Forum of Mathematics, Sigma (2019), Vol. 7, e30, 28 pages
doi:10.1017/fms.2019.25

THE CHERN–SCHWARTZ–MACPHERSON CLASS OF AN EMBEDDABLE SCHEME

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Received 27 May 2019; accepted 12 August 2019

Abstract

The Chern–Schwartz–MacPherson class of a hypersurface in a nonsingular variety may be computed directly from the Segre class of the Jacobian subscheme of the hypersurface; this has been known for a number of years. We generalize this fact to arbitrary embeddable schemes: for every subscheme X of a nonsingular variety V, we define an associated subscheme $\mathscr{Y}$ of a projective bundle $\mathscr{V}$ over V and provide an explicit formula for the Chern–Schwartz–MacPherson class of X in terms of the Segre class of $\mathscr{Y}$ in $\mathscr{V}$. If X is a local complete intersection, a version of the result yields a direct expression for the Milnor class of X.

For $V = \mathbb{P}^n$, we also obtain expressions for the Chern–Schwartz–MacPherson class of X in terms of the 'Segre zeta function' of $\mathscr{Y}$.

2010 Mathematics Subject Classification: 14C17 (primary); 14J17 (secondary)

1. Introduction

1.1. Context and preliminaries. The goal of this paper is the generalization to arbitrary subschemes of nonsingular varieties of a twenty-year old formula for the *Chern–Schwartz–MacPherson* class of hypersurfaces, in terms of the Segre class of an associated scheme. We first recall the general context and the relevant definitions; the hurried reader may want to skip ahead to Section 1.2 for the statement of the main result for subschemes of projective space.

Every nonsingular variety X has a canonically defined class in its homology, namely the total Chern class of its tangent bundle. Deligne and Grothendieck conjectured, and MacPherson proved [Mac74], that (at least in characteristic 0) this class is a manifestation of a functorial theory of Chern classes which assigns a distinguished homology class to *every* complex projective variety X. The class associated with X is called the *Chern–Schwartz–MacPherson* (CSM) class of X, $c_{\mathrm{SM}}(X)$. (Brasselet and Schwartz proved [BS81] that the class $c_{\mathrm{SM}}(X)$ agrees via Alexander duality with the class defined earlier by Schwartz, [Sch65a, Sch65b].) MacPherson's theory can be refined to give a class in the Chow group of X [Ful84, Example 19.1.7], and extended to embeddable schemes over arbitrary algebraically closed fields of characteristic 0 [Ken90, Alu06], and this is the notion we adopt in this paper.

The CSM class of X encodes interesting information about the variety X. For example, if $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a complex projective variety, then the degrees of the components of $c_{\mathrm{SM}}(X)$ carry the same information as the topological Euler characteristics of its general linear sections [Alu13]. Chern–Schwartz–MacPherson classes of classical varieties such as Schubert varieties and determinantal varieties have been studied extensively and are the objects of current research (see, for example, [PP95, AM09, AM16, RV18, Zha18, FR18, AMSS, FRW]).

In [Alu99], we proved a formula for the Chern–Schwartz–MacPherson class of a *hypersurface* in a nonsingular variety, in terms of the Segre class of its singularity subscheme. (In particular, this yields a formula for the topological Euler characteristic of arbitrary hypersurfaces of nonsingular varieties.) Applications include computations in enumerative geometry [Alu98], singularities of logarithmic foliations [CSV06], Sethi–Vafa–Witten-type formulas [AE09], and others. By inclusion–exclusion, the case of hypersurfaces suffices in order to compute CSM classes of subschemes of nonsingular varieties. This fact is at the root of most implemented algorithms for the computation of CSM classes in projective spaces and more general varieties; see [Alu03, Jos15, Hel16, Hel17a] and others. (To our knowledge, the algorithm presented in [MB12] is the *only one* currently available that does not rely on the result for hypersurfaces from [Alu99] and inclusion–exclusion.)

One obvious problem with inclusion–exclusion is that the number of needed computations grows exponentially with the number of defining equations. A second problem is that the degrees of the hypersurfaces one needs to consider also grow with the number of defining equations: if, for example, the subscheme is defined by r equations of degree d, inclusion–exclusion requires the computation of the CSM class of a hypersurface of degree rd. This is taxing for both Gröbner basis and numerical computations. For these and for more conceptual reasons, it would be desirable to have direct generalizations of the result in [Alu99] to more general schemes. Such a generalization should express the CSM class of a scheme X in terms of the Segre class of a related scheme determined by the singularities of X, without invoking inclusion–exclusion. We raised this problem in [Alu05, Section 4.1], and to our knowledge it has so far remained open in the intended generality. Fullwood [Ful14] gave an answer for global complete intersections $M_1 \cap \cdots \cap M_k$ such that $M_1 \cap \cdots \cap M_{k-1}$ is nonsingular. Complete and local complete intersections are also treated in references with different viewpoints (among these [BLSS02, MSS13, CBMS]); but a result along the lines envisioned above had, to our knowledge, not been formulated, even in the (unrestricted) complete intersection case.

The purpose of this article is to rectify this situation. For $\iota : X \hookrightarrow V$ an *arbitrary* closed embedding of a scheme X in a nonsingular variety V, we provide a formula for $\iota_* c_{SM}(X) \in A_* V$ in terms of the Segre class $s(\mathscr{Y}, \mathscr{V})$ of an associated subscheme $\mathscr{Y}$ of a projective bundle $\mathscr{V}$ over V. In the hypersurface case, this formula will agree with the result of [Alu99]. In the case of local complete intersections, it will yield an expression for (the push-forward to V of) the so-called *Milnor class* of X. In general, the formula will make no assumptions on X, other than that it can be embedded as a closed subscheme of a nonsingular variety. In fact, the formula will have nontrivial content even if X is nonsingular.

1.2. The result, in projective space. In this introduction, we present the result in the particular case in which $V = \mathbb{P}^n$. This leads to some simplifications, and is possibly the most useful in concrete computations. In Section 2, we state the formula for arbitrary nonsingular ambient varieties V.

Let $\iota : X \hookrightarrow \mathbb{P}^n$ be a closed embedding; then X may be defined by a homogeneous ideal generated by a set of forms $F_0, \ldots, F_r$ of a fixed degree d. Let $\mathscr{Y}$ denote the subscheme of $\mathbb{P}^n \times \mathbb{P}^r$ defined by the ideal

$$(F_0, \ldots, F_r) + \left(y_0 \frac{\partial F_0}{\partial x_i} + \cdots + y_r \frac{\partial F_r}{\partial x_i} \right)_{i=0,\ldots,n}.$$

Here $x_0, \ldots, x_n$ are homogeneous coordinates in $\mathbb{P}^n$, and $y_0, \ldots, y_r$ are homogeneous coordinates in $\mathbb{P}^r$. Denote by $\pi : \mathbb{P}^n \times \mathbb{P}^r \to \mathbb{P}^n$ the projection and let H, respectively, h denote the pull-backs of the hyperplane classes from $\mathbb{P}^n$, respectively, $\mathbb{P}^r$.

THEOREM 1.1. *With notation as above, assume $r \geqslant n$. Then*

$$\iota_* c_{SM}(X) = \pi_* \left(\frac{(1+H)^{n+1}(1+h)^{r+1}}{1+dH+h} (s(\mathscr{Y}, \mathbb{P}^n \times \mathbb{P}^r)^\vee \otimes_{\mathbb{P}^n \times \mathbb{P}^r} \mathscr{O}(dH+h)) \right), \tag{1.1}$$

where $s(\mathscr{Y}, \mathbb{P}^n \times \mathbb{P}^r)$ is the Segre class of $\mathscr{Y}$ in $\mathbb{P}^n \times \mathbb{P}^r$.

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(This statement uses the notation $\otimes$, $^\vee$ introduced in [Alu94, Section 2]. We recall this notation in Section 2.3.) For instance, the degree of the class on the right-hand side equals the Euler characteristic of X.

Note that the choices of the integer $d \gg 0$ and of the generators F_i are arbitrary. In particular, we could choose some of the F_i to coincide, or even to be 0, in order to guarantee that $r \geqslant n$. Every such choice leads to an expression for the CSM class of X.

The main result we present in Section 2 (Theorem 2.5) will pose no restriction on the number r of generators of a defining ideal for X. The case $r \geqslant n$ leads to a direct formula for the CSM class of X, of which Theorem 1.1 is a particular case. Another case of interest is $r + 1 = \operatorname{codim} X$, that is, the case of a global complete intersection. Recall that the *Milnor class* of a complete intersection X is the (signed) difference of its CSM class and of the Chern class of the virtual tangent bundle of X:

$$\mathscr{M}(X) = (-1)^{\dim X}(c_{\mathrm{vir}}(X) - c_{\mathrm{SM}}(X)). \tag{1.2}$$

(See, for example, [PP01]. To our knowledge, this terminology is due to Yokura, [Yok99a, Yok99b].)

THEOREM 1.2. *Let $\iota : X \hookrightarrow \mathbb{P}^n$ be a complete intersection of $r + 1$ hypersurfaces of degree d. Then with notation as above*

$$\begin{aligned}(-1)^{\dim X+1}&\iota_*\mathscr{M}(X)\\ &= \pi_*\left(\frac{(1+H)^{n+1}(1+h)^{r+1}}{1+dH+h}(s(\mathscr{Y}, \mathbb{P}^n \times \mathbb{P}^r)^\vee \otimes_{\mathbb{P}^n\times\mathbb{P}^r} \mathscr{O}(dH+h))\right).\end{aligned} \tag{1.3}$$

It is worth stressing that the right-hand sides in (1.1) and (1.3) are *identical.* The claim is that for $r \gg 0$ this formula yields the CSM class of X, while if X is a complete intersection of $r + 1$ hypersurfaces of a fixed degree d, the same formula yields the Milnor class of X (up to a sign).

