*Compositio Mathematica* **123:** 131–165, 2000.



# Deformation Quantization of Symplectic Fibrations

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(Received: 15 April 1997; accepted in final form: 14 September 1999)

**Abstract.** A symplectic fibration is a fibre bundle in the symplectic category (a bundle of symplectic fibres over a symplectic base with a symplectic structure group). We find the relation between the deformation quantization of the base and the fibre, and that of the total space. We consider Fedosov's construction of deformation quantization. We generalize the Fedosov construction to the quantization with values in a bundle of algebras. We find that the characteristic class of deformation of a symplectic fibration is the weak coupling form of Guillemin, Lerman, and Sternberg. We also prove that the classical moment map could be quantized if there exists an equivariant connection.

**Mathematics Subject Classifications (2000).** 53D55, 81S10, 53D20, 32G08.

**Key words.** deformation quantization, symplectic fibrations, moment map, Weyl algebra, Fedosov quantization.

## 1. Introduction: Statement of the Problem and the Main Theorem

Quantization is a map from functions on a (phase) space to operators on some Hilbert space. It involves a parameter (usually the Planck's constant $h$ or $\hbar = h/2\pi$). The product of two operators is given by some series in $\hbar$. The quantization map transforms the product of functions into a noncommutative product of corresponding operators. The zero degree term in the $\hbar$-decomposition of a noncommutative function (series in $\hbar$ with functional coefficients) gives a commutative product of functions. The term of degree one linearly depend on the Poisson bracket. This allows one to think of quantization as a deformation of a multiplicative structure of the algebra of functions on a manifold in the direction of the Poisson bracket. In the formal deformation quantization one does not consider questions of convergence of series in $\hbar$. However, the formal deformation turns out to be a useful tool for describing global properties of a manifold. The concept of deformation quantization was described in [1]. When we say quantization we mean deformation quantization.

Fedosov found a beautiful geometrical construction of deformation quantization [8], [10] which we use here. What follows can be considered as an exercise in his domain, it is in fact a generalization of the first part of his article [9] and owes a lot to its methods.

Quantization of a symplectic (or Poisson) manifold $M$ is a construction of a noncommutative associative product on $M$. It is called a $*$-product. A $*$-product is defined as a product on $C^\infty(M)[[h]]$. This noncommutative algebra $\mathbb{A}^\hbar$ should be a deformation of the algebra of functions on the manifold, $C^\infty(M)$. Let $(M, \omega_0)$ be a symplectic manifold. Then the symplectic form $\omega_0$ defines a Lie algebra structure on $C^\infty(M)$, called the Poisson structure. (The Jacobi identity follows from the fact that the form $\omega_0$ is closed.) For $f, g \in C^\infty(M)$ let $\{f, g\} = (df)^\sharp(g)$, where $\sharp : \mathcal{T}^*M \to \mathcal{T}M$, defined by $\omega_0$ (see Section (2.2)).

DEFINITION 1.1. Deformation quantization of a symplectic manifold $(M, \omega_0)$ is an associative algebra $\mathbb{A}^\hbar = C^\infty(M)[[\hbar]]$ with an associative product $*\colon \mathbb{A}^\hbar \times \mathbb{A}^\hbar \to \mathbb{A}^\hbar$ such that

(1) The product $*$ is local, that is, in the $*$-product $a(x, \hbar) * b(x, \hbar) = \sum_{k=0}^{\infty} \hbar^k c_k(x)$, the coefficients $c_k(x)$ depend only on $a_i, b_j$ and their derivatives $\partial^\alpha a_i, \partial^\beta b_j$ with $i + j + |\alpha| + |\beta| \leqslant k$ for any $a(x, \hbar) = \sum_{i=0}^{\infty} \hbar^i a_i(x)$ and $b(x, \hbar) = \sum_{j=0}^{\infty} \hbar^j b_j(x)$, $a_i(x), b_j(x) \in C^\infty(M)$
(2) It is a formal deformation of the commutative algebra $C^\infty(M)$: $c_0(x) = a_0(x)b_0(x)$.
(3) The correspondence principle gives

$$[a, b] := \frac{i}{\hbar}(a * b - b * a) = \{a_0(x), b_0(x)\} + \hbar\, r(a, b),$$

where $r(a, b) \in \mathbb{A}^\hbar$.
(4) There is a unit: $a(x, \hbar) * 1 = 1 * a(x, \hbar) = a(x, \hbar)$.

DeWilde and Lecomte [6] and also Fedosov [8] proved that on any symplectic manifold there exists a quantization.

The following idea lies behind the Fedosov construction (see [7]): a Koszul-type resolution is considered for $C^\infty(M)[[\hbar]]$. Each term of the resolution has a noncommutative algebraic structure, thus providing the algebra of functions with a new noncommutative product. Fedosov constructs such a resolution by using the differential forms on the manifold with values in the Weyl-algebra bundle. The main step is to find a differential on it which respects the algebra structure. This differential is called Fedosov connection and is obtained by an iteration procedure from a torsion free symplectic connection on the manifold.

It is known [20, 22] that any connection on a symplectic manifold gives rise to a torsion-free symplectic connection. Hence, one can get the Fedosov connection from any connection on a tangent bundle: first adding some tensor to make it symplectic (Section 2.2) and then applying the iteration procedure.

We introduce a notion of an $F$-manifold:

DEFINITION 1.2. An $F$-manifold $\Phi$ is the following triple (manifold, deformation of a symplectic form, a connection): $\Phi = (M, \omega, \nabla)$, where $\omega = \omega_0 + \hbar\alpha$, and $\alpha \in \Gamma(M, \Lambda^2 T^*M)[[\hbar]]$, a series in $\hbar$ with coefficients being closed 2-forms on the manifold $M$.

In a recent article [13] a similar object is called a Fedosov manifold, namely, a symplectic manifold together with a symplectic connection. Indeed, these three objects $(M, \omega_0, \nabla)$ define the first three terms in the $*$-product:

- Algebraically a manifold is described by the algebra of functions on it, that is $M$ defines the structure of the commutative product.
- A symplectic form defines the Poisson structure and, hence, the term at $\hbar$.
- A connection defines the term at $\hbar^2$ (as follows from [20]).

It turns out that these three terms determine the higher terms in the $*$ product. The deformation quantization theorem ([8]) can be stated as follows

THEOREM 1.3 *(Fedosov). An $F$-manifold $(M, \omega, \nabla)$ uniquely determines a $*$-product on the underlying manifold $M$.*

Deligne [5] and Nest and Tsygan [25] showed that the class of isomorphisms of quantizations of a symplectic manifold $M$ is determined by the class of the form $\omega$ in $H^2(M)[[\hbar]]$. It is called a *characteristic class of deformation*.

We study the deformation of the *twisted products* of two $F$-manifolds $(B, \omega_B, \nabla^B)$ and $(F, \sigma, \nabla^F)$. Our question is the following: how to define the product of two $F$-manifolds and what the $*$–product on the total space is, in other words, what a 'twisted product of quantizations' is.

We show that under certain assumptions a twisted product of two $F$-manifolds is again an $F$-manifold. So we want to relate the $*$-products on these three manifolds.

One can regard the product $M = B \ltimes F$ as a fibre bundle $M \to B$ over a symplectic base $B$ with a symplectic fibre $F$. Obviously, the product depends on how twisted the symplectic fibration $M = B \ltimes F$ is. This can be described by a connection on $M \to B$ which should be compatible with the symplectic form on $M$.

When the structure group of the bundle $M \to B$, $G$, acts by symplectomorphisms on fibres this bundle is a symplectic fibration ([15], see Section 3). The total space $M$ is symplectic and the fibres are symplectic submanifolds of it, provided that the action be Hamiltonian and $G$ be a finite-dimensional Lie group.

The case of $B$ being a cotangent bundle of some differentiable manifold $X$ was first developed by Sternberg [27]. His construction describes the movement of a 'classical particle' in Yang–Mills field for any gauge group $G$ and any differentiable $X$. Weinstein gave a general construction of the symplectic form on $M$ in [29].

The quantization of a symplectic fibration is a quantization with coefficients in the auxiliary bundle of fiberwise quantizations. This leads us to a more general case of quantization with coefficients in some bundle of algebras (some examples of such bundles are considered in [10]). Our case is more difficult than the quantization with values in most other auxiliary bundles. Fibres of the auxiliary bundle obtained from the symplectic fibration structure are noncommutative algebras. This non-commutativity of fibres makes the quantization procedure more complicated.

We believe that it is useful to understand that the quantization of symplectic fibrations respects a fibre bundle structure in order to see that the quantization is a fundamental notion like some homology theory.

We make a new definition:

DEFINITION 1.4. An $F$-bundle (with an underlying manifold $B$) is a triple $\Psi = (\Phi, \mathbb{A}, \nabla^{\mathbb{A}})$, where

- $\Phi = (B, \omega^B, \nabla^B)$ is an $F$-manifold,
- $\mathbb{A}$ is an auxiliary bundle of algebras over $B$,
- $\nabla^{\mathbb{A}}$ is a covariant derivative on $\mathbb{A}$, which respects the algebra structure on the fibres.

We construct an $F$-bundle from a symplectic fibration $M \to B$. Each fibre of $M \to B$ is an $F$-manifold, so we can quantize the fibres. Provided the connection $\nabla^F$ be $G$-invariant we can construct a new bundle $\mathbb{A} \to B$, the bundle of algebras of quantized functions on fibres. The fibre of $\mathbb{A}$ over a point $b$ is the quantization of the fibre of $M \to B$ at that point:

$$\mathbb{A}_b = \mathbb{A}^\hbar(M_b). \tag{1}$$

The bundle $\mathbb{A}$ is defined by (1). A connection on the bundle $M \to B$ determines a covariant derivative $\nabla^{\mathbb{A}}$ on the bundle $\mathbb{A}$. Its construction is carried out in section (5.1) in the way that it respects the algebraic structure, that is this covariant derivative is a derivation of the $*$-product on $\mathbb{A}$.

An $F$-bundle corresponds to an $F$-manifold modeled on $M$, the total space of the symplectic fibration, and hence provides a quantization of the total space. Our main theorem is:

THEOREM 1.5. *Consider a symplectic fibration $M \to B$ with a standard fibre being an $F$-manifold $(F, \sigma, \nabla^F)$ and the base an $F$-manifold $\Phi = (B, \omega^B, \nabla^B)$. An $F$-bundle $(\Phi, \mathbb{A}, \nabla^{\mathbb{A}})$ with fibres $\mathbb{A}_b = \mathbb{A}^\hbar(M_b)$ gives a quantization of the underlying manifold $B$ with values in the auxiliary bundle. This also defines a quantization of the total space $M$.*

The main claim of this theorem is that a quantization of the base with values in the auxiliary bundle (1) corresponds to a certain $F$-manifold $(M, \omega, \nabla)$, with $\omega$ being a series in $\hbar$ starting from a symplectic form on $M$.

To carry out the program first of all one has to construct a symplectic form on $M$. There is a one-parameter family of symplectic forms on the total space. The construction involves the notion of the weak coupling limit of Guillemin, Lerman and Sternberg [15]. The behavior of the $*$-product when the parameter tends to zero gives us a way to understand the relation between quantizations of the base and the fibre with the quantization of the total space.

Our main Theorem (1.5) can be reformulated as a statement about the solutions of two equations given in Theorem (5.5).

Fedosov's quantization procedure is discussed in Section 2. The classical setup for symplectic fibrations is discussed in Section 3 and the quantization of the moment map is presented in Section 4. Main results about the quantization of symplectic fibrations are given in Section 5, examples are discussed in the last section.

NOTATIONS

Repeated indices assume summation.

Grading and filtration of the Weyl algebra bundle are $\mathbb{Z}$-grading and $\mathbb{Z}$-filtration, we do not use the natural $\mathbb{Z}_2$-grading on the differential forms.

For any bundle $\mathcal{E} \to M$, $\mathcal{A}^n(M, \mathcal{E})$ denotes $C^\infty$-sections of $n$-form bundle with values in the bundle $\mathcal{E}$,

$$\mathcal{A}^k(M, \mathcal{E}) = \Gamma(M, \Lambda^k \mathcal{T}^* M \otimes \mathcal{E}); \quad \mathcal{A}(M, \mathcal{E}) = \oplus_{k=0}^{\infty} \mathcal{A}^k(M, \mathcal{E}).$$

$\mathcal{A}^n(M)$ denotes the bundle of $n$-forms on $M$.

The term 'connection' is used in two senses:

- for a covariant derivative on any vector bundle, usually denoted by $\nabla$
- for a connection on a fibre bundle that is a splitting of the tangent bundle to the total space of a fibre bundle into a sum of a vertical and a horizontal subbundles.

## 2. Generalities on Deformation Quantization

The subject of this section becomes nowadays fairly standard (see for example an excellent introduction to Fedosov quantization [19]).

