Bull. Aust. Math. Soc. **108** (2023), 428–437
doi:10.1017/S0004972723000114

SIMPLICITY OF LEAVITT PATH ALGEBRAS VIA GRADED RING THEORY

PATRIK LUNDSTRÖM and JOHAN ÖINERT

(Received 22 November 2022; accepted 11 January 2023; first published online 3 March 2023)

Abstract

Suppose that R is an associative unital ring and that $E = (E^0, E^1, r, s)$ is a directed graph. Using results from graded ring theory, we show that the associated Leavitt path algebra $L_R(E)$ is simple if and only if R is simple, E^0 has no nontrivial hereditary and saturated subset, and every cycle in E has an exit. We also give a complete description of the centre of a simple Leavitt path algebra.

2020 *Mathematics subject classification*: primary 16S88; secondary 16W50.

Keywords and phrases: Leavitt path algebra, graded ring, simple ring, centre.

1. Introduction

The Leavitt path algebra of a row-finite graph, over a field, was introduced in [2, 5] and has since then been successively generalised (see, for example, [3, 20]). The Leavitt path algebra of an arbitrary directed graph, over a unital ring, was introduced in [12]. For an account of the development of the field of Leavitt path algebras, we refer the reader to [1]. Here is our first main result.

THEOREM 1.1. *Suppose that R is an associative unital ring and that $E = (E^0, E^1, r, s)$ is a directed graph. The Leavitt path algebra $L_R(E)$ is simple if and only if R is simple, E^0 has no nontrivial hereditary and saturated subset, and every cycle in E has an exit.*

Characterisations of simple Leavitt path algebras over fields have previously been established in [19, Theorem 6.18], [3, Theorem 3.1] and [11, Theorem 3.5]. Theorem 1.1 generalises all of those results, and also partially generalises [20, Theorem 7.20]. Our second main result, stated below, completely describes the centre of a simple Leavitt path algebra. It generalises [6, Theorem 4.2] from the case where R is a field and E is a row-finite graph.

THEOREM 1.2. *Suppose that R is an associative unital ring and that $E = (E^0, E^1, r, s)$ is a directed graph. Furthermore, suppose that $L_R(E)$ is a simple Leavitt path algebra. The following assertions hold.*

(a) *If $L_R(E)$ is not unital, then $Z(L_R(E)) = \{0\}$.*
(b) *If $L_R(E)$ is unital, then $Z(L_R(E)) = Z(R) \cdot 1_{L_R(E)}$.*

Whereas earlier proofs of Theorems 1.1 and 1.2 (when R is a field) use *ad hoc* arguments, specifically designed for graph algebras, we use the general theory of *graded rings* to obtain our results. This makes our proofs shorter and, we believe, clearer. Indeed, we show that $L_R(E)$ is graded simple if and only if R is simple and E^0 has no nontrivial hereditary and saturated subset (see Proposition 3.6). We also show that every cycle in E has an exit if and only if the centre of each corner subring of $L_R(E)$ at a vertex has degree zero (see Proposition 3.14).

We point out that there are various generalisations of Leavitt path algebras in the literature (see, for example, [1, Section 5] and [9, 18]). A simplicity result for Steinberg algebras was obtained in [8], and when translated to Leavitt path algebras, one recovers Theorem 1.1 in the special case where R is a commutative unital ring. Note that [6, Theorem 4.2] was generalised to Kumjian–Pask algebras in [7], and in [10], Steinberg algebra techniques were used to give a complete description of the centre of a general (not necessarily simple) Leavitt path algebra $L_R(E)$, where R is a commutative unital ring.

2. Simple $\mathbb{Z}$-graded rings

Let $\mathbb{Z}$ denote the rational integers and write $\mathbb{N} := \{1, 2, 3, \ldots\}$. Suppose that S is a ring. By this, we mean that S is associative but not necessarily unital. If S is unital, then we let 1_S denote the multiplicative identity of S. Furthermore, we let $Z(S)$ denote the centre of S, that is, the set of all $s \in S$ satisfying $st = ts$ for every $t \in S$. Recall that S is said to be $\mathbb{Z}$-*graded* if, for each $n \in \mathbb{Z}$, there is an additive subgroup S_n of S such that $S = \oplus_{n\in\mathbb{Z}} S_n$ and $S_n S_m \subseteq S_{n+m}$, for all $n, m \in \mathbb{Z}$. In that case, each element $s \in S$ may be written as $s = \sum_{n\in\mathbb{Z}} s_n$, where $s_n \in S_n$ is zero for all but finitely many $n \in \mathbb{Z}$. The *support of s* is defined as the finite set $\mathrm{Supp}(s) := \{n \in \mathbb{Z} \mid s_n \neq 0\}$. An ideal I of a $\mathbb{Z}$-graded ring S is said to be *graded* if $I = \oplus_{n\in\mathbb{Z}}(I \cap S_n)$. If $\{0\}$ and S are the only graded ideals of S, then S is said to be *graded simple*.

We recall some properties of graded rings.