1.3. One example. The formulas stated above can be implemented easily in Macaulay2 [GS], using the package `CharacteristicClasses.m2` written by Helmer and Jost [HJ] to compute the relevant Segre class: this package can handle Segre classes of subschemes of products of projective spaces, and computing the push-forward amounts to simply extracting the coefficient of h^r. The same package also implements the computation of CSM classes (by the inclusion–exclusion method mentioned above), so it may be used as an independent verification of results obtained by applying Theorems 1.1 and 1.2. We illustrate the application of Theorem 1.2 to the complete intersection of the singular hypersurfaces

$$Z_1 : \{x_1 x_2 x_3 = 0\}, \quad Z_2 : \{x_0 x_1^2 + x_2^3 = 0\}$$

in $\mathbb{P}^6$. The scheme $X = Z_1 \cap Z_2$ consists of three components of codimension 2, one of which (supported on a linear subspace) is nonreduced. The following Macaulay2 session implements the computation of the Segre class $s(\mathscr{Y}, \mathbb{P}^6 \times \mathbb{P}^1)$ for the scheme $\mathscr{Y}$ associated with X. (We omit inessential output.)

```
i1 : load ("CharacteristicClasses.m2");

i2 : R=MultiProjCoordRing({6,1});

i3 : r= gens R

o3 = {x , x , x , x , x , x , x , x , x }
       0   1   2   3   4   5   6   7   8

i4 : Y=ideal(r_1*r_2*r_3,r_1^2*r_0+r_2^3,r_8*r_1^2,r_7*r_2*r_3+r_8*2*r_0*r_1,

             r_7*r_1*r_3+r_8*3*r_2^2, r_7*r_1*r_2)

                         2     3    2                                 2
o4 = ideal (x x x , x x  + x , x x , x x x  + 2x x x , x x x  + 3x x , x x x )
             1 2 3   0 1    2   1 8   2 3 7     0 1 8   1 3 7     2 8   1 2 7

i5 : Segre(Y)

          6         6        5         5       4        4       3      3      2      2
o5 = 181h h  - 240h  - 167h h  + 72h  + 69h h  - 16h  - 19h h  + h  + 3h h  + h
         1 2       1        1 2      1      1 2      1      1 2    1     1 2    1
```

Here $H = \mathrm{h}_1$ and $h = \mathrm{h}_2$. The result is that

$$\begin{aligned} s(\mathscr{Y}, \mathbb{P}^6 \times \mathbb{P}^1) = (&H^2 + 3H^2 h + H^3 - 19H^3 h - 16H^4 \\ &+ 69H^4 h + 72H^5 - 167H^5 h - 240H^6 + 181H^6 h) \cap [\mathbb{P}^6 \times \mathbb{P}^1] \end{aligned}$$

(after push-forward to the ambient space). It is then straightforward to compute

$$\begin{aligned} \frac{(1+H)^7(1+h)^2}{1+3H+h} &(s(\mathscr{Y}, \mathbb{P}^6 \times \mathbb{P}^1)^\vee \otimes_{\mathbb{P}^6 \times \mathbb{P}^1} \mathscr{O}(3H+h)) \\ &= (H^2 - 3H^3 + H^4 - 17H^5 + 42H^6) \\ &\quad - (4H^2 - 9H^3 + 29H^4 - 107H^5 + 363H^6)h. \end{aligned}$$

According to Theorem 1.2,

$$\iota_* \mathscr{M}(X) = (4H^2 - 9H^3 + 29H^4 - 107H^5 + 363H^6) \cap [\mathbb{P}^6]$$

is the Milnor class of X. Since X is a complete intersection of two hypersurfaces of degree 3,

$$\iota_* c_{\mathrm{vir}}(X) = \frac{(1+H)^7}{(1+3H)^2} \cap [\mathbb{P}^6] = (9H^2 + 9H^3 + 54H^4 - 90H^5 + 369H^6) \cap \mathbb{P}^6.$$

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It follows that

$$\begin{aligned}\iota_* c_{\mathrm{SM}}(X) &= \iota_*(c_{\mathrm{vir}}(X) - (-1)^{\dim X}\mathscr{M}(X)) \\ &= (5H^2 + 18H^3 + 25H^4 + 17H^5 + 6H^6) \cap [\mathbb{P}^6].\end{aligned}$$

(This can be confirmed independently by [HJ].)

Note that Z_1 and Z_2 are both singular, and their singular loci have nonempty intersection. It follows that, in this example, the complete intersection X cannot be represented as a hypersurface in a nonsingular subvariety of $\mathbb{P}^6$; therefore, it does not satisfy the hypotheses of [Ful14, Hel17b].

The requirement in Theorem 1.2 that the degrees of the defining hypersurfaces coincide leads to the particularly explicit formula (1.3). The more general result presented in Section 2 (Corollary 2.6) will dispense of this requirement; an expression for the Milnor class will be obtained for every local complete intersection represented as the zero-scheme of a regular section of a vector bundle on a nonsingular variety.

1.4. Organization of the paper. The paper is organized as follows. In Section 2, we provide a full statement of the main result (Theorem 2.5) and give several illustrating examples, including the derivation of Theorems 1.1 and 1.2. In Section 3, we prove the main result. The proof relies on the hypersurface case given in [Alu99], on calculus of constructible functions, and on intersection-theoretic computations. A key ingredient in the proof is the construction of an auxiliary hypersurface, an idea we borrow from [CBMS]. As we show in Section 3.2, the scheme $\mathscr{Y}$ is the singularity subscheme of this hypersurface. In [CBMS], this hypersurface is constructed in the local complete intersection case, and it is also used to obtain formulas for Milnor classes (see Section 3.6). We note here that the scheme $\mathscr{Y}$ was also considered by Ohmoto [Ohm] and Liao [Lia].

As an application of the main result, we expand in Section 4 on the case of subschemes of projective space. Recent results on Segre classes lead to alternative, and in some way more efficient, formulations of the result in this case.

1.5. Another example. The result implies that expressions such as (1.1) (or the more general version (2.4) given in Section 2) are independent of the choices leading to these expressions: in the case of subschemes X of $\mathbb{P}^n$ these choices are the degree $d \gg 0$ of the generators of a defining ideal, the number $r \geqslant n$ of generators, or in fact the generators themselves. We do not know a more direct proof of this independence. In fact, X may be replaced by *any* scheme with

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the same support as X without affecting these expressions. While this fact is an immediate consequence of the main result, it seems quite nontrivial in itself.

We illustrate this fact with an example. Let X be the scheme with ideal (x_0^2, x_0x_1) in $\mathbb{P}^2$; so X is supported on a line $\mathbb{P}^1$, with an embedded component at the point $x_0 = x_1 = 0$. We choose the generators $F_0 = x_0^2$, $F_1 = x_0x_1$, $F_2 = 0$ for the ideal of X, which determine as described above the subscheme $\mathscr{Y}$ of $\mathbb{P}^2 \times \mathbb{P}^2$ with ideal

$$(x_0^2, x_0x_1, 2x_0y_0 + x_1y_1, x_0y_1).$$

According to `CharacteristicClass.m2`,

$$s(\mathscr{Y}, \mathbb{P}^2 \times \mathbb{P}^2) = (Hh + H^2 - Hh^2 - 2H^2h + 3H^2h^2) \cap [\mathbb{P}^2 \times \mathbb{P}^2];$$

it follows that the class appearing on the right-hand side of (1.1) is

$$(H^2 + (H - H^2)h + (H + 2H^2)h^2) \cap [\mathbb{P}^2 \times \mathbb{P}^2].$$

Performing the same computation using the reduced $\mathbb{P}^1$, with generators $F_0 = x$, $F_1 = 0$, $F_2 = 0$, yields the class

$$((H + H^2)h + (H + 2H^2)h^2) \cap [\mathbb{P}^2 \times \mathbb{P}^2].$$

The classes differ, but the coefficient of h^2, that is, their push-forward to $\mathbb{P}^2$, agree (and yield $c_{\mathrm{SM}}(\mathbb{P}^1) = c(T\mathbb{P}^1) \cap [\mathbb{P}^1]$ as prescribed by Theorem 1.1).

The results of this paper will prove that CSM classes of schemes X with the same support agree as classes in the Chow group of every nonsingular variety containing X. It would be desirable to have a direct proof that classes obtained in this fashion are independent of all choices as classes in the Chow group A_*X_{red}.

1.6. Relations with other results, and possible extensions. As mentioned above, a different formula for the Milnor class of a complete intersection is given in [**CBMS**], using a construction similar to the one used in this paper; see Section 3.6.

The formula for CSM classes of hypersurfaces in [**Alu99**] may be seen as a manifestation of an identity of characteristic cycles; see [**PP01**] for this point of view and an alternative proof of the formula in [**Alu99**]. It is a natural project to provide a characteristic cycle version of the generalization obtained in this note.

To our knowledge, the hypersurface formula is not implied by the motivic theory for characteristic classes of hypersurfaces, as presented in [**CMSS10**] (and extended to complete intersections in [**MSS13**]). *A fortiori,* a direct relation between the generalization presented here and the theory of motivic Hirzebruch

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classes would be surprising and very interesting. Equally interesting would be a connection with Yokura's 'motivic Milnor classes' [Yok10].