### 2.1. WEYL ALGEBRA OF A VECTOR SPACE

Let $E$ be a vector space with a nondegenerate skew-symmetric form $\omega$. The algebra of polynomials on $E$ is the algebra of symmetric powers of $E^*$, $S(E^*)$, and it has a

https://doi.org/10.1023/A:1002452002677 Published online by Cambridge University Press

skew-symmetric form on it which is dual to $\omega$. Let $e$ be a point in $E$ and $\{e^i\}$ denote its linear coordinates in $E$ with respect to some fixed basis. Then $\{e^i\}$ define a basis in $E^*$. Let $\omega^{ij}$ be the matrix for the skew-symmetric form on $E^*$. Let us consider the power series in $\hbar$ with values in $S(E^*)$:

DEFINITION 2.1. The Weyl algebra $W(E^*)$ of a vector space $E^*$ is an associative algebra

$$W(E^*) = S(E^*)[[\hbar]] : \quad a(e, \hbar) = \sum_{k \geqslant 0} a_k(e)\hbar^k,$$

given by the Moyal–Vey product:

$$a \circ b(e, \hbar) = \exp\left\{-\frac{i\hbar}{2}\omega^{kl}\frac{\partial}{\partial x^k}\frac{\partial}{\partial z^l}\right\} \; a(x, \hbar)\; b(z, \hbar)\Big|_{x=z=e}. \tag{2}$$

The Lie bracket is defined with respect to this product. We can look at this algebra as at a completion of the universal enveloping algebra of the Heisenberg algebra on $E^* \oplus \hbar\mathbb{C}$, namely, the algebra with relations

$$e^i \circ e^j - e^j \circ e^j = -i\hbar\omega^{ij} \tag{3}$$

where $\omega^{ij} = \omega(e^i, e^j)$ defines a Poisson bracket on $E^*$.

Hence, one can define the Weyl algebra as

$$W(E^*) = U(E^* \oplus \hbar\mathbb{C}).$$

Let us consider the product of the Weyl algebra and the exterior algebra of the space $E^*$: $W(E^*) \otimes \Lambda E^*$. Let $dx^i$ be the basis in $\Lambda E^*$ corresponding to $e^i$ in $W(E^*)$.

There is a decreasing filtration on the Weyl algebra $W(E^*)$: $W_0 \supset W_1 \supset W_2 \supset \ldots$ given by the degree of generators. The generators $e$'s have degree 1 and $\hbar$ has degree 2, so that the condition (3) is homogeneous of degree 2:

$$W_p = \{\text{elements with degree} \geqslant p\}.$$

One can define a grading on $W$ as follows

$$gr_i W = \{\text{elements with degree} = i\}.$$

it is isomorphic to $W_i/W_{i+1}$. One can see that the product (2) preserves the grading.

DEFINITION 2.2. An operator on $W(E^*) \otimes \Lambda E^*$ is said to be of degree $k$ if it maps $W_i \otimes \Lambda E^*$ to $W_{i+k} \otimes \Lambda E^*$ for all $i$.

Such an operator defines maps $gr_i W \otimes \Lambda E^*$ to $gr_{i+k} W \otimes \Lambda E^*$ for all $i$.

DEFINITION 2.3. Derivation on $W(E^*) \otimes \Lambda E^*$ is a linear operator which satisfies the Leibniz rule:

$$D(ab) = (Da)b + (-1)^{\tilde{a}\tilde{D}} a(Db)$$

where $\tilde{a}$ and $\tilde{D}$ are corresponding degrees. It turns out that all $\mathbb{C}[[\hbar]]$-linear derivations are inner.

LEMMA 2.4. *Any linear derivation $D$ on $W(E^*) \otimes \Lambda E^*$ is inner, namely there exists such $v \in W(E^*)$ so that $Da = i/\hbar[v, a]$ for any $a \in W(E^*)$*

*Proof.* Indeed, $\partial/\partial e^i a = i/2\hbar[\omega_{ij} e^j, a]$.

So for any derivation one can get a formula:

$$Da = \frac{i}{\hbar}\left[\frac{1}{2}\omega_{ij} e^i De^j, a\right]. \qquad \square$$

One can define two natural operators on the algebra $W(E^*) \otimes \Lambda E^*$: $\delta$ and $\delta^*$ of degree $-1$ and 1 correspondingly. The operator $\delta$ is the lift of the 'identity' operator

$$u : e^i \otimes 1 \to 1 \otimes dx^i$$

and $\delta^*$ is the lift of its inverse. On monomials $e^{i_1} \otimes \ldots \otimes e^{i_m} \otimes dx^{j_1} \wedge \ldots \wedge dx^{j_n} \in W^m(E^*) \otimes \Lambda^n E^*$ $\delta$ and $\delta^*$ can be written as follows:

$$\delta : e^{i_1} \otimes \ldots \otimes e^{i_m} \otimes dx^{j_1} \wedge \ldots \wedge dx^{j_n} \mapsto \sum_{k=1}^{m} e^{i_1} \otimes \ldots \widehat{e^{i_k}} \ldots \otimes e^{i_m} \otimes dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_n}$$

$$\delta^* : e^{i_1} \otimes \ldots \otimes e^{i_m} \otimes dx^{j_1} \wedge \ldots dx^{j_n} \mapsto \sum_{l=1}^{n} (-1)^l e^{j_l} \otimes e^{i_1} \otimes \ldots \otimes e^{i_m} \otimes dx^{j_1} \wedge \ldots \widehat{dx^{j_l}} \ldots \wedge dx^{j_n}$$

LEMMA 2.5. *Operators $\delta$ and $\delta^*$ have the following properties:*

$$\delta a = dx^j \frac{\partial a}{\partial e^j} = \left[-\frac{i}{\hbar}\omega_{kl}\, e^k dx^l,\ a\right], \quad \delta^* a = e^j \iota_{\frac{\partial}{\partial x^j}} a, \quad \delta^2 = \delta^{*2} = 0$$

*On monomials $e^{i_1} \otimes \ldots \otimes e^{i_m} \otimes dx^{j_1} \wedge \ldots \wedge dx^{j_n}$*

$$\delta\delta^* + \delta^*\delta = (m+n)Id,$$

*where $Id$ is the identity operator. Any element $a \in gr_m W(E^*) \otimes \Lambda^n E^*$ has a decomposition:*

$$a = \frac{1}{m+n}(\delta\delta^* a + \delta^*\delta a) + a_0,$$

*where $a_0$ is a projection of $a$ to the center of the algebra, that is the summands in $a$ which do not contain e-s.*

### 2.2. SYMPLECTIC CONNECTIONS (COVARIANT DERIVATIVES)

The term *symplectic connection* in this section in fact must be changed to *symplectic covariant derivative* to avoid confusion with another symplectic connection notion in the next Section. However there is already an established practice to call a covariant derivative a connection which we decided to follow here. We hope that one can get used to distinguish one from the other from the context.

Let us consider connections on a manifold $M$.

PROPOSITION 2.6. *Let $\omega$ be a skew-symmetric 2-form on $\mathcal{T}M$. Then there exists a torsion-free connection $\nabla$ preserving this form only if $\omega$ is closed.*

*Proof.* The skew-symmetry of $\omega$ is the following condition: $\omega(X, Y) = -\omega(Y, X)$. The connection $\nabla$ is torsion-free when $\nabla_X Y - \nabla_Y X = [X, Y]$. Suppose such $\nabla$ exists. Then it preserves the form $\omega$ when $\nabla\omega = 0$. This means that for all $X, Y, Z \in \mathcal{T}M$:

$$\nabla_X(\omega(Y, Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) \tag{4}$$

Since $\omega(Y, Z)$ is a function $\nabla_X(\omega(Y, Z)) = X\omega(Y, Z)$. Then,

$$\begin{aligned} X\omega(Y, Z) - Y\omega(X, Z) + Z\omega(X, Y) \\ &= \omega(\nabla_X Y, Z) - \omega(\nabla_X Z, Y) - \omega(\nabla_Y X, Z) + \\ &\quad + \omega(\nabla_Y Z, X) + \omega(\nabla_Z X, Y) - \omega(\nabla_Z Y, X) \\ &= \omega([X, Y], Z) - \omega([X, Z], Y) + \omega([Y, Z], X) \end{aligned}$$

which is exactly the condition $d\omega = 0$. □

*Remark* 2.7. Here we want to make an analogy with a Riemannian case. The Riemannian metric is a symmetric two-form and there is a unique torsion free connection compatible with it (the Levi-Civita connection).

The statement of the uniqueness of Levi-Civita connection in the Riemannian case is substituted by the requirement for the form to be closed in the skew-symmetric setting:

- Symmetric: A torsion-free compatible connection always exists and unique (Levi-Civita connection).
- Skew-symmetric: A torsion-free compatible connection exists if the form is closed and, in general, not unique.

Let $M$ be a symplectic manifold with a symplectic form $\omega$, a closed and nondegenerate skew-symmetric form 2-form on $\mathcal{T}M$. There are many different torsion free connections preserving it.

DEFINITION 2.8. A connection which preserves a symplectic form is called a symplectic connection.

Any connection on a symplectic manifold gives rise to a symplectic connection:

PROPOSITION 2.9 ([20, 22]). *Let $\omega$ be a closed nondegenerate 2-form. Then for every connection $\nabla$ there exists a 3-tensor $S$, such that*

$$\tilde{\nabla} = \nabla + S$$

*be a connection on $\mathcal{T}M$ compatible with $\omega$. Then*

$$\hat{\nabla}_X Y = \tilde{\nabla}_X Y - \frac{1}{2} Tor(X, Y); \quad X, Y \in \mathcal{T}M$$

*defines a torsion-free connection compatible with the form $\omega$. The 2-form Tor is the torsion of the connection $\tilde{\nabla}$: $Tor(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - \tilde{\nabla}_{[X,Y]}$*

The tensor $S$ is defined as follows:

$$S_X Y = \frac{1}{2}\{(\nabla_X \omega)(Y, .)\}^{\sharp},$$

where $\sharp : \mathcal{T}^*M \to \mathcal{T}M$ is the inverse to $\flat : \mathcal{T}M \to \mathcal{T}^*M \quad u^{\flat} = \omega(u, .)$ *for* $u \in \mathcal{T}M$. Symplectic connections form an affine space with the associated vector space $\mathcal{A}^1(M, sp(2n))$, the Lie algebra $sp(2n)$ valued 1-forms on $M$.

## 2.3. DEFORMATION QUANTIZATION OF A SYMPLECTIC MANIFOLD

Let $M^{2n}$ be a symplectic manifold with a symplectic form $\omega$. In local coordinates at a point $x$:

$$\omega = \omega_{ij} dx^i \wedge dx^j.$$

The symplectic form on a manifold $M$ defines a Poisson bracket on functions on $M$. For any two functions $u, v \in C^{\infty}(M)$:

$$\{u, v\} = \omega^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \tag{5}$$

where $(\omega^{ij}) = (\omega_{ij})^{-1}$.

We can define the bundle of Weyl algebras $\mathcal{W}_M$, with the fibre at a point $x \in M$ being the Weyl algebra of $\mathcal{T}_x^*M$. Let $\{e^1, \dots e^{2n}\}$ be $2n$ generators in $\mathcal{T}_x^*M$, corresponding to $dx^i$. The form $\omega^{ij}$ defines pointwise Moyal–Vey product.

The filtration and the grading in $\mathcal{W}_M$ are inherited from $W(\mathcal{T}_x^*M)$ at each point $x \in M$. Denote by $\mathcal{W}^i$ the $i$th graded component in $\mathcal{W}_M$:

$$\mathcal{W}_M = \oplus_i \mathcal{W}^i.$$

https://doi.org/10.1023/A:1002452002677 Published online by Cambridge University Press

A symplectic connection $\nabla$ acts on any symmetric power of the cotangent bundle by the Leibniz rule. Moreover, the cotangent bundle $\mathcal{T}^*M \cong \mathcal{W}^1$, hence $\nabla$ acts naturally on the sections $\Gamma(M, \mathcal{W}^i)$ with values in $\Gamma(M, \mathcal{W}^i \otimes \mathcal{T}^*M)$.

It preserves the grading, in other words it is an operator of degree zero. In general this connection is not flat: $\nabla^2 \neq 0$. Fedosov's idea is that for $\mathcal{W}_M$ bundle one can add to the initial symplectic connection some operators not preserving the grading so that the sum gives a flat connection on the Weyl bundle.

THEOREM 2.10 (Fedosov). *There is a unique set of operators* $r_k : \Gamma(M, \mathcal{W}^i) \to \Gamma(M, \mathcal{T}^*M \otimes \mathcal{W}^{i+k})$ *such that*

$$D = -\delta + \nabla + r_1 + r_2 + \dots \tag{6}$$

*is a flat connection and* $\delta^* r_i = 0$. *There is a one-to-one correspondence between formal series in* $\hbar$ *with coefficients in smooth functions* $C^\infty(M)$ *and horizontal sections of this connection:*

$$Q : C^\infty(M)[[\hbar]] \to \Gamma_{flat}(M, \mathcal{W}_M). \tag{7}$$

Main idea of the proof is to use the following complex:

$$0 \to \Gamma(M, \mathcal{W}) \xrightarrow{\delta} \mathcal{A}^1(M, \mathcal{W}) \xrightarrow{\delta} \mathcal{A}^2(M, \mathcal{W}) \xrightarrow{\delta} \dots. \tag{8}$$

This complex is exact since $\delta$ is homotopic to the identity by $\delta^*$. An equation for $r_i$ for each $i > 1$ has the form

$$\delta(r_i) = \text{function}(\nabla, r_1, \dots, r_{i-1}). \tag{9}$$

It is not difficult to show that this function is in the kernel of $\delta$ hence $r_i$ exists.

First few terms in the recursive construction of $D$, its flat sections and $*$-product in coordinates are given in [18].

The noncommutative structure on the Weyl bundle determines a $*$-product on functions by this correspondence, namely for two functions $f, g \in \mathbb{C}^\infty(M)[[\hbar]]$;

$$f * g = Q^{-1}(Q(f) \circ Q(g)). \tag{10}$$

In fact, the equation $D^2 = 0$ is just the Maurer–Cartan equation for a flat connection. One can see the analogy with the Kazhdan connection [12] on the bundle of algebras of formal vector fields. Notice that $\delta = dx^i \partial/\partial e^i$ is of degree $-1$. The flatness of the connection is given by the recurrent procedure, namely starting from the terms of degree $-1$ and 0 one can get other terms step by step. While Kazhdan connection does not have a parameter involved it has the same structure – it starts with known $-1$ and 0 degree terms. Other terms are of higher degree and can be recovered one by one.