LEMMA 2.1. *Suppose that S is a unital $\mathbb{Z}$-graded ring.*

(a) *The ring $Z(S)$ is $\mathbb{Z}$-graded with respect to the grading $Z(S)_n$ which is defined by $Z(S)_n := Z(S) \cap S_n$ for $n \in \mathbb{Z}$.*
(b) *If S is a field, then $S = S_0$.*

PROOF. Item (a) is [15, page 15, Exercise 8] and item (b) is [15, Remark 1.3.10]. □

https://doi.org/10.1017/S0004972723000114 Published online by Cambridge University Press

Next, we state a special case of [17, Theorem 1.2] and [13, Theorem 5]. For the convenience of the reader, we include a shortened version of the proof from these sources adapted to the situation at hand.

PROPOSITION 2.2. *Suppose that S is a unital $\mathbb{Z}$-graded ring. Then, the following assertions are equivalent:*

(i) *S is simple;*
(ii) *S is graded simple and $Z(S)$ is a field;*
(iii) *S is graded simple and $Z(S) \subseteq S_0$.*

PROOF. (i)⇒(ii) is clear and (ii)⇒(iii) follows from Lemma 2.1. Now we show that (iii)⇒(i). Suppose that S is graded simple and that $Z(S) \subseteq S_0$. Let I be a nonzero ideal of S. We wish to show that $1_S \in I$. Amongst all nonzero elements of I, choose s such that $|\mathrm{Supp}(s)|$ is minimal. Take $m \in \mathrm{Supp}(s)$. Since S is graded simple, there are $n \in \mathbb{N}$ and homogeneous elements $p_1, \ldots, p_n, q_1, \ldots, q_n \in S$, such that $\sum_{i=1}^{n} p_i s_m q_i = 1_S$ and $p_i s_m q_i \in S_0 \setminus \{0\}$ for every $i \in \{1, \ldots, n\}$. Write $t := \sum_{i=1}^{n} p_i s q_i$. Note that $t \in I$, $t_0 = 1_S$ and $|\mathrm{Supp}(t)| \leq |\mathrm{Supp}(s)|$. Take $z \in \mathbb{Z}$ and $x \in S_z$. Then, $tx - xt \in I$ and, since $t_0 = 1_S$, it follows that $|\mathrm{Supp}(tx - xt)| < |\mathrm{Supp}(t)|$. By the assumptions on s, we get $|\mathrm{Supp}(tx - xt)| = 0$ and hence $xt = tx$. Thus, $t \in Z(S) \subseteq S_0$. We conclude that $1_S = t_0 = t \in I$. □

Let S be a ring. Recall from [4] (see also [16]) that a set U of idempotents in S is called a *set of local units* for S if for every $n \in \mathbb{N}$ and all $s_1, \ldots, s_n \in S$, there is some $e \in U$ such that $es_i = s_i e = s_i$ for every $i \in \{1, \ldots, n\}$.

REMARK 2.3. Suppose that S is a $\mathbb{Z}$-graded ring. If $e \in S_0$ is an idempotent, then the corner subring eSe inherits a natural $\mathbb{Z}$-grading defined by $(eSe)_n := eS_n e$ for $n \in \mathbb{Z}$.

For future reference, we recall the following two results.

PROPOSITION 2.4. *Suppose that S is a $\mathbb{Z}$-graded ring equipped with a set of local units $U \subseteq S_0$. Then, S is (graded) simple if and only if, for every $f \in U$, the ring fSf is (graded) simple.*

PROOF. First we show the 'only if' statement. Suppose that S is (graded) simple and that $f \in U$. Let J be a nonzero (graded) ideal of fSf. By (graded) simplicity of S, it follows that $SJS = S$. Thus, $fSf = fSJSf = (fSf)J(fSf) \subseteq J$ and hence $J = fSf$. Next, we show the 'if' statement. Suppose that fSf is (graded) simple for every $f \in U$. Let I be a nonzero (graded) ideal of S. Take a nonzero (homogeneous) $x \in S$. Take a nonzero (homogeneous) $y \in I$ and $f \in U$ with $fx = xf = x$ and $fy = yf = y$. By (graded) simplicity of fSf, it follows that $I \supseteq fSySf = fSf \ni x$. Thus, $I = S$. □

PROPOSITION 2.5. *Suppose that S is a $\mathbb{Z}$-graded ring equipped with a set of local units and that $f \in S_0$ is a nonzero idempotent. If S is graded simple and fSf is simple, then S is simple.*

https://doi.org/10.1017/S0004972723000114 Published online by Cambridge University Press