Finally, we note that Fullwood and Wang have proposed a conjectural generalization of the hypersurface formula [FW], in terms of a blow-up construction; they prove that this formulation is correct for certain complete intersections. It is straightforward to express our results in this note in terms of the blow-up along the scheme $\mathscr{Y}$, as this blow-up may be used to compute the Segre class of $\mathscr{Y}$. It would be interesting to relate the center of the blow-up in [FW] to $\mathscr{Y}$.

2. Statement

2.1. Preliminaries. We work over an algebraically closed field k of characteristic 0. (This requirement is needed for Chern–Schwartz–MacPherson classes.) Throughout the paper, X will denote a k-scheme which can be embedded as a closed subscheme of a nonsingular variety V.

The *Chern–Schwartz–MacPherson* (CSM) class of X may be defined as an element in the Chow group A_*X of X. It is determined by the requirement that if X is nonsingular, then $c_{\mathrm{SM}}(X) = c(TX) \cap [X]$ and by a specific behavior with respect to proper morphisms, which we now recall.

We can associate with each X the group of *constructible functions* $F(X)$, that is, integer-valued functions on X which may be obtained as finite linear combinations of indicator functions on subvarieties of X: $\varphi = \sum_W \mathbb{1}_W$, where the sum ranges over finitely many closed subvarieties W of X, and $\mathbb{1}_W(p) = 1$ if $p \in W$, $\mathbb{1}_W(p) = 0$ if $p \notin W$. The assignment $X \rightsquigarrow F(X)$ defines a *covariant functor* to the category of abelian groups, if we prescribe the following push-forward for proper maps: if $f : X \to Y$ is a proper morphism, a homomorphism $f_* : F(X) \to F(Y)$ is defined by requiring $f_*(\mathbb{1}_W)$ to be the function

$$f_*(\mathbb{1}_W)(p) = \chi(f^{-1}(p) \cap W),$$

where χ denotes the topological Euler characteristic if $k = \mathbb{C}$, and a suitable analogue over more general fields (see, for example, [Alu13, Section 2.1]).

According to a theorem of MacPherson [Mac74] and extensions of this result to the context used here, there exists a natural transformation from F to the Chow group functor A_*, such that the indicator function $\mathbb{1}_X$ is sent to $c(TX) \cap [X]$ if X is nonsingular. The class $c_{\mathrm{SM}}(X)$ is the image of $\mathbb{1}_X$ in A_*X, regardless of the singularities of X. More generally, we denote by $c_{\mathrm{SM}}(\varphi)$ the image of $\varphi \in F(X)$ in A_*X. With this notation, if $f : X \to Y$ is a proper map, then

$$c_{\mathrm{SM}}(f_*\varphi) = f_*c_{\mathrm{SM}}(\varphi).$$

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This covariance property implies easily that the natural transformation is unique: indeed, by resolution of singularities the CSM class of any scheme X as above is determined by the CSM classes of a suitable selection of nonsingular varieties mapping to X. Also note that if the Euler characteristic of the fibers of a proper morphism $f : X \to Y$ is a constant χ, then covariance implies that

$$f_* c_{\mathrm{SM}}(X) = \chi \cdot c_{\mathrm{SM}}(Y). \tag{2.1}$$

By abuse of language, if $X \subseteq V$, then we may denote by $c_{\mathrm{SM}}(X)$ the class $c_{\mathrm{SM}}(\mathbb{1}_X)$ in the Chow group *of* V. With this convention, the CSM class satisfies a basic inclusion–exclusion principle: for $X, Y \subseteq V$, we have

$$c_{\mathrm{SM}}(X \cup Y) = c_{\mathrm{SM}}(X) + c_{\mathrm{SM}}(Y) - c_{\mathrm{SM}}(X \cap Y).$$

This is often useful in concrete computations.

2.2. The scheme $\mathscr{Y}$. We now fix a nonsingular variety V, and a closed subscheme $X \subseteq V$. We denote by ι the inclusion map $X \hookrightarrow V$.

We may view X as the zero-scheme of a section of a vector bundle E. Indeed, we may choose $E = \mathrm{Spec}(\mathrm{Sym}\,\mathscr{E})$, where $\mathscr{E}$ is any locally free sheaf surjecting onto the ideal sheaf $\mathscr{I}_{X,V}$ of X in V; the composition $s^\vee : \mathscr{E} \twoheadrightarrow \mathscr{I}_{X,V} \hookrightarrow \mathscr{O}_V$ corresponds to a section $s : V \to E$, such that $X = Z(s)$. (Cf. [Ful84, B.8.2].)

Recall that we have an exact sequence

$$\mathscr{I}_{X,V}/\mathscr{I}^2_{X,V} \longrightarrow \Omega_V|_X \longrightarrow \Omega_X \longrightarrow 0$$

[Har77, Proposition 8.12]. Restricting the surjection $\mathscr{E} \twoheadrightarrow \mathscr{I}_{X,V}$ to X and composing with the first morphism in this sequence determines a morphism of locally free sheaves

$$\mathscr{E}|_X \longrightarrow \Omega_V|_X,$$

or equivalently a morphism of vector bundles on X:

$$\phi : E^\vee|_X \longrightarrow T^*V|_X. \tag{2.2}$$

Now consider the projective bundle (of lines) $\rho : \mathbb{P}(E^\vee|_X) \to X$. Composing the pull-back of (2.2) with the inclusion of the tautological subbundle, we obtain a morphism

$$\sigma_{\mathscr{Y}} : \mathscr{O}(-1) \to \rho^* E^\vee|_X \to \rho^* T^*V|_X$$

of vector bundles over $\mathbb{P}(E^\vee|_X)$.

DEFINITION 2.1. With notation as above, we define $\mathscr{Y} \subseteq \mathbb{P}(E^\vee|_X)$ to be the zero-scheme of $\sigma_{\mathscr{Y}} : \mathscr{Y} = Z(\sigma_{\mathscr{Y}})$. ⌟

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Set-theoretically, $\mathscr{Y}$ consists of points $(\underline{e}, x)$, with $\underline{e}$ in the fiber of $\mathbb{P}(E^\vee|_X)$ at $x \in X$, such that $\underline{e} \in \ker \phi_x$, where ϕ is the morphism in (2.2).

In local analytic coordinates $(x_1, \ldots, x_n)$ for V at x, $s^\vee$ describes the ideal of X in terms of a choice of generators $f_0, \ldots, f_r \in k[[x_1, \ldots, x_n]]$, where $r+1 = \operatorname{rk} E$. The morphism ϕ is given by the $n \times (r+1)$ matrix

$$\begin{pmatrix} \frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}$$

and $\mathscr{Y}$ is defined as a subscheme of $\mathbb{P}(E^\vee|_X)$ by the vanishing

$$\begin{pmatrix} \frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial x_n} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} e_0 \\ \vdots \\ e_r \end{pmatrix} = 0 \tag{2.3}$$

with $\underline{e} = (e_0 : \cdots : e_r)$. Thus, $\mathscr{Y}$ detects linear relations among differentials of the chosen generators for X.

Remark 2.2. As Terry Gaffney pointed out, $\mathscr{Y}$ may therefore be viewed as a 'Tyurina transform' associated with the morphism ϕ. ⌟

One source of such relations are the singularities of X.

Example 2.3. Assume that $r = 0$, so that X is the hypersurface in V with local equation $f_0 = 0$. Then $\mathscr{Y} \subseteq X \times \mathbb{P}^0 \cong X$ is locally defined by the vanishing of the partials of f_0; that is, in this case $\mathscr{Y}$ is the singularity subscheme of X. ⌟

In the case of Example 2.3, $\mathscr{Y}$ is empty if X is nonsingular. More generally, $\mathscr{Y}$ is empty if X is a smooth complete intersection realized as the zero-scheme of a regular section of a bundle E of rank equal to $\operatorname{codim} X$; in this case, ϕ_x has full rank for all $x \in X$.

However, this is not typical. Linear relations among differentials of the generators may be due to reasons other than the singularities of X. For example, two generators may coincide or one of the generators may be identically 0.

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EXAMPLE 2.4. Let $X \subseteq V$ be a smooth hypersurface, given as the zero-scheme of a section f of a line bundle L. With notation as above, let $E = L^{\oplus r+1}$, with $r > 0$, and let $s = (f, \ldots, f)$. Then $\mathscr{Y}$ is a $\mathbb{P}^{r-1}$ bundle over X. ⌟

2.3. The main theorem. Let $\mathscr{V} = \mathbb{P}(E^\vee)$, and let $\pi : \mathscr{V} \to V$ denote the projection:

$$\begin{tikzcd} \mathbb{P}(E^\vee|_X) \arrow[r, hook] \arrow[d, "\rho"'] & \mathscr{V} \arrow[d, "\pi"] \\ X \arrow[r, hook] & V \end{tikzcd}$$

With this notation, $\mathbb{P}(E^\vee|_X) = \pi^{-1}(X)$.

It will be useful to view $\mathscr{Y}$ as a subscheme of $\mathscr{V}$; as such, the ideal of $\mathscr{Y}$ is generated by the pull-back of $\mathscr{I}_{X,V}$ and by the relations (2.3). The closed embedding $\mathscr{Y} \subseteq \mathscr{V}$ determines the *Segre class* $s(\mathscr{Y}, \mathscr{V}) \in A_*\mathscr{Y}$ [**Ful84**, Ch. 4]. We implicitly often view this class as a class in $A_*\mathscr{V}$, omitting the evident push-forward notation.

