Solution by L. Carlitz.

$$x^{13} + x + 90 \equiv x^{13} + x - 1 \qquad (\mathrm{mod}\ 7\cdot 13)$$

$$x^{13} + x - 1 \equiv (x - \tfrac{1}{2}) \{ (x - \tfrac{1}{2})^{12} + 1 \} \qquad (\mathrm{mod}\ 13).$$

Since

$$y^{12} + 1 \equiv (y^2 - 2)(y^2 - 5)(y^2 - 6)(y^2 - 7)(y^2 - 8)(y^2 - 11) \qquad (\mathrm{mod}\ 13)$$

(the numbers 2, 5, 6, 7, 8, 11 are the quadratic non-residues (mod 13)) we get the quadratic factors

$$x^2 - x + c, \quad c \equiv \pm 5, 4, 3, 2, -1 \qquad (\mathrm{mod}\ 13).$$

Next, if $f(x) = x^{13} + x - 1$, then

$$f(\tfrac{1}{2}) \equiv f'(\tfrac{1}{2}) \equiv 0 \qquad (\mathrm{mod}\ 7),$$

so that $f(x)$ is divisible (mod 7) by $(x - \frac{1}{2})^2$, which is congruent to $x^2 - x + 2$. Since this polynomial occurs among the quadratics (mod 13) found above, it is a likely candidate. By division we find that

$$x^{13} + x + 90 = (x^2 - x + 2)(x^{11} + x^{10} - x^9 - 3x^8 - x^7 + 5x^6$$
$$+ 7x^5 - 3x^4 - 17x^3 - 11x^2 + 23x + 45).$$

It would be interesting to know whether the second factor is irreducible. (Also solved by the proposer.)

SEQUENCE AND SERIES TRANSFORMATIONS

M. S. Macphail

The summability methods

$$A:\ t_n = \sum_{k=0}^{\infty} a_{nk} s_k, \qquad B:\ T_n = \sum_{k=0}^{\infty} b_{nk} u_k,$$

where $b_{nk} = a_{nk} + a_{n,k+1} + \ldots$, are regarded as the sequence-to-sequence and series-to-sequence forms of the same method, and if $s_k = u_0 + \ldots + u_k$, we speak of the series $\sum u_k$ or the

https://doi.org/10.4153/CMB-1960-011-0 Published online by Cambridge University Press

sequence $\{s_k\}$ indifferently, as summable A or B. We have by partial summation

(1) $b_{no}u_o + \ldots + b_{nk}u_k = a_{no}s_o + \ldots + a_{n,k-1}s_{k-1} + b_{nk}s_k$;

so in order that $B \supset A$ (every A-summable sequence is B-summable to the same sum) it is necessary and sufficient that $\lim_n \lim_k b_{nk}s_k = 0$ for every A-summable sequence $\{s_k\}$, and in order that $A \supset B$ it is necessary and sufficient that the same holds for every B-summable $\{s_k\}$.

The purpose of this note is to give simple sufficient conditions depending on the coefficients b_{nk} alone.

THEOREM 1. In order that $B \supset A$, it is sufficient that for each n = 0, 1, ... there is a positive constant R_n such that $|1 - b_{n,k+1}/b_{nk}| > R_n$ $(k = 0, 1, \ldots)$.

Proof. For $B \supset A$, it is plainly sufficient that T_n exists and equals t_n, for every $\{s_k\}$ such that t_n exists; or, from (1), that $A_n^* \supset A_n$, where

$$A_n^* = \begin{bmatrix} b_{no} \\ a_{no}\ b_{n1} \\ a_{no}\ a_{n1}\ b_{n2} \\ a_{no}\ a_{n1}\ a_{n2}\ b_{n3} \\ . \quad . \quad . \quad . \quad . \quad . \end{bmatrix}, \quad A_n = \begin{bmatrix} a_{no} \\ a_{no}\ a_{n1} \\ a_{no}\ a_{n1}\ a_{n2} \\ a_{no}\ a_{n1}\ a_{n2}\ a_{n3} \\ . \quad . \quad . \quad . \quad . \quad . \end{bmatrix}$$

We easily find that $A_n^* A_n^{-1}$ has for its k-th row

$$(0, 0, \ldots, 0, 1 - b_{nk}/a_{nk}, b_{nk}/a_{nk}).$$

Applying the Toeplitz conditions for regularity, we have at once that the column limits are zero and the row-sum limit is 1. The row-norm condition reduces to $|b_{nk}/a_{nk}| < M_n$, which is equivalent to the condition stated in the theorem.

THEOREM 2. In order that $A \supset B$, it is sufficient that for each n = 0, 1, ... there is a constant M_n such that

(2) $|b_{n,k+1}| \sum_{r=o}^{k} |b_{n,r+1}^{-1} - b_{nr}^{-1}| < M_n$ $(k = 0, 1, \ldots)$,

and $\lim_k b_{nk} = 0$.

This may be proved by a similar method, after writing (1) in the modified form

$$a_{no}s_o + a_{n1}s_1 + \dots + a_{nk}s_k$$

$$= b_{no}u_o + b_{n1}u_1 + \dots + b_{nk}u_k - b_{n,k+1}s_k$$

$$= (b_{no} - b_{n,k+1})u_o + (b_{n1} - b_{n,k+1})u_1 + \dots + (b_{nk} - b_{n,k+1})u_k.$$

Or we may use a theorem of Kronecker [1, p. 129-130], to show that $\lim_k b_{nk}s_k = 0$.

THEOREM 3. In order that $A \supset B$, it is sufficient that for each $n = 0, 1, \dots$ there is a constant C_n $(0 < C_n < 1)$, such that $|b_{n,k+1}/b_{nk}| < C_n$ $(k = 0, 1, \dots)$. For real b_{nk} it is sufficient that for each n, $b_{nk} \to 0$ monotonically from a certain k on.

Proof. The second condition is obviously sufficient for (2). For the first, we observe that

$$|b_{n,k+1}| \sum_{r=o}^{k} |b_{n,r+1}^{-1} - b_{nr}^{-1}| < 2|b_{n,k+1}| \sum_{r=o}^{k+1} |b_{nr}^{-1}| .$$

Denoting the right hand side by $2B_{nk}$, we find

$$B_{n,k+1} = |b_{n,k+2}/b_{n,k+1}| \ B_{nk} + 1 ,$$

whence we see inductively that B_{nk} is bounded, under our hypothesis.

We may illustrate with the well-known "circle method":

$$b_{nk} = \begin{cases} \binom{k}{n} t^{k-n} (1-t)^n & (k \geq n) \\ 0 & (k < n) . \end{cases}$$

This is in the customary series-to-series form. We easily obtain from Theorems 1 and 3 the known results [2, p.549; 3, p. 141] that (with $a_{nk} = b_{nk} - b_{n,k+1}$) we have $B \supset A$ for all $t \neq 1$ and $A \supset B$ for $|t| < 1$; here A is a sequence-to-series method which is equivalent to the corresponding sequence-to-sequence method. It is easily proved [2] that the condition $|t| < 1$ is necessary for $A \supset B$.

REFERENCES

1. K. Knopp, Theory and Application of Infinite Series (London, 1928).

2. P. Vermes, Series to series transformations and analytic continuation by matrix methods. Amer. J. Math. 71 (1949), 541-562.

3. K. Zeller, Theorie der Limitierungsverfahren (Berlin, 1958).

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https://doi.org/10.4153/CMB-1960-011-0 Published online by Cambridge University Press