PROOF. Suppose that S is graded simple and that fSf is simple. Let I be a nonzero ideal of S. Take a nonzero $s \in I$ and write $s = \sum_{n \in \mathrm{Supp}(s)} s_n$. Fix $m \in \mathrm{Supp}(s)$ and define $J := S s_m S$. Then, J is a nonzero graded ideal of S. By graded simplicity of S, it follows that $J = S$ and, in particular, that $f \in J$. Note that $f \in fJf$. Since $f \neq 0$, it follows that there exist nonzero homogeneous $y, z \in S$ such that $f y s_m z f$ is nonzero and $\deg(y) + \deg(z) = -m$. Now, define $s' := fyszf$. By the construction of s', it follows that $s' \in I \cap fSf$ and that s' is nonzero. In particular, $I \cap fSf \neq \{0\}$. Hence, by simplicity of fSf, we see that $I \cap fSf = fSf$. Thus, $f \in I$. Note that SfS is a nonzero graded ideal of S. Hence, by graded simplicity of S, we have $I \supseteq SfS = S$. This shows that $I = S$. □

3. Simple Leavitt path algebras

Let R be an associative unital ring and let $E = (E^0, E^1, r, s)$ be a directed graph. Recall that r (range) and s (source) are maps $E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 are called *edges*. The elements of E^1 are called *real edges*, while for $f \in E^1$, we call f^* a *ghost edge*. The set $\{f^* \mid f \in E^1\}$ will be denoted by $(E^1)^*$. A *path* μ in E is a sequence of edges $\mu = \mu_1 \dots \mu_n$ such that $r(\mu_i) = s(\mu_{i+1})$ for $i \in \{1, \dots, n-1\}$. In that case, $s(\mu) := s(\mu_1)$ is the *source* of μ, $r(\mu) := r(\mu_n)$ is the *range* of μ and $|\mu| := n$ is the *length* of μ. If $\mu = \mu_1 \dots \mu_n$ is a (real) path in E, then we let $\mu^* := \mu_n^* \dots \mu_1^*$ denote the corresponding *ghost path*. For any vertex $v \in E^0$, we put $s(v) := v$ and $r(v) := v$. We let $r(f^*)$ denote $s(f)$ and we let $s(f^*)$ denote $r(f)$. For $n \geq 2$, we define E^n to be the set of paths of length n and $E^* := \bigcup_{n \geq 0} E^n$ the set of all finite paths.

Following Hazrat [12], we make the following definition.

DEFINITION 3.1. The *Leavitt path algebra of E with coefficients in R*, denoted by $L_R(E)$, is the algebra generated by the sets $\{v \mid v \in E^0\}$, $\{f \mid f \in E^1\}$ and $\{f^* \mid f \in E^1\}$ with the coefficients in R, subject to the relations:

(1) $uv = \delta_{u,v} v$ for all $u, v \in E^0$;
(2) $s(f)f = fr(f) = f$ and $r(f)f^* = f^* s(f) = f^*$ for all $f \in E^1$;
(3) $f^* f' = \delta_{f,f'} r(f)$ for all $f, f' \in E^1$;
(4) $\sum_{f \in E^1, s(f) = v} f f^* = v$ for every $v \in E^0$ for which $s^{-1}(v)$ is nonempty and finite.

Here, elements of the ring R commute with the generators.

REMARK 3.2. (a) The Leavitt path algebra $L_R(E)$ carries a natural $\mathbb{Z}$-grading. Indeed, put $\deg(v) := 0$ for each $v \in E^0$. For each $f \in E^1$, we put $\deg(f) := 1$ and $\deg(f^*) := -1$. By assigning degrees to the generators in this way, we obtain a $\mathbb{Z}$-grading on the free algebra $F_R(E) = R\langle v, f, f^* \mid v \in E^0, f \in E^1 \rangle$. Moreover, the ideal coming from relations (1)–(4) in Definition 3.1 is graded. Using this, it is easy to see that the natural $\mathbb{Z}$-grading on $F_R(E)$ carries over to a $\mathbb{Z}$-grading on the quotient algebra $L_R(E)$.

https://doi.org/10.1017/S0004972723000114 Published online by Cambridge University Press

(b) The set $\{\sum_{v\in F} v \mid F$ is a finite subset of $E^0\}$ is a set of local units for $L_R(E)$. If E^0 is finite, then $L_R(E)$ is unital and $1_{L_R(E)} = \sum_{v\in E^0} v$.

(c) Motivated by Definition 3.1(2), for $u \in E^0$, we write $u^* := u$.

DEFINITION 3.3. Let $E = (E^0, E^1, r, s)$ be a directed graph. A subset $H \subseteq E^0$ is said to be *hereditary* if $s(f) \in H$ implies $r(f) \in H$ for any $f \in E^1$. A hereditary subset $H \subseteq E^0$ is called *saturated* if $\{r(f) \in H \mid f \in E^1$ and $s(f) = v\} \subseteq H$ implies $v \in H$ whenever $v \in E^0$ satisfies $0 < |s^{-1}(v)| < \infty$.

REMARK 3.4. Note that $\varnothing$ and E^0 are always hereditary and saturated subsets of E^0. They are referred to as *trivial*.