We need the following notation from [**Alu94**, Section 2]. Let M be an ambient variety, and let Z be a subscheme of M. Further, let $\mathscr{L}$ be a line bundle on Z. For $\alpha \in A_*Z$, write $\alpha = \sum_i \alpha^{(i)}$, where $\alpha^{(i)}$ is the component of α with codimension i *in* M. We define

$$\alpha \otimes_M \mathscr{L} := \sum_i c(\mathscr{L})^{-i} \cap \alpha^{(i)}, \quad \alpha^\vee := \sum_i (-1)^i \alpha^{(i)}.$$

The subscript M may be omitted in context (and the notation $^\vee$ must be understood in context, since it also depends on the dimension of the ambient variety M). This notation satisfies simple compatibility properties with the notion of dual of vector bundles and of tensor product of vector bundles by line bundles, in terms of their effect on Chern classes. Further, it is an *action* in the sense that if $\mathscr{L}_1$ and $\mathscr{L}_2$ are line bundles on Z, then $\alpha \otimes (\mathscr{L}_1 \otimes \mathscr{L}_2) = (\alpha \otimes \mathscr{L}_1) \otimes \mathscr{L}_2$. (See [**Alu94**, Propositions 1 and 2].)

The following is our main result.

THEOREM 2.5. *Let V be a nonsingular variety, and let $\iota : X \to V$ be a closed subscheme. Assume $X = Z(s)$ for a section s of a vector bundle E on V, and construct $\mathscr{Y}, \mathscr{V}$ as above. Then*

$$\iota_* c_{\mathrm{SM}}(X) - \frac{c(TV)}{c(E)} c_{\mathrm{top}}(E) \cap [V] = c(TV) \cap \pi_* \left(\frac{c(\pi^* E^\vee \otimes \mathscr{O}(1))}{c(\mathscr{O}(1))} \cap (s(\mathscr{Y}, \mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1)) \right). \quad (2.4)$$

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The proof of Theorem 2.5 is given in Section 3. We record here the following consequence and several special cases illustrating the statement.

COROLLARY 2.6. *With notation as above, let* $\operatorname{rk} E = r + 1$. *Then*

- *If* $r \geqslant \dim V$, *then*

$$\iota_* c_{\mathrm{SM}}(X) = c(TV) \cap \pi_*\left(\frac{c(\pi^* E^\vee \otimes \mathscr{O}(1))}{c(\mathscr{O}(1))} \cap (s(\mathscr{Y}, \mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))\right).$$

- *If* X *is a local complete intersection in* V *and* $r + 1 = \operatorname{codim} X$, *then*

$$\iota_* \mathscr{M}(X) = (-1)^{\dim X+1} c(TV) \cap \pi_*\left(\frac{c(\pi^* E^\vee \otimes \mathscr{O}(1))}{c(\mathscr{O}(1))} \cap (s(\mathscr{Y}, \mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))\right),$$

where $\mathscr{M}(X)$ *denotes the Milnor class of* X.

Proof. If $r \geqslant \dim V$, then $\operatorname{rk} E > \dim V$, hence $c_{\mathrm{top}}(E) = 0$ for dimensional reasons. The first formula then follows immediately from Theorem 2.5.

Concerning the second formula: if X is a local complete intersection, X is the zero-scheme of a section of a vector bundle E, and $\operatorname{rk} E = \operatorname{codim} X$, then $E|_X \cong N_X V$, and $(TV|_X)/(E|_X)$ is the virtual tangent bundle of X. Further, $c_{\mathrm{top}}(E) \cap [V] = \iota_*[X] \in A_* V$. Therefore,

$$\frac{c(TV)}{c(E)} c_{\mathrm{top}}(E) \cap [V] = \iota_* c_{\mathrm{vir}}(X)$$

in this case. By definition of Milnor class (1.2), we have

$$\iota_* c_{\mathrm{SM}}(X) - \frac{c(TV)}{c(E)} c_{\mathrm{top}}(E) \cap [V] = \iota_*(c_{\mathrm{SM}}(X) - c_{\mathrm{vir}}(X)) = (-1)^{\dim X+1} \iota_* \mathscr{M}(X),$$

and the second formula follows from Theorem 2.5. □

EXAMPLE 2.7. Let $V = \mathbb{P}^n$. Every $X \subseteq V$ may be defined by a homogeneous ideal generated by forms of degree d, if $d \gg 0$. Choose such a d, and choose generators $F_0, \ldots, F_r$ of $H^0(\mathbb{P}^n, \mathscr{I}_{X,\mathbb{P}^n}(d))$. View $(F_0, \ldots, F_r)$ as a section of $E = \mathscr{O}(d)^{\oplus(r+1)}$. We have

$$\mathbb{P}(E^\vee) = \mathbb{P}(\mathscr{O}(-dH)^{\oplus(r+1)}) \cong \mathbb{P}^n \times \mathbb{P}^r,$$

where H denotes the hyperplane class in $\mathbb{P}^n$ (and its pull-back). Denoting by h the hyperplane class in $\mathbb{P}^r$, we have $c_1(\mathscr{O}_{\mathbb{P}(E^\vee)}(1)) = dH + h$. Therefore,

$$
\begin{aligned}
&c(TV)\cap\pi_*\left(\frac{c(\pi^*E^\vee\otimes\mathscr{O}(1))}{c(\mathscr{O}(1))}\cap(s(\mathscr{Y},\mathscr{V})^\vee\otimes_{\mathscr{V}}\mathscr{O}(1))\right)\\
&\quad=(1+H)^{n+1}\cap\pi_*\Bigg(\frac{(1-dH+(dH+h))^{r+1}}{1+dH+h}\\
&\qquad\cap(s(\mathscr{Y},\mathscr{V})^\vee\otimes_{\mathscr{V}}\mathscr{O}(dH+h))\Bigg),
\end{aligned}
$$

and the formulas in Corollary 2.6 specialize to Theorems 1.1 and 1.2. ⌟

EXAMPLE 2.8. Let X be a hypersurface in V, given as the zero-scheme of a section f of the line bundle $E=\mathscr{O}(X)$. As noted in Example 2.3, $\mathscr{Y}$ equals the singularity subscheme JX of X in this case. We have $\mathscr{V}=\mathbb{P}(E^\vee)=\mathbb{P}(\mathscr{O}(-X))\cong V$, and π is the identity under this identification. The line bundle $\mathscr{O}(1)$ is tautologically isomorphic to $\mathscr{O}(X)$. Since $\operatorname{rk}E=\operatorname{codim}X$, the second formula in Corollary 2.6 applies, giving

$$
\iota_*\mathscr{M}(X)=(-1)^{\dim X}c(TV)\cap\left(\frac{c(\mathscr{O}(-X)\otimes\mathscr{O}(X))}{c(\mathscr{O}(X))}\cap(s(JX,V)^\vee\otimes_V\mathscr{O}(X))\right),
$$

that is,

$$
\iota_*(c_{\mathrm{SM}}(X)-c_{\mathrm{vir}}(X))=c(TV)\cap(c(\mathscr{O}(X))^{-1}\cap(s(JX,V)^\vee\otimes_V\mathscr{O}(X))).
$$

This is the main result of [**Alu99**] (after push-forward by ι_* to the ambient nonsingular variety V). ⌟

EXAMPLE 2.9. To illustrate the dependence of the result on the rank of E in a particularly transparent case, let $X\subseteq V$ be a smooth hypersurface. We can view X as the zero-scheme of a section of $E=\mathscr{O}(X)^{\oplus(r+1)}$, of the form (for example) $(f,\dots,f)$. As observed in Example 2.4, $\mathscr{Y}\subseteq\mathbb{P}(E^\vee|_X)$ is then a $\mathbb{P}^{r-1}$ bundle over X. We can identify $\mathscr{V}=\mathbb{P}(E^\vee)$ with $V\times\mathbb{P}^r$; let h be the pull-back of the hyperplane class from the second factor, and π the projection onto the first factor. Then $\mathscr{O}_{\mathbb{P}(E^\vee)}(1)\cong\mathscr{O}(h+\pi^*X)$, and $\mathscr{Y}$ is a complete intersection of $\mathbb{P}(E^\vee|_X)=\pi^{-1}(X)$ and a hyperplane in the second factor (with equation $e_0+\cdots+e_r=0$). We have

$$
s(\mathscr{Y},\mathscr{V})=\frac{h\cdot\pi^*X}{(1+h)(1+\pi^*X)}\cap[\mathscr{V}]
$$

as a class in $A_*\mathscr{V}$, hence

$$
\begin{aligned}
s(\mathscr{Y},\mathscr{V})^\vee\otimes_{\mathscr{V}}\mathscr{O}(1)&=\frac{h\cdot\pi^*X}{(1-h)(1-\pi^*X)}\otimes_{\mathscr{V}}\mathscr{O}(h+\pi^*X)\cap[\mathscr{V}]\\
&=\frac{h\cdot\pi^*X}{(1+\pi^*X)(1+h)}\cap[\mathscr{V}].
\end{aligned}
$$

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(Using [Alu94, Proposition 1].) Therefore, omitting evident pull-backs,

$$\begin{aligned}
&\frac{c(E^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))}{c(\mathscr{O}(1))} \cap (s(\mathscr{Y},\mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))\\
&= \frac{(1+h)^{r+1}}{1+h+X} \cdot \frac{h\cdot X}{(1+X)(1+h)} \cap [\mathscr{V}]\\
&= \frac{(1+h)^r}{(1+h+X)(1+X)} \cap [\mathscr{V}] = (1+h)^r \left(\frac{X}{1+X} - \frac{X}{1+h+X}\right) \cap [\mathscr{V}].
\end{aligned}$$