Let us also mention here that locally the connection $D$ can be written as a sum of two terms – one being a derivation along the manifold, the usual differential $d$, and the other, denote it $\Gamma$, being an endomorphism of a fibre of the Weyl algebra bundle. Since all endomorphisms are inner, one can write it as an adjoint action with respect to the Moyal–Vey product. $\Gamma$ acts adjointly by an operator from $\Gamma(M, \mathcal{W})$ to $\Gamma(M, \mathcal{T}^*M \otimes \mathcal{W})$.

$$D = d + \Gamma = d + \frac{i}{\hbar}[\gamma, \cdot]_\circ, \tag{11}$$

where $\gamma \in \Gamma(M, \mathcal{T}^*M \otimes \mathcal{W})$. Then the equation $D^2 = 0$ becomes

$$d\Gamma + \frac{1}{2}[\Gamma, \Gamma]_\circ = 0.$$

The same equation for $\gamma$ then is as follows:

$$\omega + d\gamma + \frac{i}{\hbar}\frac{[\gamma, \gamma]_\circ}{2} = 0, \tag{12}$$

where $\omega$ is a central 2-form. This equation states that $D^2$ is given by an adjoint action of a central element, so it is zero. However, it turns out to be very important which exactly form $\omega$ is given in the center by the connection $D$. Inner automorphisms of the Weyl algebra are given by the adjoint action of elements of the algebra (Lemma 2.4). Its central extension gives the whole algebra. Curvature of the Fedosov connection is zero, however its lift to the central extension is nonzero and defines the isomorphism class of quantization.

DEFINITION 2.11. The characteristic class of the deformation quantization is the cohomology class of the form $[\omega] \in \frac{1}{\hbar}H^2(M)[[\hbar]]$.

## 2.4. QUANTIZATION WITH VALUES IN A BUNDLE OF ALGEBRAS

This subject was discussed at length in the book [9], but here we want to look at it from a slightly different angle. Given an $F$-manifold $(B, \omega, \nabla^B)$ we know how to construct a map $C^\infty(B) \to \Gamma(B, \mathcal{W}_B)$. Let $\mathcal{L} \to B$ be a bundle of $\mathbb{C}[[\hbar]]$-algebras. Now we want to generalize the problem of quantization and obtain a map:

$$Q^{\mathcal{L}} : \Gamma(B, \mathcal{L}) \to \Gamma(B, \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathcal{L}).$$

In order to do that we need a connection $\nabla^{\mathcal{L}}$ on $\mathcal{L}$. Let $R^{\mathcal{L}} \in \mathcal{A}^2(B, \mathcal{L})$ be its curvature. Also the sections of $\mathcal{L}$ must commute with sections of the Weyl algebra bundle $\mathcal{W}_B$. Then we can define a connection on $\mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathcal{L}$ as the sum of connections.

$$\nabla = \nabla^B \otimes 1 + 1 \otimes \nabla^{\mathcal{L}}$$

with the curvature

$$R = R^B \otimes 1 + 1 \otimes R^{\mathcal{L}}.$$

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We define a grading on $\mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathcal{L}$ as on $\mathcal{W}_B$. The operator $R^{\mathcal{L}}$ could have a degree if it changes the power of $\hbar$, and since the degree of $\hbar$ is 2 it could only be even: $R^{\mathcal{L}} = \sum R^{\mathcal{L}}_{2k},\ k \geqslant 0$ where

$$R^{\mathcal{L}}_{2k} : \mathcal{A}^q(B, gr_{\cdot}(\mathcal{L})) \to \mathcal{A}^{q+2}(B, gr_{\cdot+2k}(\mathcal{L})).$$

One should add an extra term in the equation on $r_i$ (9) for each even $i = 2k$

$$\delta(r_{2k}) = \text{function}(\nabla, r_1, \ldots, r_{2k-1}) + R^{\mathcal{L}}_{2k}.$$

Then like in Theorem 2.10 we can consider a complex similar to (8)

$$0 \to \Gamma(B, \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathcal{L}) \overset{\delta}{\to} \mathcal{A}^1(B, \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathcal{L}) \overset{\delta}{\to} \mathcal{A}^2(B, \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathcal{L}) \overset{\delta}{\to} \ldots$$

with $\delta$ acting only in $\mathcal{W}_B$. This complex is still exact (the $\otimes$-product is over a field $\mathbb{C}[[\hbar]]$), since $\delta R^{\mathcal{L}} = 0$ one can find a preimage of $R^{\mathcal{L}}$ and the reasoning is exactly as before. The flat connection and the corresponding flat sections are constructed similarly to [18].

However, in the case when $\mathcal{L}$ has a Lie algebra structure and $R^{\mathcal{L}}$ is an inner action of the form $R^{\mathcal{L}} = i/\hbar\, \mathrm{ad}H$ for some $H \in \mathcal{A}^2(B, \mathcal{L})$ the procedure changes! The adjoint action might start from the degree $-2$ term and one has to change not only the equations (9), but also the initial $\delta$ to balance it. This is exactly what happens in the case of symplectic fibrations and what makes it more interesting.

## 3. Symplectic Forms on Symplectic Fibrations

In this section we collect the facts known about symplectic fibrations: we give a definition and construct a one-parameter family of symplectic forms on the total space. One can find a nice exposition in the sixth chapter of the book [21], see also [15].

The meaning of the term *symplectic connection* used in this section is different from Definition 2.8. It is a connection which preserves the symplectic structure on fibres (see Definition 3.3) while in Section 2 the symplectic connection was in fact a symplectic covariant derivative preserving a symplectic form on a symplectic manifold.

### 3.1. SYMPLECTIC FIBRATIONS

DEFINITION 3.1. A symplectic fibration is a locally trivial fibration $\pi : M \to B$ with a symplectic fibre $(F, \sigma)$ whose structure group preserves the symplectic form $\sigma$ on $F$. This means that

(1) There is an open cover $U_\alpha$ of $B$ and a collection of diffeomorphisms $\phi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}U_\alpha & \xrightarrow{\phi_\alpha} & U_\alpha \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle pr} & \\ & U_\alpha & \end{array}$$

(2) For the fibre over $b \in B$, $F_b = \pi^{-1}(b)$, let $\phi_\alpha(b)$ denote the restriction of $\phi_\alpha$ to $F_b$ followed by projection onto $F$, $\phi_\alpha(b) : F_b \to F$. Then

$$\phi_{\beta\alpha}(b) = \phi_\beta(b) \circ \phi_\alpha(b)^{-1} \quad \in Symp(F, \sigma)$$

for all $\alpha$, $\beta$ and $b \in U_\alpha \bigcap U_\beta$.

If $\pi : M \to B$ is a symplectic fibration then each fibre $F_b$ carries a symplectic structure $\sigma_b \in \mathcal{A}^2(F_b)$ defined by

$$\sigma_b = \phi_\alpha(b)^*\sigma$$

for $b \in U_\alpha$. The symplectomorphism class of the form is independent of $\alpha$ as follows from the definition. Also, if there is a $G$-invariant symplectic torsion-free connection $\nabla^F$ on $F$ it defines a symplectic torsion-free connection $\nabla_b$ on each fibre $F_b$.

DEFINITION 3.2. A symplectic form $\omega$ on the total space $M$ of a symplectic fibration is called compatible with the fibration $\pi$ if each fibre $(F_b, \sigma_b)$ is a symplectic submanifold of $(M, \omega)$, with $\sigma_b$ being the restriction of $\omega$ to $F_b$.

## 3.2. SYMPLECTIC CONNECTIONS

Each symplectic form on $M$ compatible with the symplectic fibration $\pi : M \to B$ defines a connection on it, that is a choice of splitting of the following short exact sequence of vector bundles:

$$0 \to \mathcal{V}M \to \mathcal{T}M \to \pi^*\mathcal{T}B \to 0.$$

Here $\mathcal{V}M$ is the canonically defined bundle of vertical tangent vectors, that is those fields which vanish on functions coming from the base.

DEFINITION 3.3. A connection on a fibre bundle $M \to B$ is a splitting of the tangent bundle of $M$ into a sum of subbundles

$$\Gamma : \quad \mathcal{T}M = \mathcal{H}M \oplus \mathcal{V}M, \tag{13}$$

where $\mathcal{H}M = \pi^*\mathcal{T}B$. The connection is compatible with a symplectic form on $M$, $\omega$ if at each point $x \in M$

$$\mathcal{H}_xM := \{X \in \mathcal{T}_xM | \quad \omega(X, V) = 0 \quad \text{for all } V \in \mathcal{V}_xM\}.$$

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Each symplectic form such that its restriction to fibres is nondegenerate defines a compatible connection. Namely, the horizontal subbundle consists of all vector fields which are perpendicular to the vertical ones with respect to the symplectic form.

### 3.3 INGREDIENTS: A CONNECTION ON A PRINCIPAL BUNDLE AND A HAMILTONIAN ACTION ALONG THE FIBRES

Symplectic fibrations are associated fibre bundles to the principal bundles with a structure group being the group of symplectomorphisms of the fibre, so we have a principal $G$-bundle and a symplectic manifold $(F, \sigma)$ to start with.

Let us first consider a principal $G$-bundle, that is a smooth manifold $P$ with a smooth action $P \times G \longrightarrow P$ which is free and transitive. Then the quotient $P/G = B$ is a manifold.

For a principal bundle a connection can be defined by a so-called connection 1-form. Namely, the fibres of a vertical subbundle $\mathcal{V}P$ are naturally identified with $\mathfrak{g}$ under the map: $\mathfrak{g} \to Vect(P)$ given by the infinitesimal action of $G$ on $P$.

$$X \in \mathfrak{g} \mapsto \hat{X} \in Vect(P).$$

Hence the horizontal subbundle $\mathcal{H}P$ can be described not only as a kernel of the projection operator $Pr : \mathcal{T}P \to \mathcal{V}P$, but also as a kernel of a connection 1-form:

$$\lambda : \mathcal{T}P \to \mathfrak{g}.$$

It is a $G$–invariant form on the principal $G$-bundle $P$ with values in the Lie algebra $\mathfrak{g}$, such that

$$\iota_{\hat{X}}\lambda = X, \quad \text{for} \quad X \in \mathfrak{g}.$$

Now let $G$ act on a symplectic manifold $(F, \sigma)$ by symplectomorphisms, that is there is a group homomorphism

$$G \to Symp(F, \sigma) : \quad g \mapsto \psi_g.$$

The infinitesimal action determines the Lie algebra homomorphism

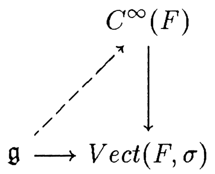
$$\mathfrak{g} \to Vect(F, \sigma) : X \mapsto \hat{X}, \quad \text{defined by} \quad \hat{X} = \frac{d}{dt}\Big|_{t=0} \psi_{exp(tX)} \tag{14}$$

A symplectic form determines a correspondence between functions and certain vector fields, called Hamiltonian vector fields:

$$C^\infty(F) \to Vect(F, \sigma) : H \mapsto X^H, \quad \text{defined by} \quad \iota_{X^H}\sigma = dH. \tag{15}$$

DEFINITION 3.4. The action of $G$ on $F$ is called Hamiltonian if

(1) There is a lift $\mathfrak{g} \to C^\infty(F): \quad X \mapsto H_X$



This means that there is a Hamiltonian function $H_X$ so that $\iota_{\hat{X}}\sigma = dH_X$.

(2) This map is a Lie algebra homomorphism $H_{[X,Y]} = \{H_X, H_Y\}$. $[X, Y]$ is the Lie bracket in $\mathfrak{g}$ and $\{H_X, H_Y\}$ is the Poisson bracket in $C^\infty(F)$.

(If a group action satisfies only first condition it is called weakly Hamiltonian.) Let us also mention the following equality:

$$H_{[X,Y]} = \hat{X} H_Y - \hat{Y} H_X \tag{16}$$

Hamiltonian action determines a map $\mu : F \to \mathfrak{g}^*$, for each point $x \in F$ defined by $\langle \mu(x) | X \rangle = H_X(x)$ for any $X \in \mathfrak{g}$, where $\langle \cdot | \cdot \rangle$ is the pairing: $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$. This map is usually called a moment map, however for our purposes of quantization we will call a map $\mathfrak{g} \to C^\infty(F)$ a moment map as well. It is this map of algebras which we are going to quantize.

### 3.4. WEAK COUPLING: CONNECTION $\longleftrightarrow$ SYMPLECTIC FORM

The following proposition in its present form is an adaptation for our purposes of a theorem about weak coupling form from [15].

PROPOSITION 3.5. *Let $G \to Symp(F, \sigma) : g \mapsto \psi_g$ be a Hamiltonian action on $(F, \sigma)$ with a moment map $\mu^F : \mathfrak{g} \to C^\infty(F)$. Then every connection $\Gamma$ on the principal $G$-bundle $P \to B$ over a symplectic manifold $(B, \omega^B)$, gives rise to a one-parameter family of symplectic forms on the associated bundle $M = P \times_G F \to B$, which restricts to the forms $\sigma_b$ on the fibres:*

$$\Omega_\varepsilon = \varepsilon^2 \omega^\Gamma + \pi^* \omega^B, \tag{17}$$

*where $\varepsilon$ is a small parameter and $\omega^\Gamma$ is the coupling form. This coupling form at each point $x \in M$, $\pi(x) = b$ is*

$$\omega^\Gamma = \sigma_b + H_T, \tag{18}$$

*where $T \in \mathcal{A}^2(B, \mathcal{V}M)$: $T(X, Y) = -Pr([X^H, Y^H])$, $X, Y \in \mathcal{T}B$. $Pr$ is the projection to the vertical subbundle of $\mathcal{T}M$, $\mathcal{V}M$. The Hamiltonian function $H_T$ is defined as follows: $\iota_T \sigma_b = dH_T$*

*Remark* 3.6. Notice that $\sigma_b$ is nonzero only on vertical vectors, while $H_T$ is nonzero only on horizontal ones. This extra term $H_T$ is needed for the form to be closed, the small parameter $\varepsilon$ insures that the form $\Omega_\varepsilon$ is nondegenerate.