LEMMA 3.5. *Every element in $E^0 \cup E^1 \cup (E^1)^*$ is nonzero in $L_R(E)$, and the set of real (respectively ghost) paths is linearly independent in the left R-module $L_R(E)$ and in the right R-module $L_R(E)$.*

PROOF. The proof of [20, Proposition 4.9] immediately carries over to the case where R is a noncommutative unital ring. The same holds for the proof of [20, Proposition 3.4] in case E^0 and E^1 are countable sets. Otherwise, the proof may be adapted by taking $\aleph$ to be an infinite cardinal at least as large as $\mathrm{card}(E^0 \cup E^1)$ and defining $Z := \oplus_{\aleph} R$ (with the notation of [20, Proposition 3.4]). □

PROPOSITION 3.6. *The Leavitt path algebra $L_R(E)$ is graded simple if and only if R is simple and E^0 has no nontrivial hereditary and saturated subset.*

PROOF. First we show the 'if' statement. Suppose that R is simple and that E^0 has no nontrivial hereditary and saturated subset. Let I be a nonzero graded ideal of $L_R(E)$. Consider the set $H_I := \{v \in E^0 \mid kv \in I$ for some nonzero $k \in R\}$. By the same argument as in [20, Lemma 5.1], H_I is nonempty. Furthermore, since R is simple, it follows that $H_I = \{v \in E^0 \mid v \in I\}$. We wish to show that H_I is hereditary and saturated. To this end, take $v \in H_I$. Suppose that $e \in E^1$ with $s(e) = v$. Then, $r(e) = e^*e = e^*ve \in I$. Thus, H_I is hereditary. Now, take $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$, and suppose that $r(s^{-1}(v)) \subseteq H_I$. For each $e \in s^{-1}(v)$, we have $r(e) \in H_I$ and hence $ee^* = er(e)e^* \in I$. Thus, $v = \sum_{e\in s^{-1}(v)} ee^* \in I$ and $v \in H_I$. Therefore, H_I is saturated. By our assumption, $H_I = E^0$. This shows that I must contain all the local units of $L_R(E)$ and thus $I = L_R(E)$. Hence, $L_R(E)$ is graded simple.

Now, we show the 'only if' statement. Suppose that $L_R(E)$ is graded simple. Let J be a proper ideal of R. We wish to show that $J = \{0\}$. To this end, let $L_J(E)$ denote the graded ideal of $L_R(E)$ consisting of all elements of $L_R(E)$ with coefficients coming from J. Consider the natural ring homomorphism $\varphi : L_R(E) \to L_{R/J}(E)$. Clearly, φ is well defined. Note that $L_J(E) \subseteq \ker(\varphi)$. Choose some $u \in E^0$. By Lemma 3.5, applied to $L_{R/J}(E)$, it follows that $u \notin \ker(\varphi)$ and hence $u \notin L_J(E)$. Thus, $L_J(E)$ is a proper graded ideal of $L_R(E)$. By graded simplicity of $L_R(E)$, it follows that $L_J(E) = \{0\}$. Thus, in particular, $Ju = \{0\}$. By Lemma 3.5 applied to $L_R(E)$, we see that $J = \{0\}$.

Let H be a proper hereditary and saturated subset of E^0. Following [2, 3], we let $F := (F^0, F^1, r, s)$ be the graph consisting of all vertices not in H and all edges whose range

https://doi.org/10.1017/S0004972723000114 Published online by Cambridge University Press

is not in H. For $v \in E^0$, define $\Psi(v) := v$ if $v \in F^0$, and $\Psi(v) := 0$ otherwise. For $e \in E^1$, define $\Psi(e) := e$ if $e \in F^1$, and $\Psi(e) := 0$ otherwise. Furthermore, define $\Psi(e^*) := e^*$ if $e^* \in (F^1)^*$, and $\Psi(e^*) := 0$ otherwise. The argument in [2, 3] shows that this yields a well-defined ring homomorphism $\Psi : L_R(E) \to L_R(F)$. Clearly, Ψ is graded. Thus, the ideal $I := \ker(\Psi)$ of $L_R(E)$ is graded. Note that F^0 is nonempty, because H is proper, and hence $I \neq L_R(E)$. By our assumption, $I = \{0\}$. By the construction of Ψ, it follows that $H \subseteq I$. Thus, $H = \emptyset$. □

Definition 3.7. Define an additive map $\mathcal{L} : L_R(E) \to L_R(E)$ by requiring that $\mathcal{L}(\lambda\alpha\beta^*) = \lambda\beta\alpha^*$ for all $\lambda \in R$ and $\alpha, \beta \in E^*$.

Remark 3.8. The map $\mathcal{L}$ is an isomorphism of additive groups such that $\mathcal{L}((L_R(E))_N) = (L_R(E))_{-N}$ for every $N \in \mathbb{Z}$.