The push-forward of this class to $\mathscr{V}$ is determined by the coefficient of h^r in this expression. It is easy to verify that this equals

$$\left(\frac{X}{1+X} - \frac{X^{r+1}}{(1+X)^{r+1}}\right) \cap [V]$$

and it follows that

$$\begin{aligned}
c(TV) \cap \pi_*&\left(\frac{c(\pi^* E^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))}{c(\mathscr{O}(1))} \cap (s(\mathscr{Y},\mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))\right)\\
&= c(TV) \cap \left(\frac{X}{1+X} - \frac{X^{r+1}}{(1+X)^{r+1}}\right) \cap [V]\\
&= c(TX) \cap [X] - c(TV)\frac{c_1(\mathscr{O}(X))^{r+1}}{c(\mathscr{O}(X))^{r+1}} \cap [V]
\end{aligned}$$

in agreement with Theorem 2.5. ⌟

EXAMPLE 2.10. If $X = V$, we can represent X as the zero-scheme of the zero-section of any vector bundle E on V. In this case $\mathscr{Y} = \mathscr{V}$, so that

$$s(\mathscr{Y},\mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1) = [\mathscr{V}]^\vee \otimes_{\mathscr{V}} \mathscr{O}(1) = [\mathscr{V}].$$

Theorem 2.5 reduces then to the statement that

$$\pi_*\left(\frac{c(\pi^* E^\vee \otimes \mathscr{O}(1))}{c(\mathscr{O}(1))} \cap [\mathscr{V}]\right) = \left(1 - \frac{c_{\mathrm{top}}(E)}{c(E)}\right) \cap [V]. \tag{2.5}$$

This statement will in fact be an ingredient in the proof of Theorem 2.5, and will be (independently) proven in Section 3.3. ⌟

3. Proof

3.1. Roadmap. The proof of Theorem 2.5 relies on several ingredients. In Section 3.2, we give an alternative description of the scheme $\mathscr{Y}$ defined in

Section 2.2, as the singularity subscheme of a hypersurface $\mathscr{X}$ in $\mathbb{P}(E^\vee)$. In Section 3.3, we compute the push-forward of $c_{\text{vir}}(\mathscr{X})$, by standard techniques in intersection theory. In Section 3.4, we compute the push-forward of $c_{\text{SM}}(\mathscr{X})$ by applying the functoriality of CSM classes, and in Section 3.5, we use the main result of [**Alu99**] to establish Theorem 2.5.

In Section 3.6, we comment on related work of Callejas-Bedregal, Morgado, and Seade concerning Milnor classes of local complete intersections [**CBMS**]. The hypersurface $\mathscr{X}$ we use in the proof of Theorem 2.5 was to our knowledge first introduced in [**CBMS**] (in the local complete intersection case).

3.2. Alternative description of $\mathscr{Y}$.

With notation as in Section 2.2, dualize the inclusion of the tautological subbundle $\mathscr{O}(-1) \to \pi^*(E^\vee)$ to obtain a canonical morphism $\epsilon : \pi^*(E) \to \mathscr{O}(1)$. Composing with the pull-back of s gives a section of $\mathscr{O}(1)$ on $\mathscr{V}$:

$$\sigma_{\mathscr{X}} : \mathscr{V} \xrightarrow{\pi^* s} \pi^*(E) \xrightarrow{\epsilon} \mathscr{O}(1). \tag{3.1}$$

We let $\mathscr{X}$ denote the hypersurface of $\mathscr{V}$ defined as the zero-scheme of $\sigma_{\mathscr{X}} = \epsilon \circ \pi^* s$.

LEMMA 3.1. *The scheme $\mathscr{Y}$ is the singularity subscheme of $\mathscr{X}$.*

Proof. In local analytic coordinates as above, $\mathscr{X}$ is given by the equation

$$y_0 f_0 + \cdots + y_r f_r = 0, \tag{3.2}$$

whose Jacobian ideal defines the singularity subscheme of $\mathscr{X}$. From this and the coordinate description of $\mathscr{Y}$ given in Section 2.2, the statement is clear. More intrinsically, the ideal of $\mathscr{X}$ in $\mathscr{V}$ is $\mathscr{O}(-1)$; hence we have a canonical morphism

$$\mathscr{O}(-1)|_{\mathscr{X}} \longrightarrow \Omega_{\mathscr{V}}|_{\mathscr{X}},$$

and equivalently (tensor by $\mathscr{O}(1)$) a section

$$\sigma_{J\mathscr{X}} : \mathscr{X} \longrightarrow T^*\mathscr{V}|_{\mathscr{X}} \otimes \mathscr{O}(1)$$

of a twist of the cotangent bundle to $\mathscr{V}$. By definition, the singularity subscheme of $\mathscr{X}$ is the zero-scheme of this section. Now, we have the dual Euler exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (T^*_{\mathscr{V}/V}) \otimes \mathscr{O}(1) \arrow[r] & \pi^* E \arrow[r, "\epsilon"] & \mathscr{O}(1) \arrow[r] & 0 \\ & & \mathscr{V} \arrow[u, "\pi^* s"] \arrow[ur, "\sigma_{\mathscr{X}}"'] & & \end{tikzcd}$$

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where $T^*_{\mathscr{V}/V}$ is the relative cotangent bundle. As $\mathscr{X}$ is the zero-scheme of $\sigma_{\mathscr{X}}$, we obtain a section

$$\sigma' : \mathscr{X} \to (T^*_{\mathscr{V}/V}) \otimes \mathscr{O}(1)|_{\mathscr{X}}$$

which is seen to be compatible with $\sigma_{J\mathscr{X}}$: the diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi^* T^* V \otimes \mathscr{O}(1)|_{\mathscr{X}} \arrow[r] & T^*\mathscr{V} \otimes \mathscr{O}(1)|_{\mathscr{X}} \arrow[r] & (T^*_{\mathscr{V}/V}) \otimes \mathscr{O}(1)|_{\mathscr{X}} \arrow[r] & 0 \\ & & \mathscr{X} \arrow[u, "\sigma_{J\mathscr{X}}"] \arrow[ur, "\sigma'"'] & & \end{tikzcd}$$

commutes. The singularity subscheme of $\mathscr{X}$, that is, the zero-scheme $Z(\sigma_{J\mathscr{X}})$, is contained in $Z(\sigma') = Z(\pi^* s) = \pi^{-1}(X) = \mathbb{P}(E^\vee|_X)$. Restricting to $\mathbb{P}(E^\vee|_X)$, $\sigma_{J\mathscr{X}}$ induces a section

$$\sigma'' : \mathbb{P}(E^\vee|_X) \longrightarrow (\pi^* T^* V \otimes \mathscr{O}(1))|_{\mathbb{P}(E^\vee|_X)} = \rho^* T^* V|_X \otimes \mathscr{O}(1),$$

such that the singularity subscheme of $\mathscr{X}$ equals $Z(\sigma'')$. It is now easy to check that σ'' agrees with $\sigma_{\mathscr{Y}} \otimes \mathscr{O}(1)$, and it follows that the singularity subscheme of $\mathscr{X}$ coincides with $Z(\sigma_{\mathscr{Y}}) = \mathscr{Y}$. □

3.3. The push-forward of $c_{\mathrm{vir}}(X)$. Theorem 2.5 will follow from the computation of push-forwards of characteristic classes of $\mathscr{X}$. In this subsection we compute $\pi_*(c_{\mathrm{vir}}(X))$. For this purpose it will be useful to prove identity (2.5): as noted in Section 2.10, this simple statement is a particular case of Theorem 2.5, and it turns out that it is in fact one of the ingredients in its proof.

LEMMA 3.2. *Let E be a vector bundle on a variety V, and let $\pi : \mathbb{P}(E^\vee) \to V$ be the projective bundle (of lines) of its dual $E^\vee$. Then*

$$\pi_*\left(\frac{c(\pi^* E^\vee \otimes \mathscr{O}(1))}{c(\mathscr{O}(1))} \cap [\mathbb{P}(E^\vee)]\right) = \left(1 - \frac{c_{\mathrm{top}}(E)}{c(E)}\right) \cap [V].$$

Proof. Let $\mathrm{rk}(E) = r + 1$. Using [**Ful84**, Remark 3.2.3(b)],

$$\frac{c(\pi^* E^\vee \otimes \mathscr{O}(1))}{c(\mathscr{O}(1))} = c(\mathscr{O}(1))^r + \sum_{i=1}^{r} c_i(\pi^* E^\vee) c(\mathscr{O}(1))^{r-i} + \frac{c_{r+1}(\pi^* E^\vee)}{c(\mathscr{O}(1))}. \quad (3.3)$$

We have

$$\pi_*(c(\mathscr{O}(1))^r \cap [\mathbb{P}(E^\vee)]) = \pi_*(c_1(\mathscr{O}(1))^r \cap [\mathbb{P}(E^\vee)]) = [V]:$$

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indeed, the other terms in the expansion of $(1+c_1(\mathscr{O}(1)))^r$ push forward to zero by [Ful84, Proposition 3.1(a)(i)], and the term $c_1(\mathscr{O}(1))^r \cap [\mathbb{P}(E^\vee)]$ pushes forward to $[V]$ by [Ful84, Proposition 3.1(a)(ii)].

