

A THERMOELASTIC CRACK PROBLEM FOR AN ANISOTROPIC SLAB

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Abstract

The problem of determining the temperature, displacement and stress fields around a single crack in an anisotropic slab is considered. The problem is reduced to Fredholm integral equations which may be solved numerically.

1. Introduction

It is now widely accepted that the behaviour of some anisotropic materials closely models the behaviour of certain fibre-reinforced composites (see, for example, Spencer [10]). This has caused a renewed interest in anisotropic materials in recent years, and a number of new solutions to boundary-value problems have been obtained. Specifically, Clements [4, 5] has presented solutions to problems involving cracks in anisotropic slabs and anisotropic layered materials while Tauchert and Aköz [13], Aköz and Tauchert [1], Clements [6], Clements and Toy [8], Atkinson and Clements [2] and Chang [3] have solved various thermostatic and thermoelastic problems for anisotropic materials. References to various other works in this area may be found in the review article by Tauchert [12].

In the present paper the problem of determining the temperature, displacement and stress fields around a crack in an anisotropic slab is considered. The bounding planes of the slab are subjected to an arbitrary temperature distribution and arbitrary tractions. Over the crack the heat flux and the tractions are prescribed. For these boundary conditions the problem is reduced to a number of Fredholm integral equations which may be solved numerically. Numerical results are obtained for a particular transversely isotropic material and the results are used to make some qualitative comments about the nature of the stress near the crack tip.

2. Statement of problem and basic equations

Take Cartesian coordinates x_1, x_2, x_3 in a homogeneous anisotropic elastic slab. Contained in the slab is a crack in the region $x_2 = 0, |x_1| < a, -\infty < x_3 < \infty$ (Fig. 1). On the faces of the crack the heat flux and the tractions are prescribed. On the faces of the slab at $x_2 = \pm h$ the temperature and tractions are prescribed. It is required to find the displacement and stress fields throughout the slab.

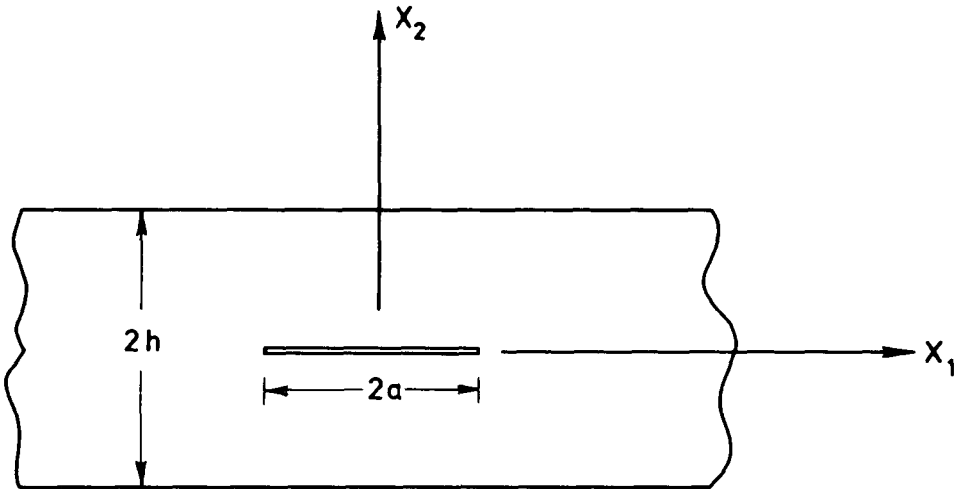


Fig. 1. Location of the crack in the slab.

In Clements [6] some representations are derived for the temperature T , displacement u_k and stress σ_{ij} in an anisotropic material. These representations take the form

$$T = \frac{1}{\pi} \Re \int_0^\infty A(p) \exp(ipz') dp, \tag{2.1}$$

$$u_k = \frac{1}{\pi} \Re \int_0^\infty \{ \sum_\alpha A_{k\alpha} E_\alpha(p) \exp(ipz_\alpha) + C_k A(p) p^{-1} \exp(ipz') \} dp, \tag{2.2}$$

$$\sigma_{ij} = \frac{1}{\pi} \Re \int_0^\infty \{ \sum_\alpha L_{ij\alpha} E_\alpha(p) ip \exp(ipz_\alpha) + (iN_{ij} - \beta_{ij}) A(p) \exp(ipz') \} dp, \tag{2.3}$$

where \Re denotes the real part of a complex number and $A(p)$ and $E_\alpha(p), \alpha = 1, 2, 3$, are functions to be determined from boundary conditions. Also, in (2.1)–(2.3) the constants τ (in $z' = x_1 + \tau x_2$), p_α (in $z_\alpha = x_1 + p_\alpha x_2$), $A_{k\alpha}, C_k, L_{ij\alpha}$ and N_{ij} are related to the coefficients of heat conduction λ_{ij} , the elastic constants c_{ijkl} , and the stress-temperature coefficients β_{ij} in a way which is indicated in Clements [6].