*Proof.* Sketch (for the full proof see [15] or [21]). The main idea is to use the so called Weinstein universal phase space – $W = P \times \mathfrak{g}^*$.

Given a connection on $P$ the space $W$ could be identified with the vertical subbundle of the cotangent bundle

$$W = P \times_G \mathcal{T}^*G = \mathcal{V}^*P.$$

A connection is a splitting $\Gamma : \mathcal{T}P = \mathcal{H}P \oplus \mathcal{V}P$ and $\mathcal{V}^*P$ is defined as 1-forms which vanish on horizontal vectors: $\mathcal{V}^*P = (\mathcal{H}P)^\perp$. Hence, it has a $G$-equivariant symplectic form coming from the canonical symplectic form on $\mathcal{T}^*P$. Moreover, the action of the group $G$ on $W$ is Hamiltonian.

The moment map $\mu^W : W \to \mathfrak{g}^*$ is given by the projection

$$pr_{\mathfrak{g}^*} : W \to \mathfrak{g}^*.$$

Then the symplectic reduction of $W \times F$ at 0 value of the moment map $\mu = \mu^W + \mu^F$ is exactly $M = P \times_G F$, and the symplectic form on $M$ is inherited from $W$.

The explicit formula is obtained in the following way. Let the connection $\Gamma$ be given by a connection 1-form, $\lambda_p : \mathcal{T}_pP \to \mathfrak{g}$. It determines a horizontal subbundle in $\mathcal{T}P$ by

$$\mathcal{H}_pP = \{v \in \mathcal{T}_pP \mid \lambda_p(v) = 0\}.$$

$\mathcal{V}^*P = (\mathcal{H}P)^\perp$ is also defined by $\lambda$. The connection $\lambda : \mathcal{T}P \to \mathfrak{g}$ together with the action $\rho : \mathfrak{g} \to \mathcal{V}P$ define the injection

$$\iota_\lambda : \mathcal{V}^*P \hookrightarrow \mathcal{T}^*P.$$

By definition of the connection 1-form this injection is equivariant under the action of $G$ and hence the 2-form

$$\omega_\lambda = \iota_\lambda^* \, \omega_{can} \in \mathcal{A}^2(\mathcal{V}^*P)$$

is invariant under the action of $G$. This pull-back of the canonical symplectic form on $\mathcal{T}^*P$ gives a closed 2-form on $\mathcal{V}^*P$.

The canonical 1-form $\alpha$ on $\mathcal{T}^*P$ is defined as follows. Let $(p, s_p)$ be a point in $\mathcal{T}^*{}_pP$, let also $v$ be a tangent vector field in the tangent bundle $\pi : \mathcal{T}(\mathcal{T}^*P) \to \mathcal{T}^*P$, then at a point $(p, s_p)$, $\langle \alpha | v \rangle_{(p,s_p)} = -\langle s_p | \pi_* v \rangle_p$. Since the pullback of the canonical 1-form to $\mathcal{V}^*P$ is $\langle \iota_\lambda^* \alpha | v \rangle_{(p,s_p)} = -\langle pr_{\mathfrak{g}^*} s_p | \lambda(\pi_* v) \rangle_p$ so

$$\omega_\lambda = -d\langle pr_{\mathfrak{g}^*} | \lambda \rangle. \tag{19}$$

The form on $W \times F$ is $\omega_\lambda + \sigma$. The form on the reduced space $M = (W \times F)//G$ at a regular value of the moment map $\mu^W + \mu^F = 0$ is obtained as follows. Since $\mu^W$ is

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given by the projection to $\mathfrak{g}^*$, $\mu^W = pr_{\mathfrak{g}^*}$ the form on $M$ becomes

$$\omega^\Gamma = d\langle \mu^F | \lambda \rangle + \sigma.$$

The 1-form $\langle \mu^F | \lambda \rangle$ can be rewritten as a Hamiltonian $H_\lambda$. It should be understood in the following way. The connection 1-form $\lambda : \mathcal{T}P \to \mathfrak{g}$ defines a connection on the associated bundle $M = P \times_G F$. The horizontal subbundle in $\mathcal{T}M$ is the image of $\mathcal{H}P$ under the map $P \times F \to M$. By abuse of notation we call the map $\mathcal{T}M \to \mathfrak{g}$ also $\lambda$. From now on $\lambda$ is a 1-form on $M$ with values in the Lie algebra $\mathfrak{g}$. Hence $H_\lambda$ is a 1-form on $M$.

Its differential gives a two form $d\langle \mu^F | \lambda \rangle = d(H_\lambda)$. Applied to two vectors $V, W \in \mathcal{T}M$ it gives

$$d(H_\lambda)(V, W) = VH_\lambda(W) - VH_\lambda(W) - H_\lambda([V, W]).$$

Using (16) we get that it is nonzero only on horizontal vectors and gives the Hamiltonian of the curvature which in this case is the commutator $[V, W]$. Also we see that $\omega^\Gamma$ restricted to fibres gives the symplectic form on the fibres. □

This construction is quite general [15]: the symplectic fibrations with a connection constructed this way turn out to include all symplectic fibre bundles with a connection for which the holonomy group is a finite dimensional Lie group.

### 3.5. LOCAL COORDINATES

Let us take a point $x \in M$. One can introduce a local frame $\{f_\alpha\}$ of vertical tangent bundle $\mathcal{V}M$ and a local frame $\{e_i\}$ in $\mathcal{T}B$ at a point $b = \pi(x)$ of $B$, with dual frames $\{f^\alpha\}$ and $\{e^i\}$. Using a connection we obtain a local frame on the tangent bundle $\mathcal{T}M = \pi^*\mathcal{T}B \oplus \mathcal{V}M$ at a point $x$. Then the form can be written as a block matrix:

$$\Omega_\varepsilon = \begin{vmatrix} \pi^*\omega^B + \varepsilon^2 H_T & 0 \\ 0 & \varepsilon^2 \sigma_b \end{vmatrix}. \tag{20}$$

Hence, the corresponding Poisson bracket is also a block matrix:

$$\begin{vmatrix} (\pi^*\omega^B + \varepsilon^2 H_T)^{-1} & 0 \\ 0 & \varepsilon^{-2}\sigma_b^{-1} \end{vmatrix}.$$

We see that the Moyal product with respect to this form is a product of those on the base and on the fibres.

Locally $M = B \times F$ and the connection can be written as $\nabla = d_x + A$, where $A$ is a 1-form on $B$ with values in $\mathfrak{g}$. If the we consider the map: $\mathfrak{g} \to C^\infty(F)$, $A$ becomes the 1-form on $B$ with values in $C^\infty(F)$, that is a $(1, 0)$ form on $M$. $H_T$ in this local form becomes just $dA$.

## 4. Quantum Moment Map

Let $(F, \Sigma, \nabla)$ be an $F$-manifold. Here $\Sigma$ is a deformation of the symplectic form $\sigma$:

$$\Sigma = \sigma + \hbar\sigma_1 + \hbar^2\sigma_2 + \cdots,$$

$\sigma_i$ being closed 2–forms on $F$. Let $\mathbb{A}^\hbar(F)$ be the corresponding quantization of $F$ with the characteristic class $[\Sigma] \in H^2(F)[[\hbar]]$. $\mathbb{A}^\hbar(F)$ is a noncommutative algebra of formal series in $\hbar$ with coefficients being smooth functions on $F$. The $*$-product on $\mathbb{A}^\hbar(F)$ defines the Lie algebra structure:

$$[f, g]_* = \frac{i}{\hbar}(f * g - g * f), \quad \text{for } f, g \in \mathbb{A}^\hbar(F).$$

Let $G$ be a group acting on $F$ so that the action is Hamiltonian (Definition 3.4). Let $\mathfrak{g} \to C^\infty(F)$ be its moment map. There is an induced action of $G$ on $\mathbb{A}^\hbar(F)$. We want to quantize the moment map, namely, get a Lie algebra map from the algebra $\mathfrak{g}$ to the quantized algebra $\mathbb{A}^\hbar(F)$. However it is possible only up to a two-cocycle in $C^\infty(F)[[\hbar]]$, so we get a projective representation, otherwise we should consider a central extension of $\mathbb{A}^\hbar(F)$. We could also slightly change the definition of a quantum moment map. We eliminate central elements by considering a map into the adjoint representation of $\mathbb{A}^\hbar(F)$, the inner automorphisms $Inn\ \mathbb{A}^\hbar(F)$. They obviously inherit the Lie algebra structure from $\mathbb{A}^\hbar(F)$, so

DEFINITION 4.1. A quantum moment map is a map of Lie algebras $\mu^{Lie} : \mathfrak{g} \to Inn\ \mathbb{A}^\hbar(F)$. In particular it means that there is the correspondence principle: $lim_{\hbar\to 0}\, \mu^{Lie}(X)(f) = \{H_X, f\}$ for $X \in \mathfrak{g}$ and $f \in \mathbb{A}^\hbar(F)$.

*Remark* 4.2. This definition could be reformulated through a homomorphism of associative algebras $\mu : U(\mathfrak{g}) \to \mathbb{A}^\hbar(F)$, such that on vector fields it gives $\mu(X)^{Lie}(f) = [\mu(X), f]_*$.

PROPOSITION 4.3. *Consider a Hamiltonian action of a group $G$ on $(F, \sigma)$, with the Hamiltonian function $X \mapsto H_X$ for $X \in \mathfrak{g}$. Let*

$$\left(F, \Sigma = \sigma + \sum_{i=1}^{\infty} \hbar^i \sigma_i, \nabla\right)$$

*be an $F$-manifold such that $\nabla$ is a $G$-invariant connection. Assume also that one can define functions $H^i_X$, $i = 1, 2, \cdots$ as follows*

$$\iota_{\hat{X}}\sigma_i = dH^i_X. \tag{21}$$

*Let $\mathbb{A}^\hbar(F)$ be the algebra of quantized functions on $F$. Then the quantum moment map*

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*is given by*

$$\mu(X) = H_X + \sum_{i=1}^{\infty} \hbar^i H^i{}_X \tag{22}$$

*Also* $\mu^{Lie} : \mathfrak{g} \to Inn\, \mathbb{A}^{\hbar}(F)$, $\mu(X)^{Lie}(f) = [\mu(X), f]_*$ *is a homomorphism of Lie algebras:*

$$\mu^{Lie}([X, Y]) = [\mu^{Lie}(X), \mu^{Lie}(Y)]_*. \tag{23}$$

*Moreover, there are no higher terms in* $\hbar$ :

$$[\mu(X), f]_* = [H_X, f]_* = \{H_X, f\}. \tag{24}$$

*Proof.*★ We are going to prove (22–24) by lifting the Hamiltonian $H_X$ to a section of the Weyl algebra bundle, since the $*$-product on $C^\infty(F)$ is defined by the Moyal–Vey product on the Weyl algebra bundle (7),(10). Namely, let $\Gamma_{\text{D-flat}}(F, \mathcal{W}_F)$ be the space of flat sections of the Fedosov connection $D$ constructed from $\nabla$, corresponding to the quantization of $F$ with the characteristic class $\Sigma$. Then there is a one-to-one correspondence $Q : C^\infty(F)[[\hbar]] \to \Gamma_{\text{D-flat}}(F, \mathcal{W}_F)$ which defines a product on $F$ by $f * g = Q^{-1}(Q(f) \circ Q(g))$. The structure of a Lie algebra in the Weyl algebra bundle $\mathcal{W}_F$ is defined by the fiberwise commutator

$$[\mathbf{a}, \mathbf{b}]_\circ = \mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a}, \quad \textit{for } \mathbf{a}, \mathbf{b} \in \Gamma(F, \mathcal{W}_F)$$

Recall that the map $H : \mathfrak{g} \to C^\infty(F)$ is given by the condition

$$\iota_{\hat{X}} \sigma = dH_X, \tag{25}$$

where $\hat{X}$ is a vector field corresponding to $X$ under the map $\mathfrak{g} \to Vect(F)$ (14). We find the image of a Hamiltonian $H_X$ in $\Gamma_{\text{D-flat}}(F, \mathcal{W}_F)$ by generalizing the proof of Fedosov [10, Propositions 5.8.1,2],[11] to the case of a deformed symplectic form. By construction the Fedosov connection $D$ is $G$-equivariant since any element of the group $G$ preserves the initial symplectic connection $\nabla$. An easy calculation shows also that flat sections of an equivariant connection $D$ are also equivariant. It also means that the Lie derivative of $D$ is zero with respect to any vector field $\hat{X}$ : $[L_{\hat{X}}, D]_\circ = 0$. Since $L_{\hat{X}}$ and $D_{\hat{X}} = \iota_{\hat{X}} D + D \iota_{\hat{X}}$ are first-order derivations commuting with $D$ we can find an analogue of the Cartan homotopy formula for the Lie derivative on forms with values in the Weyl algebra bundle. The difference of $L_{\hat{X}}$ and $D_{\hat{X}}$ could only be an inner automorphism of the Weyl algebra bundle, we denote it $[Q(X), \cdot]_\circ$ :

$$L_{\hat{X}} \mathbf{a} = (\iota_X D + D \iota_X)\mathbf{a} + [Q(X), \mathbf{a}]_\circ. \tag{26}$$

It is easy to see that the equality (26) is true in Darboux coordinates chosen so that the field $\hat{X}$ is just a pure derivation in the direction of only one of the coordinates.