The middle term in the right-hand side of (3.3) pushes forward to 0. Indeed, by the projection formula it is a combination of terms

$$c_i(E^\vee) \cap \pi_*(c_1(\mathscr{O}(1))^j \cap [\mathbb{P}(E^\vee)])$$

with $j<r$, and $\pi_*(c_1(\mathscr{O}(1))^j \cap [\mathbb{P}(E^\vee)]) \in A_{\dim V+r-j}(V) = (0)$ for $j<r$.

As for the last term in (3.3), recall that

$$\pi_*(c(\mathscr{O}(-1))^{-1} \cap [\mathbb{P}(E^\vee)]) = c(E^\vee)^{-1} \cap [V]:$$

indeed, this is essentially the definition of Chern class of a vector bundle according to [Ful84, Section 3.2]. It follows that

$$\pi_*(c(\mathscr{O}(1))^{-1} \cap [\mathbb{P}(E^\vee)]) = (-1)^r c(E)^{-1} \cap [V],$$

and therefore

$$\begin{aligned}\pi_*\left(\frac{c_{r+1}(\pi^* E^\vee)}{c(\mathscr{O}(1))} \cap [\mathbb{P}(E^\vee)]\right) &= \pi_*((-1)^{r+1} c_{r+1}(\pi^* E) c(\mathscr{O}(1))^{-1} \cap [\mathbb{P}(E^\vee)]) \\ &= -c_{r+1}(E) c(E)^{-1} \cap [V],\end{aligned}$$

again by the projection formula. □

The computation of the push-forward of $c_{\mathrm{vir}}(\mathscr{X})$ follows from this lemma.

PROPOSITION 3.3. *Let V be a nonsingular variety, and $X \subseteq V$ the zero-scheme of a section of a vector bundle E of rank $r+1$ on V. Let $\mathscr{V} = \mathbb{P}(E^\vee)$, and let $\mathscr{X}$ be the hypersurface of $\mathscr{V}$ defined in Section 3.2. Then*

$$\pi_*(c_{\mathrm{vir}}(\mathscr{X})) = r \cdot c_{\mathrm{SM}}(V) + \frac{c(TV)}{c(E)} c_{\mathrm{top}}(E) \cap [V]$$

in $A_* V$.

Proof. By definition, $c_{\mathrm{vir}}(\mathscr{X}) = c(T\mathscr{V}) c(\mathscr{O}(X))^{-1} \cap [V] = c(T\mathscr{V})/(1+\mathscr{X}) \cap [\mathscr{X}]$; we implicitly view this as a class in $A_*\mathscr{V}$. By the Euler sequence, the Chern class of the relative tangent bundle of $\mathscr{V} = \mathbb{P}(E^\vee)$ is given by $c(T_{\mathscr{V}/V}) = c(E^\vee \otimes \mathscr{O}(1))$; therefore,

$$c(T\mathscr{V}) = \pi^* c(TV) c(E^\vee \otimes \mathscr{O}(1)). \tag{3.4}$$

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Further, by the normalization and covariance of CSM classes (see (2.1)),

$$\pi_*(c(T\mathscr{V})\cap[\mathscr{V}])=\pi_* c_{\mathrm{SM}}(\mathscr{V})=(r+1)c_{\mathrm{SM}}(V).$$

(Exercise: Prove this from (3.4), without using covariance of CSM classes.) Using these facts and Lemma 3.2:

$$\begin{aligned}\pi_*\left(\frac{c(T\mathscr{V})}{1+\mathscr{X}}\cap[\mathscr{X}]\right)&=\pi_*\left(c(T\mathscr{V})\cap\left([\mathscr{V}]-\frac{1}{1+\mathscr{X}}\cap[\mathscr{V}]\right)\right)\\&=\pi_*(c(T\mathscr{V})\cap[\mathscr{V}])-c(TV)\cap\pi_*\left(c(T_{\mathbb{P}(E^\vee)/V})\frac{1}{1+\mathscr{X}}\cap[\mathscr{V}]\right)\\&=(r+1)\,c(TV)\cap[V]-c(TV)\cap\pi_*\left(\frac{c(E^\vee\otimes\mathscr{O}(1))}{c(\mathscr{O}(1))}\cap[\mathscr{V}]\right)\\&=r\cdot c(TV)\cap[V]+\frac{c(TV)}{c(E)}c_{\mathrm{top}}(E)\cap[V]\end{aligned}$$

as stated. □

3.4. The push-forward of $c_{\mathrm{SM}}(\mathscr{X})$.

Proposition 3.3 gives the push-forward of $c_{\mathrm{vir}}(\mathscr{X})$. Using the covariance of CSM classes, it is straightforward to obtain the push-forward of $c_{\mathrm{SM}}(\mathscr{X})$.

PROPOSITION 3.4. *Let V be a nonsingular variety, and $\iota : X \hookrightarrow V$ the zero-scheme of a section of a vector bundle E of rank $r+1$ on V. Let $\mathscr{V}=\mathbb{P}(E^\vee)$, and let $\mathscr{X}$ be the hypersurface of $\mathscr{V}$ defined in Section 3.2. Then*

$$\pi_*(c_{\mathrm{SM}}(\mathscr{X}))=r\cdot c_{\mathrm{SM}}(V)+\iota_* c_{\mathrm{SM}}(X)$$

in A_*V.

Proof. By definition of CSM class and by covariance,

$$\pi_*(c_{\mathrm{SM}}(\mathscr{X}))=\pi_* c_{\mathrm{SM}}(\mathbb{1}_{\mathscr{X}})=c_{\mathrm{SM}}(\pi_*\mathbb{1}_{\mathscr{X}}).$$

Now recall (Section 2.1) that $\pi_*\mathbb{1}_{\mathscr{X}}$ is the function assigning to $p\in V$ the Euler characteristic of the fiber of $\mathscr{X}$ over p. Use notation as above; in particular, $s: V\to E$ is the section defining X. If $p\in X$, then $s(p)=0$, and it follows that $s_{\mathscr{X}}=\pi^*s\circ\epsilon\equiv 0$ along $\pi^{-1}(p)$. That is, the fiber of $\mathscr{X}$ over $p\in X$ equals the fiber of $\mathscr{V}=\mathbb{P}(E^\vee)$, so it is an r-dimensional projective space. Therefore,

$$p\in X \implies \pi_*(\mathbb{1}_{\mathscr{X}})(p)=\chi(\mathbb{P}^r)=r+1. \tag{3.5}$$

If $p \notin X$, then $s(p) \neq 0$; $\pi^* s$ is then a fixed vector $(a_0, \ldots, a_r)$ of E_p along the fiber $\pi^{-1}(p)$. The vanishing of $s_{\mathscr{X}}$ at $(e_0 : \cdots : e_r) \in \pi^{-1}(p)$ is then equivalent to the linear equation

$$a_0 e_0 + \cdots + a_r e_r = 0.$$

It follows that the fiber of $\mathscr{X}$ over $p \notin X$ is a hyperplane $\mathbb{P}^{r-1}$ in the fiber $\pi^{-1}(p) \cong \mathbb{P}^r$. Therefore,

$$p \notin X \implies \pi_*(\mathbb{1}_{\mathscr{X}})(p) = \chi(\mathbb{P}^{r-1}) = r. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain that

$$\pi_* \mathbb{1}_{\mathscr{X}} = r \cdot \mathbb{1}_V + \mathbb{1}_X,$$

and the covariance of CSM classes concludes the proof. □

3.5. End of the proof. After these preliminaries we are ready to prove the main result.

Proof of Theorem 2.5. Applying [**Alu99**, Theorem I.4] to the hypersurface $\mathscr{X}$ gives

$$c_{\mathrm{SM}}(\mathscr{X}) = c_{\mathrm{vir}}(\mathscr{X}) + c(T\mathscr{V})c(\mathscr{O}(1))^{-1} \cap (s(\mathscr{Y}, \mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1)).$$

Here we used the fact that $c_{\mathrm{vir}}(\mathscr{X}) = c(T\mathscr{V})c(\mathscr{O}(\mathscr{X}))^{-1} \cap [X] = c(T\mathscr{V}) \cap s(\mathscr{X}, \mathscr{V})$, and the fact that $\mathscr{Y}$ is the singularity subscheme of $\mathscr{X}$, proven in Lemma 3.1. Pushing forward to V and using Propositions 3.3 and 3.4:

$$\begin{aligned}
&r \cdot c_{\mathrm{SM}}(V) + \iota_* c_{\mathrm{SM}}(X) \\
&\quad = \pi_* c_{\mathrm{SM}}(\mathscr{X}) \\
&\quad = \pi_*(c_{\mathrm{vir}}(\mathscr{X}) + c(T\mathscr{V})c(\mathscr{O}(1))^{-1} \cap (s(\mathscr{Y}, \mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))) \\
&\quad = r \cdot c_{\mathrm{SM}}(V) + \frac{c(TV)}{c(E)} c_{\mathrm{top}}(E) \cap [V] \\
&\qquad + \pi_*(c(T\mathscr{V})c(\mathscr{O}(1))^{-1} \cap (s(\mathscr{Y}, \mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))).
\end{aligned}$$

Therefore,

$$\iota_* c_{\mathrm{SM}}(X) - \frac{c(TV)}{c(E)} c_{\mathrm{top}}(E) \cap [V] = \pi_*\left(\frac{c(T\mathscr{V})}{c(\mathscr{O}(1))} \cap (s(\mathscr{Y}, \mathscr{V})^\vee \otimes_{\mathscr{V}} \mathscr{O}(1))\right).$$

The statement of Theorem 2.5 follows by applying (3.4) and the projection formula. □

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3.6. Milnor classes. Using the terminology of Milnor classes, Propositions 3.3 and 3.4 immediately imply the following statement.