At this point it is appropriate to note that the repeated suffix summation convention (summing from 1 to 3) will be used throughout the paper for Latin suffices only. Summation over Greek suffices (also from 1 to 3) will always be indicated explicitly.

The representations (2.1)–(2.3) tend to zero as $x_2 \rightarrow \infty$. The corresponding expressions for T , u_k and σ_{ij} which tend to zero as $x_2 \rightarrow -\infty$ will also be useful in the subsequent analysis. They may be derived by following the procedure used in Clements [6] and take the form

$$T = \frac{1}{\pi} \mathcal{R} \int_0^\infty A(p) \exp(-ipz') dp, \tag{2.4}$$

$$u_k = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha A_{k\alpha} E_\alpha(p) \exp(-ipz_\alpha) - C_k A(p) p^{-1} \exp(-ipz') \right\} dp, \tag{2.5}$$

$$\sigma_{ij} = -\frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha L_{ij\alpha} E_\alpha(p) ip \exp(-ipz_\alpha) - (iN_{ij} - \beta_{ij}) A(p) \exp(-ipz') \right\} dp, \tag{2.6}$$

where, as in (2.1)–(2.3), the $A(p)$ and $E_\alpha(p)$, $\alpha = 1, 2, 3$, are unknown functions which will be determined from the boundary conditions. The constants in (2.4)–(2.6) are the same as those in (2.1)–(2.3).

For the purposes of the present paper, it is convenient to write the temperature, displacement and stress as the sum of three fields. Specifically, we write

$$T = T^{(1)} + T^{(2)} + T^{(3)}, \tag{2.7}$$

$$u_k = u_k^{(1)} + u_k^{(2)} + u_k^{(3)}, \tag{2.8}$$

$$\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} + \sigma_{ij}^{(3)}, \tag{2.9}$$

where, from (2.1)–(2.3),

$$T^{(1)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty A^{(1)}(p) \exp(ipz') dp, \tag{2.10}$$

$$u_k^{(1)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha A_{k\alpha} E_\alpha^{(1)}(p) \exp(ipz_\alpha) + C_k A^{(1)}(p) p^{-1} \exp(ipz') \right\} dp, \tag{2.11}$$

$$\sigma_{ij}^{(1)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha L_{ij\alpha} E_\alpha^{(1)}(p) ip \exp(ipz_\alpha) + (iN_{ij} - \beta_{ij}) A^{(1)}(p) \exp(ipz') \right\} dp \tag{2.12}$$

and, from (2.4)–(2.6),

$$T^{(2)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty A^{(2)}(p) \exp(-ipz') dp, \tag{2.13}$$

$$u_k^{(2)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha A_{k\alpha} E_\alpha^{(2)}(p) \exp(-ipz_\alpha) - C_k A^{(2)}(p) p^{-1} \exp(-ipz') \right\} dp, \tag{2.14}$$

$$\sigma_{ij}^{(2)} = -\frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha L_{ij\alpha} E_\alpha^{(2)}(p) ip \exp(-ipz_\alpha) - (iN_{ij} - \beta_{ij}) A^{(2)}(p) \exp(-ipz') \right\} dp. \tag{2.15}$$

For $T^{(3)}$, $u_k^{(3)}$ and $\sigma_{ij}^{(3)}$ we consider the regions $0 < x_2 < h$ and $-h < x_2 < 0$ separately. Hence

$$T^{(3)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty A^+(p) \exp(ipz') dp \quad \text{for } 0 < x_2 < h, \tag{2.16}$$

$$u_k^{(3)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha A_{k\alpha} E_\alpha^+(p) \exp(ipz_\alpha) + C_k A^+(p) p^{-1} \exp(ipz') \right\} dp \tag{2.17}$$

for $0 < x_2 < h$,

$$\sigma_{ij}^{(3)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha L_{ij\alpha} E_\alpha^+(p) ip \exp(ipz_\alpha) + (iN_{ij} - \beta_{ij}) A^+(p) \exp(ipz') \right\} dp \tag{2.18}$$