★I am grateful to Boris Tsygan for pointing out a gap in the proof in an earlier version of the paper.

Then $D = D^0 = d + \delta$ and $Q(X) = Q^0(X) = central\ section(X, \hbar) - \imath_X \delta$. Let $D = D^0 + [\Delta\gamma, \cdot]_\circ$ be another flat connection, $\Delta\gamma$ being an equivariant one form in $\mathcal{W}_F$. Then since the $L_{\hat{X}}$ does not change and commutes with the new $D$ as well, the right-hand side of (26) must not change either, so one has to subtract $\imath_X \Delta\gamma$ from $Q^0(X)$.

We want to show that we could chose the *central section* $(X, \hbar)$ in $Q(X)$ to be equal to $\mu(X) = H_X + \hbar \ldots$ so that $Q(X) = \mu(X) - \imath_X \gamma$ becomes a quantization of the moment map, that is a flat section corresponding to $H_X$.

Recall that locally we can write a connection $D$ as $Df = df + i/\hbar[\gamma, f]_\circ$ (11). The equation $D^2 = 0$ becomes (12):

$$\Sigma + d\gamma + \frac{i}{\hbar}\frac{[\gamma, \gamma]_\circ}{2} = 0. \tag{27}$$

Then since $L_{\hat{X}} D = 0$, from (26)

$$0 = \left\{\imath_{\hat{X}}\left(d + \frac{i}{\hbar}[\gamma, \cdot]_\circ\right) + \left(d + \frac{i}{\hbar}[\gamma, \cdot]_\circ\right)\imath_{\hat{X}}\right\}(D) - [\imath_X\gamma, D]_\circ = (\imath_{\hat{X}} d + d\imath_{\hat{X}})(D)$$

we get

$$(\imath_{\hat{X}} d + d\imath_{\hat{X}})\gamma = 0. \tag{28}$$

Using (21),(25), and also (27),(28) we find that indeed $Q(X)$ is a flat section of $D$:

$$\begin{aligned} DQ(X) &= d\{H_X + \hbar H_X^1 + \cdots\} - d(\imath_X\gamma) + \frac{i}{\hbar}[\gamma, \{H_X + \hbar H_X^1 + \cdots\}]_\circ - \frac{i}{\hbar}[\gamma, \imath_X\gamma]_\circ \\ &= \imath_X\Sigma + \imath_X d\gamma + 0 + \imath_X\left(\frac{i}{\hbar}\frac{[\gamma, \gamma]_\circ}{2}\right) = \imath_X\left(\Sigma + d\gamma + \frac{i}{\hbar}\frac{[\gamma, \gamma]_\circ}{2}\right) = 0. \end{aligned}$$

We also get that $\mu(X) = H_X + \Sigma_{i=1}^{\infty}\hbar^i H^i{}_X$, so the commutator

$$\begin{aligned} [\mu(X), f]_* &= \frac{i}{\hbar}Q^{-1}\{[Q(X), Q(f)]_\circ\} = \frac{i}{\hbar}Q^{-1}\{[(H_X + \hbar H_X^1 + \cdots - \imath_{\hat{X}}\gamma), Q(f)]_\circ\} \\ &= \frac{i}{\hbar}Q^{-1}\{[-\imath_{\hat{X}}\gamma, Q(f)]_\circ\} = Q^{-1}\{-\imath_{\hat{X}}(D - d)Q(f)\} = Q^{-1}\{\imath_{\hat{X}} dQ(f)\} \\ &= L_{\hat{X}} f = \hat{X}(f) = \{H_X, f\}. \end{aligned}$$

Then since the action is Hamiltonian we get on the quantum level:

$$\begin{aligned} [\mu^{Lie}(X), \mu^{Lie}(Y)]_* &= ad[H_X, H_Y]_* \\ &= ad(\{H_X, H_Y\}) = ad(H_{[X,Y]}) = \mu^{Lie}([X, Y]). \qquad \square \end{aligned} \tag{29}$$

For the case $\Sigma = \sigma$, in the absence of additional terms $\hbar^i H^i$, we do not need to consider the adjoint representation. The notion of the quantum moment map repeats the classical one, it is a map from $\mathfrak{g}$ to $\mathbb{A}^\hbar(F)$:

COROLLARY 4.4. *Consider a quantization* $\mathbb{A}^\hbar(F)$ *obtained from the F-manifold* $(F, \sigma, \nabla)$. *Let a group G act on F, and the connection* $\nabla$ *be equivariant with respect to this action. If the action is Hamiltonian with the Hamiltonian function* $X \mapsto H_X$, $X \in \mathfrak{g}$, *the quantum moment map is a homomorphism of Lie algebras* $\mu : \mathfrak{g} \to \mathbb{A}^\hbar(F)$:

$$\mu(X) = H_X \quad and \quad [\mu(X), f]_* = \{H_X, f\}.$$

*Remark* 4.5. We get that $\mu^{Lie} : \mathfrak{g} \to Inn\ \mathbb{A}^\hbar(F)$ is a map of Lie algebras. This gives the positive answer to a question posed in [28] that every classical moment map can be uniquely lifted to a quantum moment map to $\mathbb{A}^\hbar(F)$.

## 5. Quantization of Twisted Products

### 5.1. AUXILIARY BUNDLE: QUANTIZATION OF THE FIBRES

In this section we consider the bundle of quantizations along the fibres of our symplectic fibration $M \to B$ and construct a covariant derivative on this quantized bundle from the connection on the bundle $M \to B$.

*Fibrewise Quantization*
Again, let $\pi : M \to B$ be a locally trivial fibration with a fibre $F$. Let $F$ be an $F$-manifold $(F, \Sigma, \nabla)$. Here we take $\Sigma$ to be arbitrary characteristic class, that is a cohomology class $[\Sigma] \in H^2(F)[[\hbar]]$. $\Sigma = \sigma + \hbar\sigma_1 + \hbar^2\sigma_2 + \cdots$, where $\sigma$ is a symplectic form on $F$.

Let $\mathbb{A}^\hbar(F)$ be a quantization of $F$, that is a noncommutative associative algebra of formal series in $\hbar$ with coefficients being $C^\infty$-functions on $F$. We are defining the bundle $\mathbb{A}$ over $B$ such that its fibres are algebras of quantized functions on the fibres of the bundle $\pi : M \to B$, that is the fibre of $\mathbb{A}$ at a point $b \in B$ is $\mathbb{A}_b = \mathbb{A}^\hbar(M_b)$. The structure group $G$ of $\pi : M \to B$ acts on $F$ by symplectomorphisms. If $\nabla$ is $G$–equivariant there is a $G$-action on $\mathbb{A}^\hbar(F)$. Since $M = P \times_G F$ is an associated bundle to a $G$–bundle $P \to B$, the auxiliary bundle $\mathbb{A}$ is also associated to $P$:

$$\mathbb{A} = P \times_G \mathbb{A}^\hbar(F). \tag{30}$$

*Covariant derivative on an auxiliary bundle*
A covariant derivative on the bundle $\mathbb{A} \to B$ respecting the algebra structure can be obtained from a connection 1-form on the principal bundle. Understanding of the formula for this covariant derivative is important for the sequel. We also find that the curvature of this covariant derivative equals to the adjoint action of the second summand in the coupling form (18).

In general, a choice of a connection on the principal $G$-bundle $P$ determines a covariant derivative on any associated vector bundle (we follow the exposition in [2]).

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Let $\lambda \in \mathcal{A}^1(P, \mathfrak{g})$ be a connection 1-form on a principal $G$-bundle $P$. Let also $G$ act on some vector space $E$, and the action be given by the map $\rho$, $\rho : G \to End(E)$. Then the bundle $\mathcal{E} = P \times_G E \to B$ is an associated bundle to the principal bundle $P$. The space of differential forms on $B$ with values in $\mathcal{E}$, $\mathcal{A}^k(B, P \times_G E)$, can be described as a subspace of the space of differential forms on $P$ with values in $E$. This subspace is a space of all basic forms with values in $E$, $\mathcal{A}(P, E)_{bas}$. A *basic differential form* on a principal bundle $P$ with a structure group $G$, taking values in the representation $(E, \rho)$ of $G$, is an invariant and horizontal differential form, that is a form $\alpha \in \mathcal{A}(P, E)$ which satisfies the conditions

(1) $g \cdot \alpha = \alpha, \quad g \in G$
(2) $\iota(X)\alpha = 0$ for any vertical field $X$ on $P$.

An element in $P \times_G E$ is defined by its representative $p \times s \in P \times E$. Let us denote it as $[p, s]$.

LEMMA 5.1. *If* $\alpha \in \mathcal{A}^q(P, E)_{bas}$, *define* $\alpha_B \in \mathcal{A}^q(B, P \times_G E)$ *by*

$$\alpha_B(\pi_* X_1, \ldots, \pi_* X_q)(b) = [p, \alpha(X_1, \ldots, X_q)(p)],$$

*where* $p \in P$ *is any point such that* $\pi(p) = b$, *and* $X_i \in \mathcal{T}_p P$. *Then* $\alpha_B$ *is well defined, and the map* $\alpha \to \alpha_B$ *is an isomorphism from* $\mathcal{A}^q(P, E)_{bas}$ *to* $\mathcal{A}^q(B, P \times_G E)$.

As a particular case, there is a representation of the sections of $\mathcal{E}$ as $G$–equivariant functions on $P$ with values in $E$. Let $C^\infty(P, E)^G$ denote the space of equivariant maps from $P$ to $E$, that is those maps $s : P \to E$ that satisfy $s(p \cdot g) = \rho(g)s(p)$. There is a natural isomorphism between $\Gamma(B, P \times_G E)$ and $C^\infty(P, E)^G$, given by sending $s \in C^\infty(P, E)^G$ to $s_B$ defined by $s_B(b) = [p, s(p)]$; here $p$ is any element of $\pi^{-1}(b)$ and $[p, s(p)]$ is the element of $\mathcal{E} = P \times_G E$ corresponding to $(p, s(p)) \in C^\infty(P, E)^G$. Infinitesimally, a function $s$ in $C^\infty(P, E)^G$ satisfies the formula:

$$(X_P \cdot s)(b) + \rho(X)s(b) = 0, \quad \text{for } X \in \mathfrak{g},$$

where we also denote by $\rho$ the differential of the representation $\rho$:

$$\rho : \mathfrak{g} \to Vect(E). \tag{31}$$

Given a connection 1–form $\lambda$ on $P$ one obtains the covariant derivative $\nabla$ on the associated vector bundle $\mathcal{E}$ from the following commutative diagram:

$$\begin{array}{ccc} C^\infty(P,E)^G & \xrightarrow{d+\rho(\lambda)} & \mathcal{A}^1(P,E)_{bas} \\ \downarrow & & \downarrow \\ \Gamma(B,\mathcal{E}) & \xrightarrow{\nabla} & \mathcal{A}^1(B,\mathcal{E}) \end{array} \tag{32}$$

In other words, the covariant derivative is defined as follows: $(\nabla_X s)(b) = [p, (X^H s_P)(p)]$, where $X \in \mathcal{T}B$ and $X^H$ is its horizontal lift to the principal bundle $P$. The formula for the covariant derivative on our auxiliary bundle bellow should be understood by means of the diagram (32).

*Covariant Derivative on* $\mathbb{A}$

Now let us return to our particular case, namely

$$\mathcal{E} = \mathbb{A} = P \times_G \mathbb{A}^\hbar(F), \text{ so } E = \mathbb{A}^\hbar(F),$$

Then (31) becomes a map $\rho : \mathfrak{g} \to Vect(\mathbb{A}^\hbar(F))$, or in other words it is given by the moment map $\mu^{Lie} : \mathfrak{g} \to Inn\ \mathbb{A}^\hbar(F)$.

PROPOSITION 5.2. *Covariant derivative on* $\mathbb{A} \to B$ *corresponding to a connection 1–form* $\lambda$ *on* $P \to B$ *is given by the formula:*

$$\nabla^{\mathbb{A}} f = df + [H_\lambda, f]_* = df + \{H_\lambda, f\}. \tag{33}$$

*Its curvature is a* 2*–form on* $B$ *with values in* $\mathbb{A}$*:*

$$R^{\mathbb{A}} f = \{H_T, f\}, \quad \text{where} \quad T(X, Y) = -Pr[X^H, Y^H], \quad Pr : \mathcal{T}P \to \mathfrak{g}.$$

*Proof.* The covariant derivative formula follows from (25) and the diagram (32). Then the general definition of the curvature of a covariant derivative $\nabla : \Gamma(B, E) \to \mathcal{A}^1(B, E)$ is

$$R(X, Y) = \nabla_{\tilde{X}}\nabla_{\tilde{Y}} - \nabla_{\tilde{Y}}\nabla_{\tilde{X}} - \nabla_{[\tilde{X},\tilde{Y}]},$$

where $X, Y \in \mathcal{T}B$, and $\tilde{X}, \tilde{Y}$ are their horizontal lifts. In our particular case the expression for the curvature follows from the formulas (24) and (29). □

## 5.2. FEDOSOV CONNECTION AND FLAT SECTIONS ON SYMPLECTIC FIBRATIONS

In this section we will show that the complex of differential forms with values in some twisted Weyl bundle gives a resolution of $\Gamma(B, \mathbb{A})$.