Lemma 3.9. *Suppose that $u \in E^0$. The map $\mathcal{L}$ restricts to an isomorphism of additive groups $\mathcal{L}|_{Z(uL_R(E)u)}: Z(uL_R(E)u) \to Z(uL_R(E)u)$. In particular, the equality $\mathcal{L}((Z(uL_R(E)u))_N) = (Z(uL_R(E)u))_{-N}$ holds for every $N \in \mathbb{Z}$.*

Proof. Let $x = \sum_{j=1}^m \lambda_j\alpha_j\beta_j^* \in Z(uL_R(E)u)$, where $\lambda_j \in R$, $\alpha_j, \beta_j \in E^*$ and $s(\alpha_j) = s(\beta_j) = u$ for $j \in \{1, \dots, m\}$. Take $r \in R$. Then, $0 = xru - rux = \sum_{j=1}^m (\lambda_j r - r\lambda_j)\alpha_j\beta_j^*$. Therefore, we have $0 = \mathcal{L}(0) = \sum_{j=1}^m (\lambda_j r - r\lambda_j)\mathcal{L}(\alpha_j\beta_j^*) = \sum_{j=1}^m (\lambda_j r - r\lambda_j)\beta_j\alpha_j^* = \mathcal{L}(x)ru - ru\mathcal{L}(x)$. Thus, $\mathcal{L}(x)ru = ru\mathcal{L}(x)$. Take $\gamma, \delta \in E^*$ with $s(\gamma) = s(\delta) = u$. Then, $0 = x\gamma\delta^* - \gamma\delta^*x = \sum_{j=1}^m \lambda_j(\alpha_j\beta_j^*\gamma\delta^* - \gamma\delta^*\alpha_j\beta_j^*)$. Therefore, we have $0 = \mathcal{L}(0) = \sum_{j=1}^m \lambda_j\mathcal{L}(\alpha_j\beta_j^*\gamma\delta^* - \gamma\delta^*\alpha_j\beta_j^*) = \sum_{j=1}^m \lambda_j(\delta\gamma^*\beta_j\alpha_j^* - \beta_j\alpha_j^*\delta\gamma^*) = \delta\gamma^*\mathcal{L}(x) - \mathcal{L}(x)\delta\gamma^*$. Thus, $\mathcal{L}(x)\delta\gamma^* = \delta\gamma^*\mathcal{L}(x)$. Finally, $\mathcal{L}(x)r\delta\gamma^* = \mathcal{L}(x)ru\delta\gamma^* = ru\mathcal{L}(x)\delta\gamma^* = ru\delta\gamma^*\mathcal{L}(x) = r\delta\gamma^*\mathcal{L}(x)$. This shows that $\mathcal{L}(x) \in Z(uL_R(E)u)$. □

Lemma 3.10. *Suppose that $u, v \in E^0$ and that $\alpha \in E^*$ is such that $s(\alpha) = u$ and $r(\alpha) = v$. If $x \in Z(uL_R(E)u)$, then $\alpha^*x\alpha \in Z(vL_R(E)v)$.*

Proof. Let $x \in Z(uL_R(E)u)$. Take $y \in vL_R(E)v$. Since $\alpha y\alpha^* \in uL_R(E)u$, it follows that $y\alpha^*x\alpha = vy\alpha^*x\alpha = \alpha^*\alpha y\alpha^*x\alpha = \alpha^*x\alpha y\alpha^*\alpha = \alpha^*x\alpha yv = \alpha^*x\alpha y$. Thus, $\alpha^*x\alpha \in Z(vL_R(E)v)$. □

Definition 3.11 [20]. Let $E = (E^0, E^1, r, s)$ be a directed graph. A *cycle* in E is a path $\mu \in E^* \setminus E^0$ such that $s(\mu) = r(\mu)$. An edge $f \in E^1$ is said to be an *exit* for the cycle $\mu = \mu_1 \dots \mu_n$ if $s(f) = s(\mu_i)$ but $f \neq \mu_i$ for some $i \in \{1, 2, \dots, n\}$.

Remark 3.12. The definition of a *cycle* in a directed graph varies in the literature on Leavitt path algebras. In contrast to the most common definition of a cycle (see [2, page 320], following [20], we allow a cycle to 'intersect' itself. In Theorem 1.1, the condition that 'every cycle in E has an exit' appears. That condition is commonly known as *Condition (L)*. It is easy to see that Condition (L) is satisfied with the first definition of a cycle [2] if and only if it is satisfied with the second definition of a cycle [20].

https://doi.org/10.1017/S0004972723000114 Published online by Cambridge University Press

REMARK 3.13. Let x be a nonzero element of $L_R(E)$. It is clear from the definition of $L_R(E)$ that x can be represented as a finite sum $x = \sum_{i=1}^{n} r_i\alpha_i\beta_i^*$, where $r_i \in R \setminus \{0\}$ and $\alpha_i, \beta_i \in E^*$. Following [20, Definition 4.8], we define the *real degree* (respectively *ghost degree*) of this representation as $\max\{\deg(\alpha_i) \mid 1 \le i \le n\}$ (respectively $\max\{\deg(\beta_i) \mid 1 \le i \le n\}$). Note that, in general, the real degree and ghost degree of x depend on the particular choice of representation. If, however, x has a representation in only real (respectively ghost) edges, that is, if $x = \sum_{i=1}^{n} r_i\alpha_i$ (respectively $x = \sum_{i=1}^{n} r_i\beta_i^*$), then, by Lemma 3.5, the real (respectively ghost) degree is independent of the choice of representation of x in real (respectively ghost) edges.

