,B}(u, v)\\ &= 0.\end{aligned}$$

It follows that Z is absolutely continuous and its characteristic function $x \mapsto \mathrm{E}(\mathrm{e}^{\mathrm{i}Bx} f_1(Ax))$ is the characteristic function of an absolutely continuous function.

Applying the Lebesgue decomposition to the distribution having characteristic function $x \mapsto \mathrm{E}\,\mathrm{e}^{\mathrm{i}Bx} f_2(Ax)$, we can write $\mathrm{E}\,\mathrm{e}^{\mathrm{i}Bx} f_2(Ax) = \alpha_3 f_3(x) + \alpha_4 f_4(x)$ with $\alpha_3, \alpha_4 \geq 0$, $\alpha_3 + \alpha_4 = 1$, and f_3 and f_4 the characteristic functions of an absolutely continuous and a singular distribution, respectively. By the uniqueness of the Lebesgue decomposition, it follows that

$$\alpha_1 f_1(x) = \alpha_1 \mathrm{E}[\mathrm{e}^{\mathrm{i}Bx} f_1(Ax)] + \alpha_2\alpha_3 f_3(x),$$

which, for $x = 0$, yields $\alpha_2\alpha_3 = 0$. Hence, $f_1(x) = \mathrm{E}[\mathrm{e}^{\mathrm{i}Bx} f_1(Ax)]$. Since $f(x) = \mathrm{E}[\mathrm{e}^{\mathrm{i}Bx} f(Ax)]$ also, it follows that $f(x) = f_1(x)$ by an easy extension of Proposition 1 of [8]. Hence, we conclude that $\mathcal{L}(X_\infty)$ is absolutely continuous.

It remains to show that if the conditional distribution of B given $A = 0$ is not absolutely continuous then $\mathcal{L}(X_\infty)$ cannot be absolutely continuous. For in that case there exists $C \in \mathcal{B}_1$ with Lebesgue measure 0 but $\mathrm{P}(B \in C \mid A = 0) > 0$. We conclude that $\mathrm{P}(B \in C,\ A = 0) > 0$ and, hence (for $X' \overset{\mathrm{D}}{=} X_\infty$ being independent of (A, B)),

$$\mathrm{P}(X_\infty \in C) = \mathrm{P}(AX' + B \in C) \geq \mathrm{P}(B \in C,\ A = 0) > 0,$$

so $\mathcal{L}(X_\infty)$ cannot be absolutely continuous.

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References

[1] Alsmeyer, G., Iksanov, A. and Rösler, U. (2009). On distributional properties of perpetuities. *J. Theoret. Prob.* **22,** 666–682.

[2] Behme, A., Lindner, A. and Maller, R. (2011). Stationary solutions of the stochastic differential equation $dV_t = V_{t-}dU_t + dL_t$ with Lévy noise. *Stoch. Process. Appl.* **121,** 91–108.

[3] Bertoin, J., Lindner, A. and Maller, R. (2008). On continuity properties of the law of integrals of Lévy processes. In *Séminaire de Probabilités XLI* (Lecture Notes Math. **1934**), eds C. Donati-Martin *et al.*, Springer, Berlin, pp. 137–159.

[4] Bichteler, K. (2002). *Stochastic Integration with Jumps*. Cambridge University Press.

[5] Goldie, C. M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Prob.* **1,** 126–166.

[6] Goldie, C. M. and Grübel, R. (1996). Perpetuities with thin tails. *Adv. Appl. Prob.* **28,** 463–480.

[7] Goldie, C. M. and Maller, R. A. (2000). Stability of perpetuities. *Ann. Prob.* **28,** 1195–1218.

https://doi.org/10.1239/aap/1316792666 Published online by Cambridge University Press

[8] Grincevičius, A. K. (1974). On the continuity of the distribution of a sum of dependent variables connected with independent walks on lines. *Theory Prob. Appl.* **19,** 163–168.

[9] Grincevičius, A. K. (1980). Products of random affine transformations. *Lithuanian Math. J.* **20,** 279–282.

[10] Jaschke, S. (2003). A note on the inhomogeneous linear stochastic differential equation. *Insurance Math. Econom.* **32,** 461–464.

[11] Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta. Math.* **131,** 207–248.

[12] Lindner, A. and Maller, R. (2005). Lévy integrals and the stationarity of generalised Ornstein–Uhlenbeck processes. *Stoch. Process. Appl.* **115,** 1701–1722.

[13] Maller, R. A., Müller, G. and Szimayer, A. (2009). Ornstein–Uhlenbeck processes and extensions. In *Handbook of Financial Time Series*, eds T. Andersen *et al.*, Springer, Berlin, pp. 421–437.

[14] Meyer, P.-A. (1978). Inégalités de normes pour les intégrales stochastiques. In *Séminaire de Probabilités XII* (Lecture Notes Math. **649**), eds A. Dold and B. Eckmann, Springer, Berlin, pp. 757–762.

[15] Protter, P. E. (2005). *Stochastic Integration and Differential Equations*, 2nd edn. Springer, Berlin.

[16] Protter, P. E. and Shimbo, K. (2008). No arbitrage and general semimartingales. In *Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz* (Inst. Math. Statist. Collect. **4**), Institute of Mathematical Statistics, Beachwood, OH, pp. 267–283.

[17] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

[18] Vervaat, W. (1979). On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. Appl. Prob.* **11,** 750–783.

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