for $0 < x_2 < h$,

$$T^{(3)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty A^-(p) \exp(-ipz') dp \quad \text{for } -h < x_2 < 0, \tag{2.19}$$

$$u_k^{(3)} = \frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha A_{k\alpha} E_\alpha^-(p) \exp(-ipz_\alpha) - C_k A^-(p) p^{-1} \exp(-ipz') \right\} dp \tag{2.20}$$

for $-h < x_2 < 0$,

$$\sigma_{ij}^{(3)} = -\frac{1}{\pi} \mathcal{R} \int_0^\infty \left\{ \sum_\alpha L_{ij\alpha} E_\alpha^-(p) ip \exp(-ipz_\alpha) - (iN_{ij} - \beta_{ij}) A^-(p) \exp(-ipz') \right\} dp \tag{2.21}$$

for $-h < x_2 < 0$.

3. Temperature field

On $x_2 = 0$, $|x_1| < a$ the heat flux is prescribed so that the boundary condition in this region is

$$\lambda_{12} \frac{\partial T}{\partial x_1} + \lambda_{22} \frac{\partial T}{\partial x_2} = -f(x_1). \tag{3.1}$$

Also, the temperature T is prescribed on the faces $x_2 = \pm h$ of the slab so that

$$T = g_1(x_1) \quad \text{on } x_2 = h \tag{3.2}$$

and

$$T = g_2(x_1) \quad \text{on } x_2 = -h \quad (3.3)$$

where g_1 and g_2 are given. We wish to find the temperature field throughout the slab.

A solution to this problem has been obtained by Clements and Tauchert [7]. The temperature is given by (2.7), (2.10), (2.13), (2.15) and (2.19) with

$$A^+(p) = (\lambda_{12} + \lambda_{22} \tau)^{-1} A(p), \quad (3.4)$$

$$A^-(p) = (\lambda_{12} + \lambda_{22} \bar{\tau})^{-1} A(p), \quad (3.5)$$

$$A^{(1)}(p) = Q(p) A(p) + U(p), \quad (3.6)$$

$$A^{(2)}(p) = Q(p) \bar{A}(p) + V(p), \quad (3.7)$$

where the bar denotes the complex conjugate and the functions $Q(p)$, $U(p)$, $V(p)$ are given by

$$Q(p) = -\{(\lambda_{12} + \lambda_{22} \tau)^{-1} \exp[ip(\tau - \bar{\tau})h] - (\lambda_{12} + \lambda_{22} \bar{\tau})^{-1}\} D, \quad (3.8)$$

$$U(p) = [B_1(p) \exp(-ip\bar{\tau}h) - B_2(p) \exp(ip\bar{\tau}h)] D, \quad (3.9)$$

$$V(p) = [-\bar{B}_1(p) \exp(ip\bar{\tau}h) + \bar{B}_2(p) \exp(-ip\bar{\tau}h)] D \quad (3.10)$$

with

$$D = \{\exp[ip(\tau - \bar{\tau})h] - \exp[-ip(\tau - \bar{\tau})h]\}^{-1} \quad (3.11)$$

and $B_1(p)$ and $B_2(p)$ may be determined from the equation

$$\frac{1}{\pi} \mathcal{R} \int_0^\infty B_i(p) \exp(ipx_1) dp = g_i(x_1) \quad \text{for } i = 1, 2. \quad (3.12)$$

The $A(p)$ in (3.4)–(3.7) is given by

$$A(p) = \int_0^a s(t) J_1(pt) dt + i \int_0^a r(t) J_0(pt) dt, \quad (3.13)$$

where $s(t)$ and $r(t)$ are obtained from the Fredholm equations

$$r(t) + t \int_0^a K^{(0)}(u, t) r(u) du = t \int_{-t}^t \frac{f_1(u) du}{(t^2 - u^2)^{\frac{1}{2}}} \quad \text{for } 0 < t < a, \quad (3.14)$$

$$s(t) + t \int_0^a K^{(1)}(u, t) s(u) du = \int_{-t}^t \frac{uf_1(u) du}{(t^2 - u^2)^{\frac{1}{2}}} \quad \text{for } 0 < t < a, \quad (3.15)$$

where

$$K^{(N)}(u, t) = \int_0^\infty S(p) J_N(pu) J_N(pt) p dp. \quad (3.16)$$

In (3.16) J_N denotes the Bessel function of order N , while

$$S(p) = 2\mathcal{R}[(\lambda_{12} + \lambda_{22} \tau) Q(p)]. \tag{3.17}$$

Finally, the $f_1(u)$ in (3.14) and (3.15) is defined by

$$f_1(u) = f(u) + \frac{1}{\pi} \mathcal{R} \int_0^\infty [U(p)(\lambda_{12} + \lambda_{22} \tau) + \bar{V}(p)(\lambda_{12} + \lambda_{22} \bar{\tau})] \exp(ipu) ip dp. \tag{3.18}$$