*Bundle of Sections*

The structure of the bundle $\pi : M \to B$ is reflected in the representation of the space of functions on $M$ as sections of a certain bundle over $B$. Namely, let $\mathcal{F}$ be the bundle

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over $B$, such that a fibre over $b \in B$ is the space of series in $\hbar$ with coefficients being functions on the fibre $M_b$ of the bundle $\pi : M \to B$:

$$\mathcal{F}_b = C^\infty(M_b)[[\hbar]].$$

Then $C^\infty(M)[[\hbar]]$ can be represented as sections of the bundle $\mathcal{F}$:

$$C^\infty(M)[[\hbar]] = \Gamma(B, \mathcal{F}). \tag{34}$$

Hence, we can try to obtain a quantization of $M$ by quantizing the bundle $\mathcal{F} \to B$. This leads us to consider the twisted Weyl algebra bundle over $B$.

Let $\mathcal{W}_M$ be the Weyl algebra bundle on $M$. Consider $\mathcal{W}_{M/B}$ — the bundle on $B$, such that its fibre over $b \in B$ be $\mathcal{W}_b$, the space of sections of the Weyl bundle on the fibre $M_b$. A symplectic connection (13) on the bundle $\pi : M \to B$ leads to an isomorphism of bundles

$$\beta : \Gamma(M, \mathcal{W}_M) \to \Gamma(B, \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \Gamma(M/B, \mathcal{W}_{M/B})) \tag{35}$$

or more generally for differential forms.

LEMMA 5.3. *A symplectic connection on an F-bundle* $M \to B$ :

$$\mathcal{T}M = \mathcal{H}M \oplus \mathcal{V}M, \quad \mathcal{H}_z M \cong \pi^* \mathcal{T}_{\pi(z)} B$$

*defines an isomorphism of bundles:*

$$\mathcal{A}^n(M, \mathcal{W}_M) \to \oplus_{p+q=n} \mathcal{A}^p(B, \mathcal{W}_B \otimes_{C[[\hbar]]} \mathcal{A}^q(M/B, \mathcal{W}_{M/B})) \tag{36}$$

*Proof.* At each point $z$ of $M$ the Weyl algebra can be defined as the universal enveloping algebra of the Heisenberg algebra of $\mathcal{T}_z^* M \oplus \hbar\mathbb{R}$. The universal enveloping algebra is by definition a quotient of the free tensor algebra by the ideal

$$I = \{e \otimes f - e \otimes f + i\hbar\omega^{-1}(e, f)\}, \quad \text{for } e, f \in \mathcal{T}_z^* M,$$

where $\omega$ is a symplectic form on $M$. The connection on $\mathcal{T}M$ splits the ideal into a sum of a horizontal and a vertical ideals:

$$I = I_H + I_V.$$

Thanks to the splitting of the symplectic form (20) vertical ideal $I_V$ is also an ideal in the tensor algebra of $\mathcal{V}^* M$, which leads to the result.

We consider fiberwise quantization as a first step in the quantization of the total space. This gives the auxiliary bundle described in Section 5.1. The quantization map along the fibres is

$$Q^{\text{fibre}} : \Gamma(B, \mathcal{F}) \to \Gamma(B, \mathbb{A}). \tag{37}$$

*Twisted Weyl Algebra Bundle*
Let $\mathcal{W}_B$ be the Weyl algebra bundle corresponding to the $F$–manifold $\{B, \omega, \nabla^B\}$. Consider a twisted bundle

$$^{\mathbb{A}}\mathcal{W}_B = \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathbb{A} \to B,$$

where $\mathbb{A}$ is a bundle of quantization along the fibres (30). Thanks to the splitting of the form (20) the sections of $\mathcal{W}_B$ and sections of $\mathbb{A}$ commute with each other.

*Remark* 5.4. $\Gamma(B, {}^{\mathbb{A}}\mathcal{W}_B)$ can be considered as a space of fiberwise flat sections of the bundle obtained by $\beta$ (35): $\Gamma(B, \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \Gamma(M/B, \mathcal{W}_{M/B}))$. Indeed, the $F$-manifold $(F, \sigma, \nabla^F)$ induces a structure of $F$-manifold on each fibre. It defines corresponding isomorphism of flat sections of Weyl algebra bundles:

$$\mathbb{A} = \Gamma_{\text{flat}}(M/B, \mathcal{W}_{M/B}) \tag{38}$$

The bundle $^{\mathbb{A}}\mathcal{W}_B$ is a bundle of graded algebras with degrees assigned as in the original $\mathcal{W}_B$ bundle, namely,

$$deg(\hbar) = 2, \quad deg(e^i) = 1, \quad e^i\text{'s being generators of } \mathcal{W}_B. \tag{39}$$

The filtration

$$({}^{\mathbb{A}}\mathcal{W}_B)_n = \{s \in {}^{\mathbb{A}}\mathcal{W}_B, \quad \text{such that} \quad deg(s) \geqslant n\}$$

also defines a grading

$$gr_n({}^{\mathbb{A}}\mathcal{W}_B) = \{s \in {}^{\mathbb{A}}\mathcal{W}_B, \quad \text{such that} \quad deg(s) = n\},$$

so that $gr_n({}^{\mathbb{A}}\mathcal{W}_B)$ is isomorphic to $({}^{\mathbb{A}}\mathcal{W}_B)_n/({}^{\mathbb{A}}\mathcal{W}_B)_{n+1}$. The Moyal–Vey product $\circ$ for $\mathcal{W}_B$ and the noncommutative product $*$ on the auxiliary bundle $\mathbb{A}$ define a pointwise noncommutative product on $^{\mathbb{A}}\mathcal{W}_B$. Let us denote this product on $^{\mathbb{A}}\mathcal{W}_B$ also by $\circ$. The noncommutative product on $\mathbb{A}$ contains terms in different degrees in $\hbar$, but does not have any other degree bearing terms. Hence, the product on $^{\mathbb{A}}\mathcal{W}_B$ is not preserving the grading anymore.

The symplectic connection on $B$ and the connection on the auxiliary bundle give rise to a connection on $^{\mathbb{A}}\mathcal{W}_B$: $\nabla = \nabla^B \otimes 1 + 1 \otimes \nabla^{\mathbb{A}}$.

In what follows we will omit the tensor signs, this should not cause a confusion. The curvature of this connection is $R = R^B + R^{\mathbb{A}}$. Following the general scheme we want to construct a flat connection $\nabla$ by adding to the existing one operators of degree $-1$ and higher.

*Main Theorem*

THEOREM 5.5. *The equation* $(D^{\mathbb{A}})^2 = 0$ *for the connection*

$$D^{\mathbb{A}} : \mathcal{A}^q(B, {}^{\mathbb{A}}\mathcal{W}_B) \to \mathcal{A}^{q+1}(B, {}^{\mathbb{A}}\mathcal{W}_B)$$

*has a solution in the form:*

$$D^{\mathbb{A}} = d + \frac{i}{\hbar}[\gamma, \cdot], \tag{40}$$

*where* $\gamma \in \mathcal{A}^1(B, Inn^{\mathbb{A}}\mathcal{W}_B)$ *is a sum of operators of degree* $\geqslant -1$, *so that* $d + i/\hbar[\gamma, \cdot] = \nabla + \delta + r$, *in particular* $deg(\delta) = -1$ *and*

$$r = \sum_{k \geqslant 1} r_k, \quad r_k \in \mathcal{A}^1(B, gr_k{}^{\mathbb{A}}\mathcal{W}_B).$$

*The solution satisfying a normalization condition* $\delta^{-1}r = 0$ *is unique.*

*Flat sections of this connection are in one-to-one correspondence with sections of the auxiliary bundle,* $\Gamma(B, \mathbb{A})$*:*

$$Q^{\mathbb{A}} : \Gamma(B, \mathbb{A}) \to \Gamma_{\text{flat}}(B, {}^{\mathbb{A}}\mathcal{W}_B). \tag{41}$$

*Proof.* The main equation is $\omega + d\gamma + i/\hbar[\gamma, \gamma] = 0$. It can be rewritten

$$R + \frac{1}{2}[\nabla, \delta] + \frac{1}{2}[\nabla, r] + \delta^2 + \frac{1}{2}[\delta, r] + r^2 = 0.$$

For each $k > 1$ we get an equation expressing $r_k$ in terms of $r_i, i \leqslant k$ :

$$[\delta, r_k] = [\nabla, r_{k-1}] + \sum_{i=1}^{k-2}[r_i, r_{k-i}]. \tag{42}$$

However, the equation on $r_1$ which must kill the curvature gives an unusual term in $-1$ degree. Namely, the equation on $r_1$: $\delta r_1 = R^B + R^{\mathbb{A}}$ gives a solution which is a sum of two terms in degrees 1 and $-1$ (sic!). Indeed in a local frame:

$$r_1 = \delta^{-1}(R^B + R^{\mathbb{A}}) = \delta^{-1}(R^B_{ijkl}e^ie^j + R^{\mathbb{A}}_{kl})dx^kdx^l = (R^B_{ijkl}e^ie^je^k + R^{\mathbb{A}}_{kl}e^k)dx^l.$$

The operator $R^{\mathbb{A}}_{kl}e^kdx^l = i/\hbar \; ad\{H_{T_{kl}}e^kdx^l\}$ acts on a section $\mathbf{s} \in \Gamma(B, {}^{\mathbb{A}}\mathcal{W}_B)$ in the following way:

$$\frac{i}{\hbar}[H_{T_{kl}}e^kdx^l, \mathbf{s}] = \frac{i}{\hbar}[e^k, \mathbf{s}]_\circ H_{T_{kl}}dx^l + \{H_{T_{kl}}, \mathbf{s}\}e^kdx^l,$$

where $T_{kl}$ is an element in $\mathfrak{g}$ on $M$ and, hence, its action on $\mathbb{A}^\hbar(F)$ is defined, so it is also defined on sections of the bundle ${}^{\mathbb{A}}\mathcal{W}_B$ : $[H_{T_{kl}}, \mathbf{s}]_* = \{H_{T_{kl}}, \mathbf{s}\} = T_{kl}\mathbf{s}$.

The operator $i/\hbar \; \text{ad}_\circ\{e^k\}H_{T_{kl}}dx^l$ is of degree $-1$. This term has to be added to $\delta$, the initial $-1$-degree operator, but as soon as it is present it changes all the equations (42) since there is not only $\delta$ in degree $-1$ anymore.

However, if we change $\delta$ the iteration method can still be applied yielding a solution for the flat connection. In the central extension the curvature of this connection

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starts from the term of degree $-2$ :

$$\frac{i}{\hbar}\mathrm{ad}_\circ\{(\omega_{kl} + H_{T_{kl}})dx^k \wedge dx^l\}.$$

We are looking for a square root of it in the form ${}^{\mathbb{A}}\delta = i/\hbar A_{ik}e^i dx^k$. That is applied twice ${}^{\mathbb{A}}\delta$ must give this central element:

$$\frac{i}{\hbar}(\omega_{kl} + H_{T_{kl}})dx^k \wedge dx^l = gr_{(-2)}\left[\frac{i}{\hbar}A_{ik}e^i dx^k, \frac{i}{\hbar}A_{jl}e^j dx^l\right].$$

Obviously, $A_{jl} \in \Gamma(B, \mathbb{A})$ should be in the form of some series in $\omega$ and $H_T$:

$$A_{ik}\omega^{ij}A_{jl} = \omega_{kl} + H_{T_{kl}},$$

or

$$A_{ik} = \omega_{im}(1_{mk} + 1/2\ \omega^{mn}H_{T_{nk}} + \ldots) = \omega_{im}\left(\sqrt{1 + \omega^{-1}H_T}\ \right)^m_k$$

We have used here the skew symmetry of the forms $\omega_{kl} = -\omega_{lk}$ and $H_{T_{kl}} = -H_{T_{lk}}$. This series converges if the ratio of $\omega$ and $H_T$ is much bigger than 1 (that is size of the fibres is very small in comparison to the base). It means that the fiberwise symplectic form should be much smaller than the one on the base.

This forces us to introduce a rescaling parameter $\varepsilon$:

$$(\omega_{kl} + H_{T_{kl}})dx^k \wedge dx^l \to (\omega_{kl} + \varepsilon^2 H_{T_{kl}})dx^k \wedge dx^l.$$

Then the $(-1)$-degree term in the flat connection should be

$${}^{\mathbb{A}}\delta = \frac{i}{\hbar}\omega_{km}\left(\sqrt{1 + \varepsilon^2\omega^{-1}H_T}\ \right)^m_l dx^l \mathrm{ad}_\circ\{e^k\}.$$

Now the equation for the flat connection $D$ should start with ${}^{\mathbb{A}}\delta$ instead of $\delta$:

$$D = {}^{\mathbb{A}}\delta + \nabla + r.$$

Let us write how ${}^{\mathbb{A}}\delta$ acts on a monomial

$$\mathbf{s} = e^{i_1} \otimes \ldots \otimes e^{i_m} \otimes f \otimes dx^{j_1} \wedge \ldots \wedge dx^{j_n} \in gr_m(\mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathbb{A}) \otimes \Lambda^n \mathcal{T}^* B,\ \ f \in gr_0(\mathbb{A}).$$

$$\begin{aligned} {}^{\mathbb{A}}\delta : e^{i_1} \otimes \ldots \otimes e^{i_m} \otimes f \otimes dx^{j_1} \wedge \ldots \wedge dx^{j_n} &\mapsto \\ \sum_{k=1}^{m} e^{i_1} \otimes \ldots \widehat{e^{i_k}} \ldots \otimes e^{i_m} \otimes f \otimes X^{i_k}_l dx^l \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_n}, \end{aligned}$$

where $X$ is the following matrix $X = \sqrt{1 + \varepsilon^2\omega^{-1}H_T}$. In order to get rid of this correction term $X$ we can 'rescale' the bundle $\mathcal{W}_B$ by changing

$$e^k \mapsto \hat{e}^k = (X^{-1}e)^k \in \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathbb{A}$$

with the following commutation relation:

$$[\hat{e}^k, \hat{e}^l]_\circ = -i\hbar(\omega + \varepsilon^2 H_T)^{kl}. \tag{43}$$

On monomials in $\hat{e}$, ${}^{\mathbb{A}}\delta$ has the same action as $\delta$ on monomials in $e$ (Lemma 2.5). We can define ${}^{\mathbb{A}}\delta^{-1}$ and the whole set up in the twisted Weyl bundle becomes a familiar data for Fedosov quantization of a symplectic manifold. This way we reduce the problem to the usual Fedosov quantization and prove the theorem.