PROPOSITION 3.5. *With notation as above,*

$$\pi_*\mathscr{M}(\mathscr{X}) = (-1)^{\dim \mathscr{X}}\left(\frac{c(TV)}{c(E)}\, c_{\mathrm{top}}(E)\cap[V] - \iota_* c_{\mathrm{SM}}(X)\right)$$

in A_*V.

In particular, paying careful attention to the signs gives:

COROLLARY 3.6. *Assume* $\iota : X \hookrightarrow V$ *is a local complete intersection, defined as the zero-scheme of a regular section of a bundle of rank* $\operatorname{codim}_X V$. *Then*

$$\pi_*\mathscr{M}(\mathscr{X}) = \iota_*\mathscr{M}(X) \tag{3.7}$$

in A_*V.

In the case of local complete intersections, the hypersurface $\mathscr{X}$ was introduced in [CBMS]. In fact, in [CBMS, Theorem 6.4], Callejas-Bedregal, Morgado, and Seade obtain a *different* expression relating the Milnor classes of $\mathscr{X}$ and X in the local complete intersection case. Comparing (3.7) and the expression from [CBMS] may lead to nontrivial identities for Chern classes of bundles associated with local complete intersections. It would be interesting to explore these consequences.

4. CSM from Segre zeta functions

4.1. Segre zeta functions. The results proven in this note draw a direct bridge between Segre classes and CSM classes. This should allow us to transfer information between these two notions; known facts about Segre classes should tell us something about CSM classes. This section is an example of this transfer.

It is known [Alu17] that Segre classes of subschemes of projective space admit the following description. Let $f_0, \dots, f_m$ be forms of degrees $a_0, \dots, a_m$ respectively, in variables $x_0, \dots, x_n$. For $N \geqslant n$, let $\iota_N : Z_N \hookrightarrow \mathbb{P}^N$ be the subscheme defined by the ideal $(f_0, \dots, f_m)$. Then there exists a rational function

$$\zeta(t) = \frac{P(t)}{(1+a_0t)\cdots(1+a_mt)},$$

with $P(t)$ a polynomial with nonnegative coefficients and leading term equal to $a_0\cdots a_m t^{m+1}$, such that

$$\iota_{N*}s(Z_N, \mathbb{P}^N) = \zeta(H)\cap[\mathbb{P}^N].$$

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Here H denotes the hyperplane class. We call $\zeta(t)$ the 'Segre zeta function' determined by the forms $f_0, \dots, f_m$.

A version of this result holds for subschemes of *products* of projective spaces [**Jor**, Section 5.2]. Let $\varphi_0, \dots, \varphi_m$ be bihomogeneous polynomials of bidegrees (a_i, b_i), $i = 0, \dots, m$, in variables $x_0, \dots, x_n$, $y_0, \dots, y_r$. For $N \geqslant n$, $R \geqslant r$, let $\iota_{N,R} : Z_{N,R} \hookrightarrow \mathbb{P}^N \times \mathbb{P}^R$ be the subscheme defined by the ideal $(\varphi_0, \dots, \varphi_m)$. Then there exists a rational function

$$\zeta(t,u) = \frac{P(t,u)}{(1+a_0t+b_0u)\cdots(1+a_mt+b_mu)},$$

with $P(t,u)$ a polynomial with leading term $\prod_i(a_it+b_iu)$, such that

$$\iota_{N,R*}s(Z_{N,R}, \mathbb{P}^N \times \mathbb{P}^R) = \zeta(H,h) \cap [\mathbb{P}^N \times \mathbb{P}^R]$$

where H, respectively, h denotes the pull-back of the hyperplane class from $\mathbb{P}^N$, respectively, $\mathbb{P}^R$.

4.2. Statement of the result. Theorem 1.1 may be expressed in terms of these two-variable zeta functions. In fact, we are going to obtain CSM classes directly in terms of the numerator of the zeta function determined by the bihomogeneous polynomials defining the scheme $\mathscr{Y}$. This may simplify the application of Theorem 1.1, and also has the advantage of simultaneously computing the CSM classes of the subschemes $X_N \subseteq \mathbb{P}^N$ defined by a choice of forms in $x_0, \dots, x_n$, for all $N \geqslant n$. (This information could be assembled in a 'Segre–Schwartz–MacPherson zeta function'.)

We use the notation introduced in Section 1.2: $F_0, \dots, F_r$ are homogeneous polynomials in $x_0, \dots, x_n$, of a fixed degree d; the corresponding subscheme $\mathscr{Y}$ of $\mathbb{P}^n \times \mathbb{P}^r$ is defined by the ideal generated by

$$F_0, \dots, F_r; \quad \text{and} \quad y_0\frac{\partial F_0}{\partial x_i} + \cdots + y_r\frac{\partial F_r}{\partial x_i}, \quad i = 0, \dots, n. \tag{4.1}$$

The bidegrees of the generators are $(d, 0)$, $(d-1, 1)$; some of the generators may vanish, in which case we view 0 as a form of the corresponding (bi)degree. The chosen generators determine a Segre zeta function for $\mathscr{Y}$:

$$\zeta(t,u) = \frac{P(t,u)}{(1+dt)^{r+1}(1+(d-1)t+u)^{n+1}}. \tag{4.2}$$

Therefore, we obtain a well-defined polynomial $P(t,u) \in \mathbb{Z}[t,u]$. This polynomial has degree $n+r+2$, and its term of highest degree is

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$(dt)^{r+1}((d-1)t+u)^{n+1}$. As we see (Remarks 4.3, 4.4), it is actually not necessary to know all terms of the polynomial $P(t,u)$ in order to apply the following result: the terms of degree $\leqslant n+1$ in t and $\leqslant r+1$ in u suffice, and these are determined by the Segre class of the subscheme $\mathscr{Y}_{n+1,r+1}$ of $\mathbb{P}^{n+1}\times\mathbb{P}^{r+1}$ defined by the ideal generated by the forms listed in (4.1).

THEOREM 4.1. *For $N\geqslant n$, let $\iota_N : X_N \hookrightarrow \mathbb{P}^N$ be the subscheme defined by the degree-d forms $F_0,\dots,F_r\in k[x_0,\dots,x_n]$. With notation as above, let $\gamma(t)$ be the coefficient of u^{r+1} in the polynomial*

$$Q(t,u) := (1+dt+u)^{n+r+2}\cdot P\left(\frac{-t}{1+dt+u},\frac{-u}{1+dt+u}\right).$$

Then

$$\iota_{N*}c_{\mathrm{SM}}(X_N) = (1+H)^{N-n}\gamma(H)\cap[\mathbb{P}^N],$$

where H is the hyperplane class in $\mathbb{P}^N$.

REMARK 4.2. The transformation

$$(t,u)\mapsto\left(\frac{-t}{1+dt+u},\frac{-u}{1+dt+u}\right)$$

is an involution, and sends $(1+dt+u)$ to $(1+dt+u)^{-1}$. It follows that the operation $P\mapsto Q$ defined in the statement of Theorem 4.1 is an involution. ⌟

REMARK 4.3. It will follow from the proof that $Q(t,u)$ is a polynomial of degree $r+1$ in u; this does not appear to be evident from the definition given in the statement. Thus, $\gamma(t)$ is actually the leading coefficient of $Q(t,u)$ viewed as a polynomial in u. Since terms in $P(t,u)$ of degree $>r+1$ in u do not contribute to the coefficient of u^{r+1} in $Q(t,u)$, $\gamma(t)$ is in fact determined by the terms of $P(t,u)$ of degree $\leqslant r+1$ in u. ⌟

REMARK 4.4. Since $Q(t,u)$ has degree $r+1$ in u,

$$v^{r+1}Q\left(t,\frac{1}{v}\right)$$

is a polynomial and $\gamma(t)$ is its constant term with respect to v. Applying the involution,

$$v^{r+1}Q\left(t,\frac{1}{v}\right)=\frac{(1+v+dtv)^{n+r+2}}{v^{n+1}}P\left(\frac{-tv}{1+v+dtv},\frac{-1}{1+v+dtv}\right),$$

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and therefore,

$$P\left(\frac{-tv}{1+v+dtv}, \frac{-1}{1+v+dtv}\right) = \gamma(t)\cdot v^{n+1} + \text{higher order terms in } v. \quad (4.3)$$

This gives an alternative computation of the term $\gamma(t)$ obtained in Theorem 2.5. It also shows that the terms of $P(t,u)$ of degree $> n+1$ in t do not affect $\gamma(t)$. (However, note that (4.3) may be affected by terms of $P(t,u)$ of degree $\geqslant r+1$ in u. This limits its applicability.)

Summarizing, only the terms of $P(t,u)$ of degrees $\leqslant n+1$ in t and $\leqslant r+1$ in u are needed in order to apply Theorem 4.1.

4.3. Proof of Theorem 4.1.

The proof of Theorem 4.1 will use the following simple observation, for which we do not have a reference.