4. The displacement and stress fields

As indicated in Section 2, the displacement and stress components are written as the sum of three separate fields $u_k^{(p)}$ and $\sigma_{ij}^{(p)}$ for $p = 1, 2, 3$. Now $u_k^{(1)} + u_k^{(2)}$ and $\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}$ are given by (2.11)–(2.15) and are continuous throughout the slab. In $0 < x_2 < h$ $u_k^{(3)}$ and $\sigma_{ij}^{(3)}$ are given by (2.17) and (2.18) while in $-h < x_2 < 0$ these quantities are given by (2.20) and (2.21). Hence the requirement that the stress σ_{i2} be continuous across $x_2 = 0$ yields

$$ip [\sum_\alpha \{L_{i2\alpha} E_\alpha^+(p) - \bar{L}_{i2\alpha} \bar{E}_\alpha^-(p)\}] + i[N_{i2} A^+(p) + \bar{N}_{i2} \bar{A}^-(p)] - \beta_{i2}[A^+(p) - \bar{A}^-(p)] = 0. \tag{4.1}$$

Equation (4.1) may be rearranged to yield

$$\sum_\alpha L_{i2\alpha} E_\alpha^+(p) + p^{-1} A^+(p) \{N_{i2} + i\beta_{i2}\} = \sum_\alpha \bar{L}_{i2\alpha} \bar{E}_\alpha^-(p) - p^{-1} \bar{A}^-(p) \{\bar{N}_{i2} - i\beta_{i2}\}. \tag{4.2}$$

Denoting these expressions by $F_i(p)$ we obtain

$$E_\alpha^+(p) = M_{\alpha i} F_i(p) - M_{\alpha i} \{N_{i2} + i\beta_{i2}\} A^+(p) p^{-1}, \tag{4.3}$$

$$E_\alpha^-(p) = M_{\alpha i} \bar{F}_i(p) + M_{\alpha i} \{N_{i2} + i\beta_{i2}\} A^-(p) p^{-1}, \tag{4.4}$$

where

$$\sum_\alpha L_{i2\alpha} M_{\alpha j} = \delta_{ij}. \tag{4.5}$$

The displacement u_k on $x_2 = 0$ outside the crack must be continuous and hence, from (2.17) and (2.20),

$$\begin{aligned} \mathcal{R} \int_0^\infty [\sum_\alpha \{A_{k\alpha} E_\alpha^+(p) - \bar{A}_{k\alpha} \bar{E}_\alpha^-(p)\} + C_k A^+(p) p^{-1} + \bar{C}_k \bar{A}^-(p) p^{-1}] \exp(ipx_1) dp \\ = 0 \quad \text{for } |x_1| > a. \end{aligned} \tag{4.6}$$

Use of (4.3) and (4.4) in (4.6) yields

$$\mathcal{R} \int_0^\infty [H_{ks} F_s(p) + R_k(p)] \exp(ipx_1) dp = 0 \quad \text{for } |x_1| > a, \tag{4.7}$$

where

$$\begin{aligned}
 H_{ks} &= B_{ks} - \bar{B}_{ks}, \quad B_{ks} = \sum_{\alpha} A_{k\alpha} M_{\alpha s}, \\
 R_k(p) &= \{-B_{ki}(N_{i2} + i\beta_{i2}) + C_k\} A^+(p) p^{-1} \\
 &\quad + \{-\bar{B}_{ki}(\bar{N}_{i2} - i\beta_{i2}) + \bar{C}_k\} \bar{A}^-(p) p^{-1}.
 \end{aligned}
 \tag{4.8}$$

On $x_2 = 0$ over the crack the tractions $\sigma_{i2} = P_i(x_1)$ are prescribed. Hence, from (2.9)–(2.21), we obtain

$$\begin{aligned}
 \frac{1}{\pi} \mathcal{R} \int_0^{\infty} \{F_i(p) ip + \sum_{\alpha} (L_{i2\alpha} E_{\alpha}^{(1)} + \bar{L}_{i2\alpha} E_{\alpha}^{(2)}) ip + (iN_{i2} - \beta_{i2}) A^{(1)}(p) \\
 + (-i\bar{N}_{i2} - \beta_{i2}) \bar{A}^{(2)}(p)\} \exp(ipx_1) dp = P_i(x_1) \quad \text{for } |x_1| < a.
 \end{aligned}
 \tag{4.9}$$