PROPOSITION 3.14. *Every cycle in E has an exit if and only if for every $u \in E^0$, the inclusion $Z(uL_R(E)u) \subseteq (uL_R(E)u)_0$ holds.*

PROOF. First, we show the 'if' statement by contrapositivity. Suppose that there is a cycle $p \in E^* \setminus E^0$ without any exit. Set $u := s(p)$ and write $p^0 := u$. Take $r \in R$ and $\alpha, \beta \in E^*$ with $s(\alpha) = s(\beta) = u$ and $r(\alpha) = r(\beta)$. Since p has no exit, there are $m, n \in \mathbb{N} \cup \{0\}$ and $\gamma \in E^*$ such that $\alpha = p^m\gamma$ and $\beta = p^n\gamma$. Note that $\gamma\gamma^* = u = pp^*$. This yields $pr\alpha\beta^* = prp^m\gamma\gamma^*(p^*)^n = rp^{m+1}(p^*)^n$ and $r\alpha\beta^*p = rp^m\gamma\gamma^*(p^*)^n p = rp^m(p^*)^n p$. If $n = 0$, then $p^{m+1}(p^*)^n = p^{m+1} = p^m(p^*)^n p$, and if $n > 0$, then $p^{m+1}(p^*)^n = p^m pp^*(p^*)^{n-1} = p^m(p^*)^{n-1}p^*p = p^m(p^*)^n p$. In either case, we get $pr\alpha\beta^* = r\alpha\beta^*p$. Thus, $p \in Z(uL_R(E)u) \setminus (uL_R(E)u)_0$.

Now we show the 'only if' statement. Suppose that every cycle in E has an exit. Take $u \in E^0$. We wish to show that $Z(uL_R(E)u) \subseteq (uL_R(E)u)_0$. By Lemma 2.1(a) and Lemma 3.9, it is enough to show that $(Z(uL_R(E)u))_N = \{0\}$ for every negative integer N.

We now adapt parts of the proof of [3, Theorem 3.1] to our situation. Take $N < 0$. Seeking a contradiction, suppose that the set

$$M := \{(u, x) \mid u \in E^0 \text{ and } x \in (Z(uL_R(E)u))_N \setminus \{0\}\}$$

is nonempty. If $(u, x), (v, y) \in M$, then we write $(u, x) \le (v, y)$ if x has a representation in $L_R(E)$ of real degree less than or equal to all real degrees of representations of y in $L_R(E)$. We write $(u, x) = (v, y)$ whenever $(u, x) \le (v, y)$ and $(v, y) \le (u, x)$. Clearly, $\le$ is a total order on M which therefore has a minimal element (u, x). Choose a minimising representation $x = \sum_{i=1}^{n} e_ia_i + b$, where $e_1, \ldots, e_n \in E^1$ are all distinct, each $a_i \in L_R(E)$ is either zero or nonzero and representable as an element of smaller real degree than that of x, and b is a polynomial (possibly zero) in only ghost paths whose source and range equals u. Take $i \in \{1, \ldots, n\}$. Write $v_i := r(e_i)$. By Lemma 3.10, $e_i^*xe_i \in (Z(v_iL_R(E)v_i))_N$. Since $e_i^*xe_i$ is of smaller real degree than x, it follows that $e_i^*xe_i = 0$. Further, since $x \in (Z(uL_R(E)u))_N$, it follows that $e_i^*x = e_i^*e_ie_i^*x = e_i^*xe_ie_i^* = 0$. Thus, $0 = e_i^*x = a_i + e_i^*b$ and hence $a_i = -e_i^*b$.

Now, $0 \ne x = (u - \sum_{i=1}^{n} e_ie_i^*)b$. Thus, $u \ne \sum_{i=1}^{n} e_ie_i^*$ and $b \ne 0$. This implies that there is some $f \in E^1 \setminus \{e_1, \ldots, e_n\}$ with $s(f) = u$. Furthermore, $f^*x = f^*b$, and, by Lemma 3.5, $f^*b \ne 0$ since it is a sum of distinct ghost paths. Write $v := r(f)$. By Lemma 3.10, it follows that $f^*xf \in (Z(vL_R(E)v))_N$. Observing that $0 \ne f^*x = f^*ff^*x = f^*xff^*$, we get $f^*xf \ne 0$. Note that the real degree of f^*xf is less than or

https://doi.org/10.1017/S0004972723000114 Published online by Cambridge University Press

equal to the real degree of x. Hence, by the assumption made on (u, x), and possibly after replacing (u, x) by (v, f^*xf), we may assume that $a_i = 0$ for every $i \in \{1, \ldots, n\}$. Therefore, suppose that $x = \sum_{j=1}^m r_j\beta_j^*$ for some nonzero $r_j \in R$ and some distinct paths $\beta_j \in E^{-N}$ with $s(\beta_j) = r(\beta_j) = u$. Take $k \in \{1, \ldots, m\}$. By Lemma 3.10, it follows that $r_k\beta_k^* = \beta_k^* x\beta_k \in Z(uL_R(E)u)$. By assumption, the cycle β_k has an exit at some $w \in E^0$. Thus, there are $\gamma, \delta \in E^*$ and $\epsilon \in E^1$ such that $\beta_k = \gamma\delta$, $r(\gamma) = s(\epsilon) = w$ and $\epsilon^*\delta = 0$. By Lemma 3.10, it follows that $r_k(\delta\gamma)^* = r_k\gamma^*\delta^*\gamma^*\gamma = \gamma^* r_k\beta_k^*\gamma \in Z(wL_R(E)w)$. We now reach a contradiction, because $0 \neq \epsilon\epsilon^* r_k(\delta\gamma)^* = r_k(\delta\gamma)^*\epsilon\epsilon^* = 0$. □

Now, we prove our main result.