LEMMA 4.5. *Let $S(h)$ be a power series with coefficients in a ring. Assume that the coefficient C of h^R in $(1+h)^R\cdot S(h)$ is nonzero and independent of R for $R \geqslant N$. Then $(1+h)^N S(h)$ is a polynomial of degree N in h, with leading coefficient C.*

Proof. The coefficient of h^N in $(1+h)^N S(h)$ is C by hypothesis. Arguing by contradiction, assume that $(1+h)^N S(h)$ is *not* a polynomial of degree N; then it must have a first nonzero term $s_M h^M$ with $M > N$. Note that then

$$(1+h)^{M-1} S(h) = s_0 + \cdots + Ch^{M-1} + s_M h^M + \cdots .$$

It follows that

$$(1+h)^M S(h) = s_0 + \cdots + (C + s_M)h^M + \cdots ,$$

so the coefficient of h^M in $(1+h)^M S(h)$ is $C + s_M \neq C$, contrary to the hypothesis. □

We are now ready to prove Theorem 4.1. Its derivation from Theorem 1.1 is a good exercise in the use of the properties of the notation $\otimes$, $^\vee$.

Proof of Theorem 4.1. Since the ideal of X_N is generated by $F_0, \ldots, F_r$, the corresponding subscheme $\iota : \mathscr{Y}_{N,R} \hookrightarrow \mathscr{V}_{N,R} := \mathbb{P}^N \times \mathbb{P}^R$ is defined by the forms listed in (4.1), for all $N \geqslant n$ and $R \geqslant r$. (We may choose $F_{r+1} = \cdots = F_R = 0$.) Therefore, the push-forward of the Segre class of $\mathscr{Y}_{N,R}$ is given by the zeta function (4.2):

$$\iota_* s(\mathscr{Y}_{N,R}, \mathscr{V}_{N,R}) = \left(\frac{P(H,h)}{(1+dH)^{r+1}(1+(d-1)H+h)^{n+1}}\right) \cap [\mathbb{P}^N \times \mathbb{P}^R].$$

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By properties of $\otimes$, $^\vee$ from [Alu94, Proposition 1],

$$\begin{aligned}
&\iota_* s(\mathscr{Y}_{N,R}, \mathscr{V}_{N,R})^\vee \otimes_{\mathscr{V}_{N,R}} \mathscr{O}(dH+h)\\
&\quad = \frac{P(-H,-h)}{(1-dH)^{r+1}(1-(d-1)H-h)^{n+1}} \otimes_{\mathscr{V}_{N,R}} \mathscr{O}(dH+h)\\
&\quad = \frac{(1+dH+h)^{n+r+2}}{(1+h)^{r+1}(1+H)^{n+1}}(P(-H,-h)\otimes_{\mathscr{V}_{N,R}} \mathscr{O}(dH+h))\\
&\quad = \frac{(1+dH+h)^{n+r+2}P(\frac{-H}{1+dH+h},\frac{-h}{1+dH+h})}{(1+h)^{r+1}(1+H)^{n+1}}.
\end{aligned}$$

Let $Q(t,u)$ be the polynomial $(1+dt+u)^{n+r+2}P(-t/(1+dt+u), -u/(1+dt+u))$. By Theorem 1.1, $\iota_{N*}c_{\mathrm{SM}}(X_N)$ is the push-forward of

$$\begin{aligned}
&\frac{(1+H)^{N+1}(1+h)^{R+1}}{1+dH+h}(s(\mathscr{Y},\mathbb{P}^n\times\mathbb{P}^r)^\vee \otimes_{\mathbb{P}^n\times\mathbb{P}^r}\mathscr{O}(dH+h))\\
&\quad = (1+H)^{N-n}(1+h)^R\frac{Q(H,h)}{(1+h)^r(1+dH+h)}\cap[\mathbb{P}^N\times\mathbb{P}^R],
\end{aligned}$$

provided $R \geqslant N$. The push-forward is obtained by capping against $[\mathbb{P}^N]$ the coefficient of h^R in

$$(1+H)^{N-n}(1+h)^R\frac{Q(H,h)}{(1+h)^r(1+dH+h)}.$$

We view this expression as a power series in h with coefficients in $A_*(\mathbb{P}^N)$, and note that the coefficient of h^R is $\iota_{N*}c_{\mathrm{SM}}(X_N)$, *independently of* $R \geqslant N$. By Lemma 4.5,

$$(1+H)^{N-n}(1+h)^N\cdot\frac{Q(H,h)}{(1+h)^r(1+dH+h)}$$

is a *polynomial* in h with coefficients in $A_*(\mathbb{P}^N)=\mathbb{Z}[H]/(H^{N+1})$, of degree N and leading coefficient $\iota_{N*}c_{\mathrm{SM}}(X_N)$. It then follows that

$$(1+H)^{N-n}Q(H,h)$$

is a polynomial of degree $r+1$ in h, with leading coefficient $\iota_{N*}c_{\mathrm{SM}}(X_N)$, and this is the statement.

Note that this argument shows that $Q(t,u)$ is a polynomial of degree $r+1$ in u modulo t^{N+1} for every $N \gg 0$. It follows that it has degree $r+1$ in u as a polynomial in $\mathbb{Z}[t,u]$. $\square$

4.4. Two examples. We give two examples illustrating Theorem 4.1.

EXAMPLE 4.6. Consider the forms $F_0 = x_1x_2$, $F_1 = x_0x_2$, $F_2 = x_0x_1$. (Thus $n = r = 2$.) The corresponding scheme in $\mathbb{P}^N$ consists of the union of three codimension 2 subspaces meeting along a common codimension 3 subspace.

The generators of the ideal of $\mathscr{Y}$ in this example are

$$x_1x_2, \quad x_0x_2, \quad x_0x_1; \quad x_2y_1 + y_2x_1, \quad y_0x_2 + y_1x_2, \quad y_0x_1 + y_1x_0.$$

We have

$$\zeta(t,u) = \frac{P(t,u)}{(1+2t)^3(1+t+u)^3};$$

the requirement that $\zeta(t,u)$ evaluates the Segre class in $\mathbb{P}^3 \times \mathbb{P}^3$ determines the terms of $P(t,u)$ of degree $\leqslant n+1 = 3$ in t and $\leqslant r+1 = 3$ in u. With the aid of the Macaulay2 package [HJ] we get

$$P(t,u) = t^3 + 6t^3u + 3t^2u^2 + 18t^3u^2 + 3t^2u^3 + 8t^3u^3 + \text{higher order terms}.$$

The polynomial Q appearing in the statement of Theorem 4.1 is therefore

$$(1+2t+u)^6 \cdot P\left(\frac{-t}{1+2t+u}, \frac{-u}{1+2t+u}\right)$$
$$\equiv -t^3 + 3t^3u + (3t^2+3t^3)u^2 + (3t^2+t^3)u^3 \bmod t^4.$$

This is necessarily a polynomial of degree $r+1 = 3$ in u (Remark 4.3), and the coefficient of u^3 is $3t^2 + t^3$. By Theorem 4.1, we can conclude that

$$\iota_{N*}c_{\mathrm{SM}}(X_N) = (1+H)^{N-2}(3H^2 + H^3) \cap [\mathbb{P}^N]. \tag{4.4}$$

For example, for $N = 6$ this gives

$$\iota_{6*}c_{\mathrm{SM}}(X_6) = 3[\mathbb{P}^4] + 13[\mathbb{P}^3] + 22[\mathbb{P}^2] + 18[\mathbb{P}^1] + 7[\mathbb{P}^0].$$

The reader may enjoy verifying independently that (4.4) holds, by using the geometric description of X_N given at the beginning of this example. ⌟

EXAMPLE 4.7. In closing, we revisit the example given in Section 1.3, consisting of the complete intersection of $x_1x_2x_3 = 0$ and $x_0x_1^2 + x_2^3 = 0$. Here $n = 3$, $r = 1$, so the needed information can be extracted from the Segre class of the subscheme defined by

$$x_1x_2x_3, \quad x_0x_1^2 + x_2^3; \quad y_1x_1^2, \quad y_0x_2x_3 + 2y_1x_0x_1, \quad y_0x_1x_3 + 3y_1x_2^2, \quad y_0x_1x_2$$

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in $\mathbb{P}^4 \times \mathbb{P}^2$. According to [HJ], this Segre class is

$$(H^2 + 3H^2h + H^3 + 2H^2h^2 - 19H^3h - 16H^4 - 30H^3h^2 \\ + 69H^4h + 240H^4h^2) \cap [\mathbb{P}^4 \times \mathbb{P}^2],$$

where H, respectively h is the pull-back of the hyperplane class from the first, respectively second factor. It follows that the polynomial $P(t, u)$ corresponding to this example must be

$$\begin{aligned} P(t, u) = t^2 + 15t^3 + 79t^4 &+ (7t^2 + 75t^3 + 258t^4)u \\ &+ (20t^2 + 132t^3 + 216t^4)u^2 \\ &+ \text{higher order terms.} \end{aligned}$$

Applying the involution defined in the statement of Theorem 4.1 gives

$$Q(t, u) \equiv (t^2 - 3t^3 - 2t^4) + (-3t^2 + 3t^3 - t^4)u + (5t^2 + 3t^3 + t^4)u^2 \mod t^5$$

and we can conclude that the CSM class of the complete intersection defined by $x_1x_2x_3 = 0$ and $x_0x_1^2 + x_2^3 = 0$ in $\mathbb{P}^N$ pushes forward to

$$(1 + H)^{N-3}(5H^2 + 3H^3 + H^4) \cap [\mathbb{P}^N].$$

The reader can verify that for $N = 6$ this is in agreement with the result obtained in Section 1.3. ⌟

Acknowledgements

This work was supported in part by NSA grant H98230-16-1-0016. This work was carried out while the author was visiting the University of Toronto. The author thanks the University of Toronto for the hospitality.

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