Also, on $x_2 = \pm h$ the tractions are prescribed. Hence

$$\begin{aligned}
 \sum_{\alpha} L_{i2\alpha} E_{\alpha}^{(1)}(p) ip \exp(ip p_{\alpha} h) + \sum_{\alpha} \bar{L}_{i2\alpha} E_{\alpha}^{(2)}(p) ip \exp(ip \bar{p}_{\alpha} h) \\
 = -[\sum_{\alpha} L_{i2\alpha} M_{\alpha j} F_j(p) ip \exp(ip p_{\alpha} h) - \sum_{\alpha} L_{i2\alpha} M_{\alpha j} \{N_{j2} + i\beta_{j2}\} \\
 \times A^+(p) i \exp(ip p_{\alpha} h) + (iN_{i2} - \beta_{i2}) A^{(1)}(p) \exp(ip\tau h) \\
 + (-i\bar{N}_{i2} - \beta_{i2}) \bar{A}^{(2)}(p) \exp(ip\bar{\tau} h) + (iN_{i2} - \beta_{i2}) A^+(p) \exp(ip\tau h)] \\
 + G_{i1}(p),
 \end{aligned}
 \tag{4.10}$$

$$\begin{aligned}
 \sum_{\alpha} L_{i2\alpha} E_{\alpha}^{(1)}(p) ip \exp(-ip p_{\alpha} h) + \sum_{\alpha} \bar{L}_{i2\alpha} E_{\alpha}^{(2)}(p) ip \exp(-ip \bar{p}_{\alpha} h) \\
 = -[\sum_{\alpha} L_{i2\alpha} \bar{M}_{\alpha j} F_j(p) ip \exp(-ip \bar{p}_{\alpha} h) + \sum_{\alpha} L_{i2\alpha} \bar{M}_{\alpha j} (\bar{N}_{i2} - i\beta_{i2}) \\
 \bar{A}^-(p) i \exp(-ip p_{\alpha} h) + (iN_{i2} - \beta_{i2}) A^{(1)}(p) \exp(-ip\tau h) \\
 + (-i\bar{N}_{i2} - \beta_{i2}) \bar{A}^{(2)}(p) \exp(-ip\bar{\tau} h) + (-i\bar{N}_{i2} - \beta_{i2}) \\
 \times \bar{A}^-(p) \exp(-ip\bar{\tau} h)] + G_{i2}(p),
 \end{aligned}
 \tag{4.11}$$

where

$$\frac{1}{\pi} \mathcal{R} \int_0^{\infty} G_{i1}(p) \exp(ipx_1) dp = q_{i1}(x_1),
 \tag{4.12}$$

$$\frac{1}{\pi} \mathcal{R} \int_0^{\infty} G_{i2}(p) \exp(ipx_1) dp = q_{i2}(x_1).
 \tag{4.13}$$

In (4.12) and (4.13), $q_{i1}(x_1)$ and $q_{i2}(x_1)$ are the specified tractions on $x_2 = h$ and $x_2 = -h$ respectively.

It is convenient at this stage to introduce the matrices

$$R = [L_{i2\alpha} \exp(-ipp_\alpha h)], \quad S = [L_{i2\alpha} \exp(ipp_\alpha h)],$$

$$M = [M_{\alpha i}], \quad G = [G_i], \quad H = [H_i], \tag{4.14}$$

$$E^{(I)} = [E_\alpha^{(I)}] \quad \text{for } I = 1, 2, \quad F = [F_\alpha].$$

Hence (4.10) and (4.11) may be written

$$SE^{(1)} + \bar{R}E^{(2)} = -SMF + G, \tag{4.15}$$

$$RE^{(1)} + \bar{S}E^{(2)} = -\bar{S}\bar{M}F + H, \tag{4.16}$$

where the precise forms for G and H may be readily obtained by comparing (4.10) and (4.15) and also (4.11) and (4.16). Equations (4.15) and (4.16) may be used to solve for $E^{(1)}$ and $E^{(2)}$. We obtain

$$E^{(1)} = QF + X, \tag{4.17}$$

$$E^{(2)} = QF + Y, \tag{4.18}$$

where

$$Q = -(\bar{R}^{-1}S - \bar{S}^{-1}R)^{-1}(\bar{R}^{-1}SM - \bar{M}), \tag{4.19}$$

$$X = (\bar{R}^{-1}S - \bar{S}^{-1}R)^{-1}(\bar{R}^{-1}G - \bar{S}^{-1}H), \tag{4.20}$$

$$Y = (\bar{R}^{-1}S - \bar{S}^{-1}R)^{-1}(\bar{R}^{-1}\bar{H} - \bar{S}^{-1}\bar{G}). \tag{4.21}$$