We construct a flat connection on the twisted bundle by iterations, $r_i$ being monomials in $\hat{e}$. Uniqueness follows from the condition $\delta^{-1}r = 0$, which gives the same restriction as ${}^{\mathbb{A}}\delta^{-1}r = 0$. The flat sections of this new connection corresponding to the sections of the auxiliary bundle can also be obtained by the standard recursive procedure. □

### 5.3. QUANTIZATION OF THE TOTAL SPACE

Here we want to prove that the deformation we got in the previous section is indeed a deformation of functions on $M$. Since by quantization of functions on $M$ we understand an isomorphism to flat sections of a flat connection of Weyl algebra bundle over $M$ we reformulate the problem as follows. Namely, there is a homomorphism of noncommutative algebras $\alpha$ such that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(M)[[\hbar]] & \longrightarrow & \Gamma(B, \mathcal{F}) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q^{\mathbb{A}}} \\ \Gamma_{flat}(M, \mathcal{W}_M) & \overset{\alpha}{\dashrightarrow} & \Gamma_{flat}(B, {}^{\mathbb{A}}\mathcal{W}_B) \end{array}$$

The first line is an isomorphism of commutative algebras (34). The left vertical arrow is a quantization of a symplectic manifold $M$ like in (7). $Q^{\mathbb{A}}$ is a lift of the quantization $Q^{\text{fibre}}$ (37) to $\Gamma(B, {}^{\mathbb{A}}\mathcal{W}_B)$ followed by the quantization in the twisted Weyl algebra bundle (41) from Theorem 5.5.

THEOREM 5.6. *Let $M \to B$ be a symplectic fibration with a symplectic fibre $F$. Let $\mathbb{A}$ be the quantization of $F$ with the characteristic class $\omega^F$* (Definition 2.11).

*There exists an isomorphism of bundles of algebras:*

$$\beta : \Gamma(M, \mathcal{W}_M) \to \Gamma(B, \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \Gamma(M/B, \mathcal{W}_{M/B})),$$

*where the Weyl algebra structure is defined on $\Gamma(M, W_M)$ from the symplectic form (17) on $M$, and $\Gamma(B, W_B)$ from the symplectic form $\omega_B$ on $B$.*

*Let $D^F$ be a flat connection on $\Gamma(M/B, \mathcal{W}_{M/B})$, such that $\Gamma_{\text{flat}}(M/B, \mathcal{W}_{M/B}) = \mathbb{A}$.*

*Then there exists a flat connection on $\mathcal{W}_M$, $D^M = \tilde{D} + D^F$, such that its flat sections are mapped by the isomorphism $\beta$ to the flat sections of $D^{\mathbb{A}}$ in ${}^{\mathbb{A}}\mathcal{W}_B = \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathbb{A}$. It leads to a homomorphism of algebras: $\mathbb{A}^\hbar(M) \cong \Gamma_{\text{flat}}(B, {}^{\mathbb{A}}\mathcal{W}_B)$. The characteristic class of the quantization of $M$ is exactly the class of the form (17).*

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*Proof.* Again we work in local frames. Let $\{f_\alpha\}$ be a frame in the vertical tangent bundle $\mathcal{V}M$ at a point $x \in M$ and $\{e_i\}$ be a frame in the tangent bundle to $B$, $\mathcal{T}B$, at a point $b = \pi(x)$ of $B$, with dual frames $\{f^\alpha\}$ and $\{e^i\}$.

Using a symplectic connection we obtain a local frame $\{\pi^* e_i, f_\alpha\}$ on the tangent bundle $\mathcal{T}M = \pi^*\mathcal{T}B \oplus \mathcal{V}M$ at a point $x$. Let us denote its dual frame as $\{\tilde{e}^i, f^\alpha\}$.

Then the form on $M$ can be written as a block matrix (20), so that in this frame

$$[f^\alpha, f^\beta] = \varepsilon^2 \sigma_b^{\alpha\beta}, \quad [\tilde{e}^k, \tilde{e}^l] = (\pi^*\omega^B + \varepsilon^2 H_T)^{kl}. \tag{44}$$

The flat connection $D^F$ defines an isomorphism of algebras of fiberwise deformed functions. The map $\mathcal{W}_M \to {}^{\mathbb{A}}\mathcal{W}_B : \tilde{e} \mapsto \hat{e}$ defines a homomorphism of algebras:

$$\Gamma_{\text{fiberwise flat}}(M, \mathcal{W}_M) \cong \Gamma(B, \mathcal{W}_B \otimes_{\mathbb{C}[[\hbar]]} \mathbb{A}).$$

Indeed although the Moyal–Vey products in the bundle of Weyl algebras are different, the $\circ$-product of $\hat{e}$'s in ${}^{\mathbb{A}}\mathcal{W}_B$ and the $\circ$-product of $\tilde{e}$'s in $\mathcal{W}_M$ give rise to the same $*$-product (see (43)), so it gives a homomorphism of algebras.

The last step is to construct a connection $\tilde{D}$ in $\Gamma_{\text{fiberwise flat}}(M, \mathcal{W}_M)$ so that flat sections of $D^{\mathbb{A}}$ (40) correspond to the flat sections of $\tilde{D}$. Substituting $\hat{e}$ with $\tilde{e}$ in the formula for $D^{\mathbb{A}}$ gives $\tilde{D}$.

The connection $\tilde{D}$ together with the Fedosov connection along the fibres $D^F$ gives rise to a connection $D^M$ on $\mathcal{W}$: $D^M = D^F + \tilde{D}$. Its flat section $s \in \Gamma(M, \mathcal{W})$ satisfy simultaneously two equations $D^F s = 0, \quad \tilde{D}s = 0$.

The characteristic class of this quantization is exactly the class of $\Omega_\varepsilon$ (following from (44)). This way we get a correspondence between the quantization of the total space and the quantization on the base with values in the auxiliary bundle $\mathbb{A}$. □

*Symplectic Fibrations versus Riemannian Fibrations*

There is some analogy with Riemannian fibrations (see [3]). Mazzeo and Melrose ([23]) gave an interpretation of Hodge–Leray spectral sequence from an analytic point of view. In particular they introduced language similar to $b$-calculus for description of Riemannian fibrations. The idea was to introduce a small parameter $\varepsilon$, so that all horizontal differential forms had the parameter in some degree. In other words, everything which came from the base was 'marked' by this small parameter. This gave the description of terms in the spectral sequence by the coefficients in the Taylor decomposition with respect to this small parameter.

Symplectic fibrations provide a somewhat similar picture. We want to introduce calculus similar to the one in [23]. Let us call it *MM-calculus*.

In the case of symplectic fibrations the parameter naturally comes in the construction of a symplectic form on the total space. Indeed, when the parameter is zero one gets a fiberwise noncommutative product while along the base it is commutative. The $*$-product then is a bidifferential expression only in vertical coordinates. The term at the first degree of the parameter gives a Poisson bracket in the horizontal direction, it is a first order bidifferential expression in the horizontal direction. If the fibration is trivial these bidifferential expressions are fiberwise constant.

Given a connection (13) on $M$: $\mathcal{T}M = \mathcal{H}M \oplus \mathcal{V}M$ one implements the splitting into the structure of the product manifold $X = M \times [0, \mathcal{V})$, where $\mathcal{V} \ll 1$ is some fixed small number. (We want it to be small enough so that $\varepsilon$ involved in the symplectic form (17) is bigger than $\mathcal{V}$.) The product $X = M \times [0, \varepsilon)$ has an induced fibration, with leaves $F$ and the base $B \times [0, \varepsilon)$. Consider the space $\mathcal{L}$ of smooth vector fields on $X$ which are tangent to the fibres, $M$, of the product structure $X \to [0, \mathcal{V})$ and which are also tangent to the fibres of the fibration $M \to B$, above $X_0 = M \times \{v = 0\}$. In local coordinates $x^j, \xi^k$ in $M$, where the $x$'s give coordinates in $B$, the elements of $\mathcal{L}$ are the vector fields of the form

$$\sum_{j=1}^{2p} a_j(x, \xi, v) v \partial_{x_j} + \sum_{k=1}^{2q} b_k(x, \xi, v) \partial_{\xi_k}.$$

Consider a vector bundle ${}^v\mathcal{T}M$ for which $\mathcal{L}$ is the set of sections $\mathcal{L} = C^\infty(X, {}^v\mathcal{T}M)$. There is a natural bundle map $\iota_v : {}^v\mathcal{T}M \to \mathcal{T}_X M$, the lift of $\mathcal{T}M$ to $X$. It is an isomorphism everywhere except over $M_0$, where its range is equal to $\mathcal{V}M$. It is important to define the dual map $\iota^v : \mathcal{T}^*_X M \to {}^v\mathcal{T}^*M$, which range over $M_0$ is a subbundle which is naturally isomorphic to the bundle of forms on fibres.

Given a connection (13) on $M$: $\mathcal{T}M = \mathcal{H}M \oplus \mathcal{V}M$, the restriction of ${}^v\mathcal{T}^*M$ to the boundary, $M_0$, of $X$ naturally splits

$${}^v\mathcal{T}^*_{M_0}M = v^{-1}\mathcal{T}^*B \oplus \mathcal{V}^*M, \quad u_0 = v^{-1}\pi^*\beta + \iota_v(\alpha).$$

The exterior powers of ${}^v\mathcal{T}^*M$ also split at the boundary so one can define a new bundle of rescaled differential forms on $X$. Namely,

$${}^v\mathcal{A}^k_x(M) = \sum_{j=0}^{k} \mathcal{A}^j_x(M/B) \oplus v^{-(k-j)} \mathcal{A}^{k-j}_{\pi(x)}(B).$$

The symmetric powers of ${}^v\mathcal{T}^*M$ have the following decomposition

$${}^vS^k\mathcal{T}^*M = \sum_{j=0}^{k} S^j(M/B) \oplus v^{-(k-j)} S^{k-j}\mathcal{T}^*B.$$

This way one can define the rescaled Weyl algebra bundle. Let $\{f_k\}$ be a local frame of vertical tangent bundle $\mathcal{V}M$ corresponding to $\partial_{\xi_k}$ and a local frame $\{e_i\}$ in $\mathcal{T}B$, corresponding to $\partial_{x_j}$ at a point $b = \pi(x)$ of $B$, with dual frames $\{f^k\}$ and $\{e^i\}$. Using the connection we obtain a local frame on the tangent bundle $\mathcal{T}M = \pi^*\mathcal{T}B \oplus \mathcal{V}M$ at a point $x$. The differential on the bundle of forms ${}^v\mathcal{A}^k(M)$ is given by

$$d = v\frac{dx^j}{v}\partial_{x_j} + d\xi^k\partial_{\xi_k},$$

so that $d : {}^{v}\mathcal{A}^{k}(M) \to {}^{v}\mathcal{A}^{k+1}(M)$. Similarly a connection on $\mathcal{T}M$ determines a symplectic connection on ${}^{v}\mathcal{T}M$.

The Fedosov connection on rescaled Weyl algebra bundle ${}^{v}\mathcal{W}_M$, ${}^{v}D$, has the Taylor decomposition in degrees of $v$. Since Fedosov connection is flat it should give an equation in each degree of $v$. The $v$-decomposition of $({}^{v}D)^2$ must give 0 in each degree of $v$. The result for quantization can be stated as follows:

PROPOSITION 5.7.

(1) *The quantization of $M$ for $v = 0$ is $C^{\infty}(B, \mathbb{A}^{\hbar}(M/B))$*
(2) *The term at the first power of $v$ is the product on the base given by the Poisson bracket with values in the quantization of the fiber.*
(3) $n$th *power of $v$ allows one to write a product on the base with values in the quantization of the fibre up to the* $n$th *power in $\hbar$.*

## 6. Examples of Symplectic Fibrations and Their Quantization

Fedosov quantization provides a way to construct a $*$-product on a symplectic manifold. In the previous section we showed how Fedosov quantization works for any symplectic fibration. However, step-by-step calculations become complicated very quickly and explicit formulas are readily available only in a few particular cases. Mostly the results which we are getting are of the type that 'under quantization some of the variables behave in a certain way'.

For the trivial symplectic fibration, that is for the direct product of two $F$-manifolds $(B, \omega^B, \nabla^B)$ and $(F, \sigma, \nabla^F)$ we get the direct product of quantizations:

$$\mathbb{A}^{\hbar}(B \times F) = \mathbb{A}^{\hbar}(B) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{A}^{\hbar}(F),$$

with the characteristic class $\omega^B + \varepsilon\sigma$, where $\varepsilon$ can be arbitrary nonzero number, since for nondegenerate $\omega$ and $\sigma$ for any nonzero $\varepsilon$ the sum is nondegenerate in the absence of the curvature term.

An example of a symplectic fibration with fibres being $\mathbb{R}^{2n}$ is considered in [10].

Good source of examples of symplectic fibrations are cotangent bundles to fibre bundles with connections (see again [15]). Indeed, consider a bundle $X$ over $B$, such that $X$ is an associated bundle to a principal $G$-bundle $P$ with a fibre $F$ : $X = P \times_G F$. Then $G$ acts by symplectomorphisms on the cotangent bundle $\mathcal{T}^*F$ equipped with the canonical symplectic form. We construct a new bundle $M = \mathcal{T}^*X$ over $\mathcal{T}^*B$, which is the pullback of $P \times_G \mathcal{T}^*F$ to $\mathcal{T}^*B$.