Proof of Theorem 1.1. First, we show the 'only if' statement. Suppose that $L_R(E)$ is simple. Then $L_R(E)$ is graded simple and hence, by Proposition 3.6, it follows that R is simple and that E^0 has no nontrivial hereditary and saturated subset. Furthermore, Proposition 2.4 implies that $uL_R(E)u$ is simple for every $u \in E^0$, and hence, by Proposition 2.2, $Z(uL_R(E)u) \subseteq (uL_R(E)u)_0$ for every $u \in E^0$. Thus, by Proposition 3.14, every cycle in E has an exit.

Now we show the 'if' statement. Suppose that R is simple, E^0 has no nontrivial hereditary and saturated subset, and every cycle in E has an exit. By Proposition 3.6, $L_R(E)$ is graded simple. Take $u \in E^0$. It follows from Proposition 3.14 that $Z(uL_R(E)u) \subseteq (uL_R(E)u)_0$. Furthermore, by Proposition 2.4, $uL_R(E)u$ is graded simple. Thus, by Proposition 2.2, $uL_R(E)u$ is simple. Hence, by Proposition 2.5, $L_R(E)$ is simple. □

4. The centre of a simple Leavitt path algebra

In this section, we prove Theorem 1.2 using results from the previous sections together with some auxiliary observations.

Remark 4.1. Let $E = (E^0, E^1, r, s)$ be a directed graph.

(a) Take $v \in E^0$. We write $w \leq v$, for $w \in E^0$, if there is $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. The set $T(v) := \{w \in E^0 \mid w \leq v\}$ is the smallest hereditary subset of E^0 containing v.

(b) Suppose that $X \subseteq E^0$. Put $T(X) := \bigcup_{x \in X} T(x)$. The *hereditary saturated closure* $\overline{X}$ of X is defined as the smallest hereditary and saturated subset of E^0 containing X. One can show (see [6, page 626] and the references therein) that $\overline{X} = \bigcup_{n=0}^{\infty} X_n$, where $X_0 := T(X)$ and $X_n := \{y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \text{ and } r(s^{-1}(y)) \subseteq X_{n-1}\} \cup X_{n-1}$ for $n \geq 1$.

The following result can be proved by induction (see [14, Proposition 14.11] and [20, Lemma 5.2]).

Proposition 4.2. *Suppose that R is an associative unital ring and that $E = (E^0, E^1, r, s)$ is a directed graph. If $a \in (L_R(E))_0$ is nonzero, then there exist $\alpha, \beta \in E^*$, $v \in E^0$ and a nonzero $k \in R$ such that $\alpha^* a\beta = kv$.*

Now, we prove our second main result.

https://doi.org/10.1017/S0004972723000114 Published online by Cambridge University Press

PROOF OF THEOREM 1.2. Write $S := L_R(E)$. If S is not unital, then it follows immediately from [21, Ch. 1, Section 3.3] that $Z(S) = \{0\}$. This proves item (a). Now, we show item (b). Suppose that S is unital, that is, E^0 is finite. Take a nonzero $x \in Z(S)$. By Proposition 2.2, it follows that $x \in S_0$. Therefore, by Proposition 4.2, there are $\alpha, \beta \in E^*$, $v \in E^0$ and a nonzero $k \in R$ such that $\alpha^* x \beta = kv$. From this equality, the grading and the fact that $x \in Z(S)$, it follows that $\alpha = \beta$ and $r(\alpha) = v$. Hence, $vx = \alpha^* \alpha x = \alpha^* x \alpha = \alpha^* x \beta = kv$. The equality $vx = kv$ implies that $k \in Z(R)$. Put $X := \{v\}$. Then, $\overline{X}$ is a nonempty hereditary and saturated subset of E^0. By Theorem 1.1, $\overline{X} = E^0$. We claim that this implies that $wx = kw$ for every $w \in E^0$. Let us assume, for a moment, that this claim holds. Then, $x = 1_S \cdot x = \sum_{w \in E^0} wx = \sum_{w \in E^0} kw = k \cdot \sum_{w \in E^0} w = k \cdot 1_S \in Z(R) \cdot 1_S$. Thus, $Z(S) \subseteq Z(R) \cdot 1_S$. Clearly, $Z(R) \cdot 1_S \subseteq Z(S)$ holds.