Substitution of (4.17) and (4.18) into (4.9) gives

$$\mathcal{R} \int_0^\infty [F_j(p) + T_{jk}(p)F_k(p)]ip \exp(ipx_1) dp = \mathcal{P}_j(x_1) \quad \text{for } |x_1| < a, \tag{4.22}$$

in which

$$\mathcal{P}_j(x_1) = \pi P_j(x_1) - \mathcal{R} \int_0^\infty \sum_\alpha [L_{j2\alpha} X_\alpha(p) + L_{j2\alpha} Y_\alpha(p)] ip \exp(ipx_1) dp$$

$$- \mathcal{R} \int_0^\infty \{ (iN_{j2} - \beta_{j2}) A^{(1)}(p) - (i\bar{N}_{j2} + \beta_{j2}) A^{(2)}(p) \} \exp(ipx_1) dp \tag{4.23}$$

and

$$T_{jk}(p) = 2\mathcal{R} \sum_\alpha L_{j2\alpha} Q_{\alpha k}(p). \tag{4.24}$$

Returning to (4.7), we note that the matrix H_{ks} is non-singular (see Stroh [11]) and hence there exists an inverse matrix U_{jk} such that

$$U_{jk} H_{ks} = \delta_{js}. \tag{4.25}$$

Thus, equation (4.7) can be written as

$$\Re \int_0^\infty [F_k(p) + U_{kj} R_j(p)] i \exp(ipx_1) dp = 0 \quad \text{for } |x_1| > a. \tag{4.26}$$

Now equation (4.22) may be rewritten as

$$\begin{aligned} \Re \int_0^\infty [F_j(p) + U_{jk} R_k(p) + T_{jk}(p) \{F_k(p) + U_{kr} R_r(p)\}] \\ \times ip \exp(ipx_1) dp = \mathcal{S}_j(x_1) \quad \text{for } |x_1| < a, \end{aligned} \tag{4.27}$$

where

$$\mathcal{S}_j(x_1) = \mathcal{P}_j(x_1) + \Re \int_0^\infty [U_{jk} R_k(p) + T_{jk}(p) U_{kr} R_r(p)] ip \exp(ipx_1) dp. \tag{4.28}$$

Let

$$\mathcal{F}_k(p) = F_k(p) + U_{kj} R_j(p). \tag{4.29}$$

Then (4.26) and (4.27) yield

$$\Re \int_0^\infty \mathcal{F}_k(p) i \exp(ipx_1) dp = 0 \quad \text{for } |x_1| > a, \tag{4.30}$$

$$\Re \int_0^\infty [\mathcal{F}_j(p) + T_{jk}(p) \mathcal{F}_k(p)] ip \exp(ipx_1) dp = \mathcal{S}_j(x_1) \quad \text{for } |x_1| < a. \tag{4.31}$$

Equation (4.30) will be satisfied if $\mathcal{F}_k(p)$ is taken in the form

$$\mathcal{F}_k(p) = \int_0^a s_k(t) J_1(pt) dt + i \int_0^a r_k(t) J_0(pt) dt, \tag{4.32}$$

where $s_k(t)$ and $r_k(t)$, $k = 1, 2, 3$, are real functions to be determined and J_0 and J_1 are Bessel functions of order zero and one respectively. Use of (4.32) in (4.31) yields

$$\begin{aligned} \int_0^\infty p \cos(px_1) dp \int_0^a r_j(t) J_0(pt) dt + \int_0^\infty T_{jk}(p) p \cos(px_1) dp \int_0^a r_k(t) J_0(pt) dt \\ = -\frac{1}{2} [\mathcal{S}_j(x_1) + \mathcal{S}_j(-x_1)] \quad \text{for } |x_1| < a, \end{aligned} \tag{4.33}$$

$$\begin{aligned} \int_0^\infty p \sin(px_1) dp \int_0^a s_j(t) J_1(pt) dt + \int_0^\infty T_{jk}(p) p \sin(px_1) dp \int_0^a s_k(t) J_1(pt) dt \\ = -\frac{1}{2} [\mathcal{S}_j(x_1) - \mathcal{S}_j(-x_1)] \quad \text{for } |x_1| < a. \end{aligned} \tag{4.34}$$