The connection, that is a splitting of $\mathcal{T}M$, is inherited from the splitting of $\mathcal{T}X$. Namely, $\mathcal{H}M$ is defined as the pullback of $\mathcal{H}X$.

The symplectic form on $M$ is the canonical symplectic form on the cotangent bundle. Due to the fibre bundle structure we can split it into two parts – one along the fibres and the other one coming from the base. There is no need to introduce the small parameter since we know that the form is already nondegenerate.

The algebraic index theorem on a cotangent bundle coincides with the analytical index theorem of Atiyah–Singer on the initial space [26]. Our example shows that this way we may obtain the relation on the index of the fibre, the base, and the total space for any fibre bundle (if the structure group is a finite dimensional Lie group).

Now let us move to other particular examples. Consider the following situation: the associated bundle $M \to B$ to the principal $G$-bundle with a fibre being $\mathcal{T}^*G$ – the cotangent bundle to $G$. This example is the inverse of the quantum reduction problem and it was discussed in the article [11], which is a generalization of [9].

The article [9] treats a symplectic fibration $M \to B$ with a symplectic fibre being a cylinder. The fibre can be represented as $\mathbb{C}^* = \mathcal{T}^*S^1$ – the cotangent bundle to the circle. Locally a point $z$ in $M$ can be described by coordinates $(x, r, \theta)$, $\theta$ being an angle coordinate on the cylinder and $r$ the height, while $x = (x_1, \ldots, x_{2n})$ denotes coordinates of the base point $\pi(z)$. Then $M = P \times_{U(1)} \mathbb{C}^*$, where $P$ is a principal $U(1)$-bundle. The symplectic form can be constructed from Proposition (3.5). Let $\lambda$ be a connection 1-form on $P$. In this particular case:

$$\lambda : \mathcal{T}P \to \mathfrak{g} = \mathbb{R}.$$

Hamiltonian of $\lambda$ is a function in $r$ only, it does not involve $\theta$. Quantization on the fibres is the Weyl quantization like in $\mathbb{R}^{2n}$:

$$\begin{aligned} a * b &= \exp\left\{-i\hbar \frac{\partial}{\partial y^k}\frac{\partial}{\partial z^{n+k}}\right\} a(y)b(z)|_{y=z=x} \\ &= ab - i\hbar \frac{\partial a}{\partial x^k}\frac{\partial b}{\partial x^{n+k}} + \frac{\hbar^2}{2}\left(\frac{\partial^2 a}{\partial x^k \partial x^l}\right)\left(\frac{\partial^2 b}{\partial x^{n+k}\partial x^{n+l}}\right) + \cdots. \end{aligned} \tag{45}$$

The symplectic connection is flat. Hence the the fiberwise flat sections are expressions in $(r + f^1)$, $(\theta + f^2)$, where $(f^1, f^2)$ are generators in the Weyl algebra bundle corresponding to $(dr, d\theta)$ in $T^*F$. The resulting flat connection in the Weyl algebra bundle on $M$ is a series in $(r + f^1)$ (it does not involve the other coordinate $(\theta + f^2)$).

Examples of symplectic fibrations are numerous. Symplectic fibrations with two–dimensional fibres are first nontrivial examples to look at, in particular fibrations over a symplectic base with a Riemannian surface as fibres (see [21]).

A nontrivial example of a symplectic fibration is provided by an $S^2$-bundle over a symplectic base. An $S^2$-bundle $\pi : M \to B$ can be considered as an associated bundle to a principal $U(1)$-bundle, $P$. We construct a symplectic form on $M = P \times_{U(1)} S^2$. The manifold $M$ is presented as the symplectic reduction of $(W, \Omega)$ at a regular value 0 of the moment map. The algebra of the group $U(1)$ is simply $\mathbb{R}$. Let $\mathcal{V}P \subset \mathcal{T}P$ be the bundle of vertical tangent vectors. The fibre at a point $p$ being $\mathcal{V}_pP \subset \mathcal{T}_pP$. A connection 1-form, $\lambda : \mathcal{T}_pP \to \mathbb{R}$, determines a horizontal subbundle by

$$\mathcal{H}_p = \{v \in \mathcal{T}_pP \mid \lambda_p(v) = 0\}.$$

This horizontal subbundle induces an injection: $\iota_\lambda : \mathcal{V}^*P \hookrightarrow \mathcal{T}^*P$, namely, a vertical cotangent vector $\xi \in \mathcal{V}^*_pP$ is a linear functional on $\mathcal{T}_pP$ which vanishes on the horizontal subspace $\mathcal{H}_pP$. The subbundle $\mathcal{V}^*P$ inherits the standard symplectic form from $\mathcal{T}_*P$, $d\alpha_{can}$. The manifold (Weinstein universal space)

$$W = \mathcal{V}^*P \times S^2$$

carries a natural symplectic form $\Omega = pr_B^*\omega_B + \iota_\lambda^* d\alpha_{can} + pr_S^*\sigma$, where $pr_B : W \to B$ and $pr_S : W \to S^2$ are the obvious projections and $\sigma$ is a $U(1)$-invariant volume form on $S^2$. $\Omega$ is invariant under the diagonal action of $U(1)$. $\mathcal{V}^*P$ is equivariantly diffeomorphic to $P \times \mathbb{R}$, so the moment map $\mu : W = P \times \mathbb{R} \times S^2 \to \mathbb{R}$ is given by

$$\mu(p, \eta, z) = h(z) - \eta$$

where $h : S^2 \to \mathbb{R}$ is the height function, it is a moment map for the action of $U(1)$ on $S^2$ by rotating about the vertical axis, and $\eta$ is a projection $P \times \mathbb{R} \to \mathbb{R}$.

The level set $\mu^{-1}(0)$ can be identified with the manifold $P \times S^2$ by the map which takes the form $\Omega$ to $pr_B^*\omega_B - d(H_\lambda) + pr_S^*\sigma$ on $P \times S^2$, where $H(p, z) = h(z)$ is the height function on $S^2$. This form is equivariant under the $U(1)$ action, to make sure it is nondegenerate we need to introduce a small number $\varepsilon$, so that the form on $M$ becomes

$$\omega = pr_B^*\omega_B - \varepsilon^2\{d(H_\lambda) + pr_S^*\sigma\}.$$

Let us consider local coordinates at a point $m$ in $M$: $(x, \xi, \theta)$, where $x = (x_1, \ldots, x_{2n})$ denotes coordinates of the base point $\pi(m)$ while $\xi$ and $\theta$ are cylindrical polar coordinates in the fibre, $\xi$ gives a height function and $\theta$ is an angle. Then the symplectic form on the fibre is

$$\sigma = d\xi \wedge d\theta.$$

The vertical vector field $\partial/\partial\theta$ has a Hamiltonian $\xi$: $H_{\frac{\partial}{\partial\theta}} = \xi$. The auxiliary bundle $\mathbb{A}$ is a bundle of quantized functions on fibres, it is associated to a $U(1)$-bundle $P$. The connection on $\mathbb{A}$ is inherited from a connection 1-form by Proposition (5.2). In coordinates it is $\nabla^{\mathbb{A}}\mathbf{s} = d\mathbf{s} + \lambda\{\xi, \mathbf{s}\}$, where $\lambda$ is a local 1-form on the base and $\{\cdot, \cdot\}$ is a fiberwise Poisson bracket, $\mathbf{s}$ being some section of $\mathbb{A}$. The curvature of this connection is

$$R^{\mathbb{A}} = \frac{i}{\hbar}\mathrm{ad}H_T = T\{\xi, \cdot\},$$

where $T$ is a 2-form on the base, the curvature of the connection $\lambda$. Since a Hamiltonian of any vector field is a linear function of just one coordinate $\xi$. A flat connection is $D^{\mathbb{A}} = \check{\delta} + \nabla + r$, where

$$\check{\delta} = \mathrm{ad}(\omega_{kl}\hat{e}^k dx^l) \quad \text{and} \quad \hat{e}^k = (\sqrt{1 + \varepsilon^2\omega^{-1}T\xi}\; e)^k.$$

As a result one gets that a flat connection in the case of a sphere bundle does not depend on the cylindrical angle coordinate $\theta$. The characteristic class of the deformation with values in the auxiliary bundle of quantization of the fibres is $\omega^B + \varepsilon^2 T\xi$.

## Acknowledgements

I am deeply grateful to Ezra Getzler, Richard Melrose, Boris Tsygan and David Vogan for help and encouragement. My thanks also go to Paul Bressler, David Ellwood, Pavel Etingof, Daniel Grieser, Dima Kaledin, Eugene Lerman, and Andras Szenes for many fruitful discussions.

This work started at mathematics department at MIT and completed during my stay at IHES as a European postdoctoral fellow. It is a pleasure to acknowledge the support of these two institutions.

## References

1. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D.: Deformation theory and quantization, I,II. *Ann. Phys.* **111** (1978), 61–151.
2. Berline, N., Getzler, E. and Vergne, M.: *Dirac Operators and Index Theorem,* Springer–Verlag, Berlin, 1992.
3. Bismut, J.-M.: The Atiyah–Singer index theorem for families of Dirac operators: Two heat equation proofs, *Invent. Math.* **83** (1986), 91–151.
4. Connes, A., Flato, M. and Sternheimer, D.: Closed star products and cyclic cohomology, *Lett. Mat. Phys.* **24** (1992), 1–12.
5. Deligne, P.: Déformations de l'algèbre des fonctions d'une varieté symplectique: comparaison entre Fedosov et De Wilde, Lecomte, *Selecta Math.* **1**(4) (1995).
6. De Wilde, M. and Lecomte, P. B. A.: Existence of star-products and of formal deformations in Poisson Lie algebra of arbitrary symplectic manifold, *Lett. Math. Phys.* **7** (1983), 487–496.
7. Donin, J.: On the quantization of Poisson brackets, Preprint, q-alg/9505028; *Internat. J. Modern Phys. A* **11** (1996), 863.
8. Fedosov, B.: A simple geometric construction of deformation quantization, *J. Differential. Geom.* **40** (1994), 213–238.
9. Fedosov, B.: Reduction and eigenstates in deformation quantization, In: *Advances in Partial Differential Equations,* Akademie-Verlag, Berlin, 1994, pp. 277–297.
10. Fedosov, B.: *Deformation Quantization and Index Theory,* Academie-Verlag, Berlin, 1996.
11. Fedosov, B.: Non-Abelian reduction in deformation quantization, *Lett. Math. Phys.* **43**(2) (1998), 137–154.
12. Gelfand, I. M., Kazhdan, D. A. and Fuks, D. B.: The action of infinite dimensional Lie algebras, *Funct. Anal. Appl.* **6** (1972), 9–13.
13. Gelfand, I., Retakh, V. and Shubin, M.: Fedosov manifolds, Preprint, dg-ga/9707024; *Adv. Math.* **136**(1) (1998), 104–140.
14. Getzler, E.: The Bargmann representation, generalized Dirac operators and the index of pseudodifferential operators on $R^n$, In: *Symplectic Geometry and Quantization (Sanda and Yokohama, 1993),* Contemp. Math. 179, Amer. Math. Soc., Providence, RI, 1994, pp. 63–81.
15. Guillemin, V., Lerman, E. and Sternberg, S.: *Symplectic Fibrations and Multiplicity Diagrams,* Cambridge Univ. Press, 1995.

16. Gutt, S.: An explicit $*$-product on the cotangent bundle of a Lie group, *Lett. Math. Phys.* **7** (1983), 249–258.
17. Kontsevich, M.: Deformation quantization of Poisson manifolds, I, Preprint, q-alg/9709040.
18. Kravchenko, O.: How to calculate the Fedosov connection. Exercises de style arXiv:math.SG/0008157, *Actes de colloque Glanon-II* (1998).
19. Lecomte, P. B. A.: Existence des star-produits selon B. Fedosov, *Séminaire 95.005*, Institut de Mathématique Université de Liège, 1995.
20. Lichnerowicz, A.: Connexions symplectiques et $\star$–produits invariants, *C.R. Acad. Sci. Paris* **291** (1980), 413–417.
21. McDuff, D. and Salamon, D.: *Introduction to Symplectic Topology,* Clarendon Press, Oxford, 1995.
22. Marsden, J. E., Ratiu, T. and Raugel, G.: Symplectic connections and the linearization of Hamiltonian systems, *Proc. Royal Soc. Edinburgh A* **117** (1991), 329–380.
23. Mazzeo, R. and Melrose, R.: The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration, *J. Differential Geom.* **31** (1990), 185–213.
24. Nest, R. and Tsygan, B.: Algebraic index theorem, *Comm. Math. Phys.* **172**(2) (1995), 223–262.
25. Nest, R. and Tsygan, B.: Algebraic index theorem for families, *Adv. Math.* **113**(2) (1995), 151–205.
26. Nest, R. and Tsygan, B.: Formal versus analytic index theorems, *Internat. Math. Res. Notices* No. 11 (1996), 557–564.
27. Sternberg, S.: Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang–Mills field, *Proc. Natl. Acad. Sci. USA* **74** (1977), 5253–5254.
28. Xu, P.: Fedosov $*$ products and quantum momentum maps, Preprint hep-th/9608006; *Comm. Math. Phys.* **197**(1) (1998), 167–197.
29. Weinstein, A.: A universal phase space for particles in Yang–Mills fields, *Lett. Math. Phys.* **2** (1978), 1–13.
30. Weinstein, A.: Deformation quantization [d'après B. Fedosov and others], *Séminaire Bourbaki,* 46ème année, 1993–94, n$^o$ 789 (1994).