Now we show the claim. We will use induction to prove that for every $n \geq 0$, the implication $w \in X_n \Rightarrow wx = kw$ holds. From this, the claim follows. Base case: $n = 0$. Suppose that $w \in X_0$, that is, $w \leq v$. Then, there is a path δ from v to w. This gives $wx = \delta^* \delta x = \delta^* v \delta x = \delta^* v x \delta = \delta^* k v \delta = k \delta^* v \delta = k \delta^* \delta = kw$. Induction step: Suppose that $wx = kw$ for every $w \in X_{n-1}$. Take $y \in X_n \setminus X_{n-1}$ and note that $0 < |s^{-1}(y)| < \infty$ and $r(s^{-1}(y)) \subseteq X_{n-1}$. Then, $yx = \sum_{e \in s^{-1}(y)} ee^* x = \sum_{e \in s^{-1}(y)} er(e)xe^* = \sum_{e \in s^{-1}(y)} ekr(e)e^* = k \sum_{e \in s^{-1}(y)} ee^* = ky$. □

References

[1] G. Abrams, 'Leavitt path algebras: the first decade', *Bull. Math. Sci.* **5**(1) (2015), 59–120.

[2] G. Abrams and G. Aranda Pino, 'The Leavitt path algebra of a graph', *J. Algebra* **293**(2) (2005), 319–334.

[3] G. Abrams and G. Aranda Pino, 'The Leavitt path algebras of arbitrary graphs', *Houston J. Math.* **34**(2) (2008), 423–442.

[4] P. N. Ánh and L. Márki, 'Morita equivalence for rings without identity', *Tsukuba J. Math.* **11**(1) (1987), 1–16.

[5] P. Ara, M. A. Moreno and E. Pardo, 'Nonstable K-theory for graph algebras', *Algebr. Represent. Theory* **10**(2) (2007), 157–178.

[6] G. Aranda Pino and K. Crow, 'The center of a Leavitt path algebra', *Rev. Mat. Iberoam.* **27**(2) (2011), 621–644.

[7] J. H. Brown and A. an Huef, 'Centers of algebras associated to higher-rank graphs', *Rev. Mat. Iberoam.* **30**(4) (2014), 1387–1396.

[8] L. O. Clark and C. Edie-Michell, 'Uniqueness theorems for Steinberg algebras', *Algebr. Represent. Theory* **18**(4) (2015), 907–916.

[9] L. O. Clark, C. Farthing, A. Sims and M. Tomforde, 'A groupoid generalisation of Leavitt path algebras', *Semigroup Forum* **89**(3) (2014), 501–517.

[10] L. O. Clark, D. Martín Barquero, C. Martín González and M. Siles Molina, 'Using the Steinberg algebra model to determine the center of any Leavitt path algebra', *Israel J. Math.* **230**(1) (2019), 23–44.

[11] D. Gonçalves, J. Öinert and D. Royer, 'Simplicity of partial skew group rings with applications to Leavitt path algebras and topological dynamics', *J. Algebra* **420** (2014), 201–216.

[12] R. Hazrat, 'The graded structure of Leavitt path algebras', *Israel J. Math.* **195**(2) (2013), 833–895.

[13] E. Jespers, 'Simple abelian-group graded rings', *Boll. Unione Mat. Ital. A (7)* **3**(1) (1989), 103–106.

[14] D. Lännström, P. Lundström, J. Öinert and S. Wagner, 'Prime group graded rings with applications to partial crossed products and Leavitt path algebras', Preprint, 2021, arXiv:2105.09224.

https://doi.org/10.1017/S0004972723000114 Published online by Cambridge University Press

[15] C. Năstăsescu and F. van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Mathematics, 1836 (Springer, Berlin–Heidelberg, 2004).
[16] P. Nystedt, 'A survey of s-unital and locally unital rings', *Rev. Integr. Temas Mat.* **37**(2) (2019), 251–260.
[17] P. Nystedt and J. Öinert, 'Simple semigroup graded rings', *J. Algebra Appl.* **14**(7) (2015), Article no. 1550102.
[18] B. Steinberg, 'A groupoid approach to discrete inverse semigroup algebras', *Adv. Math.* **223**(2) (2010), 689–727.
[19] M. Tomforde, 'Uniqueness theorems and ideal structure for Leavitt path algebras', *J. Algebra* **318**(1) (2007), 270–299.
[20] M. Tomforde, 'Leavitt path algebras with coefficients in a commutative ring', *J. Pure Appl. Algebra* **215**(4) (2011), 471–484.
[21] R. Wisbauer, *Foundations of Module and Ring Theory*, Algebra, Logic and Applications, 3 (Gordon and Breach Science Publishers, Philadelphia, PA, 1991).

PATRIK LUNDSTRÖM, Department of Engineering Science,
University West, SE-46186 Trollhättan, Sweden
e-mail: patrik.lundstrom@hv.se

JOHAN ÖINERT, Department of Mathematics and Natural Sciences,
Blekinge Institute of Technology, SE-37179 Karlskrona, Sweden
e-mail: johan.oinert@bth.se