Interchanging the order of integration and using standard results for Bessel functions we obtain

$$\begin{aligned} \frac{d}{dx_1} \int_0^{x_1} \frac{r_j(t) dt}{(x_1^2 - t^2)^{\frac{1}{2}}} + \int_0^\infty T_{jk}(p) p \cos(px_1) dp \int_0^a r_k(t) J_0(pt) dt \\ = -\frac{1}{2}[\mathcal{S}_j(x_1) + \mathcal{S}_j(-x_1)] \quad \text{for } 0 < x_1 < a, \end{aligned} \tag{4.35}$$

$$\begin{aligned} \frac{1}{x_1} \frac{d}{dx_1} \int_0^{x_1} \frac{ts_j(t) dt}{(x_1^2 - t^2)^{\frac{1}{2}}} + \int_0^\infty T_{jk}(p) p \sin(px_1) dp \int_0^a s_k(t) J_1(pt) dt \\ = -\frac{1}{2}[\mathcal{S}_j(x_1) - \mathcal{S}_j(-x_1)] \quad \text{for } 0 < x_1 < a. \end{aligned} \tag{4.36}$$

These Abel integral equations may be inverted to yield

$$r_j(t) + t \int_0^a K_{jk}^{(0)}(u, t) r_k(u) du = -\frac{t}{\pi} \int_{-t}^t \frac{\mathcal{S}_j(u) du}{(t^2 - u^2)^{\frac{1}{2}}} \quad \text{for } 0 < t < a, \tag{4.37}$$

$$s_j(t) + t \int_0^a K_{jk}^{(1)}(u, t) s_k(u) du = -\frac{1}{\pi} \int_{-t}^t \frac{u \mathcal{S}_j(u) du}{(t^2 - u^2)^{\frac{1}{2}}} \quad \text{for } 0 < t < a, \tag{4.38}$$

where standard results for Bessel function have been used to obtain $K_{ij}^{(N)}$ in the form

$$K_{jk}^{(N)}(u, t) = \int_0^\infty T_{jk}(p) J_N(pu) J_N(pt) p dp. \tag{4.39}$$

Equation (4.37) constitutes three simultaneous Fredholm integral equations for the $r_j(t)$ while (4.38) constitutes three similar equations for the $s_j(t)$. These equations may be solved numerically and then $\mathcal{F}_k(p)$ may be found through (4.32). Equations (4.29), (4.3) and (4.4) then yield $E_\alpha^+(p)$ and $E_\alpha^-(p)$ while (4.29), (4.17) and (4.18) yield $E^{(1)}(p)$ and $E^{(2)}(p)$. Hence the displacement and stress distributions throughout the slab may be calculated from (2.7)–(2.21).

5. Numerical results

The stress $\sigma_{i2}(x_1, 0)$ near the crack tip at $r = a$ takes the form

$$\sigma_{i2}(x_1, 0) \sim K_i r^{-\frac{1}{2}} \quad \text{as } r \rightarrow 0, \tag{5.1}$$

where $r = x_1 - a$ with $x_1 > a$ and the K_i are constants. In this section we use the analysis of the previous sections to determine the values of the K_i for a particular transversely isotropic material.

For transversely isotropic materials with the x_1 and x_2 axes lying in the transverse plane, the non-zero stiffness c_{ijkl} , the coefficients of linear thermal expansion

α_{ij} , and the thermal conductivities λ_{ij} are

$$\begin{aligned} c_{1111} &= c_{2222}, c_{1133} = c_{2233}, c_{1313} = c_{2323}, c_{1122}, \\ c_{3333}, c_{1212} &= (c_{1111} - c_{1122})/2, \\ \alpha_{11} &= \alpha_{22}, \alpha_{33}, \quad \lambda_{11} = \lambda_{22}, \lambda_{33}. \end{aligned}$$

