AN EXTREME POSITIVE OPERATOR ON A POLYHEDRAL CONE

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In [**2**], R. Loewy and H. Schneider studied positive linear operators on circular cones. They characterised the extremal positive operators on these cones and noticed that such operators preserve the set of extreme rays of the cone in this case. They then conjectured that this property of extremal positive operators is true in general.

Shortly afterwards R. C. O'Brien [**3**] gave a counterexample to the conjecture, using a highly non-trivial result in [**2**]. Some time later the present author remarked in [**4**, Lemma 3.1] that a certain biquadratic form of M. D. Choi and T. Y. Lam [**3**] gives rise to an extremal positive operator on the cone of positive semidefinite matrices of order 3 under which *no* extreme ray of the cone is mapped to an extreme ray. Again, the proof of the relevant result is far from easy.

The purpose of the present note is to give a very simple example which shows that the conjecture of Loewy and Schneider fails even for polyhedral cones.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ denote the usual basis vectors for $\mathbb{R}^4$ and let $\mathbf{e}_5 = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4 = (1, 1, -1, -1)$. Let K be the cone generated by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ and $\mathbf{e}_5$. It is easily verified that $\mathbf{e}_1, \ldots, \mathbf{e}_5$ are extremal in K. As usual, we shall write $\mathbf{x} \geqslant \mathbf{y}$ if $\mathbf{x} - \mathbf{y} \in K$ and $A \geqslant B$ if A and B are linear operators satisfying $A\mathbf{x} \geqslant B\mathbf{x}$ for all $\mathbf{x} \in K$.

We will need to use the fact that the cone K_{45} generated by $\mathbf{e}_4$ and $\mathbf{e}_5$ is a *face* of K. That is: if $\mathbf{x}, \mathbf{y} \in K$ and $\mathbf{x} + \mathbf{y} \in K_{45}$, then $\mathbf{x}$ and $\mathbf{y}$ are in K_{45}. In order to see this, suppose that K_{45} contains the vector

$$\sum_{j=1}^{5} \alpha_j \mathbf{e}_j = (\alpha_1 + \alpha_5,\ \alpha_2 + \alpha_5,\ \alpha_3 - \alpha_5,\ \alpha_4 - \alpha_5),$$

where $\alpha_j \geqslant 0$ $(1 \leqslant j \leqslant 5)$. Since the vector is a linear combination of $\mathbf{e}_4$ and $\mathbf{e}_5$ it follows that $\alpha_1 + \alpha_5 = \alpha_2 + \alpha_5 = -(\alpha_3 - \alpha_5)$. Hence $\alpha_1 = \alpha_2 = \alpha_3 = 0$, because α_1, α_2, α_3 are non-negative. If follows easily from this that K_{45} is a face of K, as we asserted.

We now construct an extremal positive linear operator on K which does not map extreme vectors to extreme vectors. The operator A is defined to have diagonal matrix $\operatorname{diag}(1, 1, 1, 0)$ relative to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$. Then $A\mathbf{e}_5 = (1, 1, -1, 0) = \mathbf{e}_4 + \mathbf{e}_5$, so that A is a positive operator on K and maps the extreme vector $\mathbf{e}_5$ to $\mathbf{e}_4 + \mathbf{e}_5$, which is not extreme.

Finally we claim that A is an extreme positive operator on K. For suppose that $A \geqslant B$, where B is a positive operator on K.

If $1 \leqslant j \leqslant 3$ then $\mathbf{e}_j = A\mathbf{e}_j \geqslant B\mathbf{e}_j$, so that $B\mathbf{e}_j = \beta_j \mathbf{e}_j$ for some $\beta_j \geqslant 0$. Also $B\mathbf{e}_4 = 0$ since $A\mathbf{e}_4 = 0$. Now $\mathbf{e}_4 + \mathbf{e}_5 = A\mathbf{e}_5 \geqslant B\mathbf{e}_5$. Therefore $B\mathbf{e}_5 = \beta_4 \mathbf{e}_4 + \beta_5 \mathbf{e}_5$, where β_4, $\beta_5 \geqslant 0$, since K_{45} is a face of K. Thus

$$\beta_4 \mathbf{e}_4 + \beta_5 \mathbf{e}_5 = B(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4),$$

i.e.

$$(\beta_5, \beta_5, -\beta_5, \beta_4 - \beta_5) = (\beta_1, \beta_2, -\beta_3, 0).$$

It follows that $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5$. Therefore $B = \beta_5 A$. This completes the proof.

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