If a rotation of α about the x_2 axis is followed by a rotation of θ about the x_1 axis, then the constants referred to the rotated frame are

$$c'_{ijkl} = a_{im} a_{jn} a_{kp} a_{lq} c_{mnpq}, \quad \beta'_{ij} = a_{im} a_{jn} \beta_{mn}, \quad \lambda'_{ij} = a_{im} a_{jn} \lambda_{mn},$$

where

$$[a_{ij}] = \begin{bmatrix} \cos \alpha & -\sin \alpha \sin \theta & -\sin \alpha \cos \theta \\ 0 & \cos \theta & \sin \theta \\ -\sin \alpha & -\cos \alpha \sin \theta & \cos \alpha \cos \theta \end{bmatrix}.$$

For illustrative purposes we consider the constants for a crystal of zinc. Referred to symmetry axes with the x_3 axis normal to the transverse plane the constants are

$$\begin{aligned} c_{1111} &= 16.5, \quad c_{1122} = 3.1, \quad c_{1133} = 5, \quad c_{3333} = 6.2, \\ c_{1313} &= 3.92, \quad 10^6 \alpha_{11} = 60.8, \quad 10^6 \alpha_{33} = 14.3, \quad \lambda_{11}/\lambda_{33} = 1.17. \end{aligned}$$

If the elastic constants are multiplied by 10^{11} then the units for these constants are dynes/cm², while the coefficients of thermal expansion are for a temperature increase of one degree centigrade.

Consider a slab with $h/a = 20$. Further let $a = 1$ and suppose the crack faces and slab faces $x_2 = \pm h$ are traction free. Also suppose the temperature on the slab faces is zero while over the crack the heat flux is constant so that, in (3.1),

$$f(x_1) = f_0,$$

where f_0 is a constant. The values of K_i for various combinations of the angles α and θ are given in Table 1. The quantity K_0 in the table is K_1 calculated for $\alpha = 0$ and $\theta = \pi/2$.

TABLE 1

α	0	0	$\pi/4$	$\pi/4$	$\pi/3$	$\pi/2$
θ	$\pi/2$	$\pi/4$	0	$\pi/4$	$\pi/6$	0
K_1/K_0	1	1.46	1.19	-1.72	-1.52	0.85
K_2/K_0	0	0	0	1.35	1.22	0
K_3/K_0	0	0	-2.02	0.85	1.12	0

When $\alpha = 0$ and $\theta = \pi/2$, each of the planes $x_i = 0$, $i = 1, 2, 3$, is a plane of elastic symmetry and it is clear from Table 1 and equation (5.1) that, in this case, the stress σ_{12} is singular in the plane of the crack but σ_{22} and σ_{32} do not exhibit singular behaviour.

When $\alpha = 0$ and $\theta = \pi/4$, the $x_1 = 0$ plane is a plane of elastic symmetry but the $x_2 = 0$ and $x_3 = 0$ planes are not planes of the crack is similar to the above case when $\alpha = 0$ and $\theta = \pi/2$.

When $\alpha = \pi/4$ and $\theta = 0$, the $x_2 = 0$ plane is a plane of elastic symmetry while the $x_1 = 0$ and $x_3 = 0$ planes are not planes of elastic symmetry. In this case both σ_{12} and σ_{32} exhibit singular behaviour in the plane of the crack while σ_{22} does not exhibit singular behaviour.

When $\alpha = \pi/4$ and $\theta = \pi/4$, or $\alpha = \pi/3$ and $\theta = \pi/6$, none of the planes $x_i = 0$, $i = 1, 2, 3$, are planes of elastic symmetry and all the stresses σ_{i2} , $i = 1, 2, 3$, exhibit singular behaviour.

When $\alpha = \pi/2$ and $\theta = 0$, each of the planes $x_i = 0$, $i = 1, 2, 3$, is a plane of elastic symmetry and the singular behaviour of the stress in the plane of the crack is similar to the first case considered above when $\alpha = 0$ and $\theta = \pi/2$.

It is of interest to compare these results with those obtained by Sih [9] for the thermal stress singularity at the tip of a crack in an isotropic material. Sih's results show that a constant heat flux over the crack face gives rise to a singular stress σ_{12} in the plane of the crack while the stresses σ_{22} and σ_{23} remain non-singular. The results obtained here indicate that this situation also holds true for anisotropic material provided the $x_1 = 0$ plane is a plane of elastic symmetry. However, if the $x_1 = 0$ plane is not a plane of elastic symmetry, then at least one of the stresses σ_{22} and σ_{23} is singular in the plane of the crack.

Finally, we note that the displacements induced by purely mechanical loads applied to the surfaces of the anisotropic slab with the crack stress-free have been studied in detail in a previous paper (Clements and Tauchert [7]). An analysis of temperature-induced displacements can be carried out in a similar fashion using the present formulation.

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