

MODULI SPACES OF FRAMED PERVERSE INSTANTONS ON \mathbb{P}^3

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Abstract. We study moduli spaces of framed perverse instantons on \mathbb{P}^3 . As an open subset, it contains the (set-theoretical) moduli space of framed instantons studied by I. Frenkel and M. Jardim in [9]. We also construct a few counter-examples to earlier conjectures and results concerning these moduli spaces.

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1. Introduction. A mathematical instanton is a torsion-free sheaf E on \mathbb{P}^3 such that $H^1(E(-2)) = H^2(E(-2)) = 0$ and there exists a line on which E is trivial. It is conjectured that the moduli space of locally free instantons of rank 2 is smooth and irreducible but this is known only for very small values of the second Chern class c (see [6, 20] for proof of this conjecture for $c \leq 5$ and history of the problem).

Originally, instantons appeared in physics as anti-selfdual connections on the four-dimensional sphere. Later, they were connected by the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction to mathematical instantons on \mathbb{P}^3 with some special properties. But it was Donaldson who realised that there is a bijection between physical instantons on the 4-sphere with framing at a point and vector bundles on a plane framed along a line (see [8]). The correspondence can be seen using Wards' construction and restricting vector bundles from \mathbb{P}^3 to a fixed plane containing the line corresponding to the point of the sphere. Using this interpretation, Donaldson was able to conclude that the moduli space of physical instantons is smooth and irreducible.

In [9] (see also [18]) Frenkel and Jardim started to investigate the moduli space of mathematical instantons framed along a line, hoping that this moduli space is easier to handle than the moduli space of instantons. Since an open subset of the moduli space of framed instantons is a principal bundle over the moduli space of instantons, it is sufficient to consider the conjecture in the framed case. In fact, Frenkel and Jardim conjectured that their framed moduli space is smooth and irreducible even at non-locally free framed instantons. We show that this conjecture is false (see Subsection 7.3). On the other hand, we also show that the moduli space of locally free framed instantons is smooth for low ranks and values of the second Chern class

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(see Corollary 9.7). We also use Tyurin's idea to show that in some case the restriction map embeds the moduli space of instantons as a Lagrangian submanifold into the moduli space of sheaves on a quartic in \mathbb{P}^3 .

One of the main aims of this paper is the study of moduli spaces of perverse framed instantons on \mathbb{P}^3 (see Definition 3.8). In particular, we use perverse instantons to introduce partial compactifications of Gieseker and Donaldson-Uhlenbeck type of the moduli space of framed instantons and study the morphism between these moduli spaces. The picture that we get is quite similar to the one known from the plane case (see [28]) or from the study of a similar morphism for sheaves on surfaces (see e.g. [15, Remark 8.2.17]). However, this is the first case when a similar morphism is described for moduli spaces of sheaves on a three-dimensional variety.

In the two-dimensional case the Donaldson-Uhlenbeck compactification has a stratification by products of moduli spaces of locally free sheaves for a smaller second Chern class and symmetric powers of a plane \mathbb{A}^2 . In our case, the situation is quite similar but more complicated: we get a stratification by products of moduli spaces of regular perverse instantons and moduli spaces of perverse instantons of rank 0.

Perverse instantons of rank 0 are sheaves E of pure dimension 1 on \mathbb{P}^3 such that $H^0(E(-2)) = H^1(E(-2)) = 0$. The moduli space of such sheaves (with fixed second Chern class) has a similar type as a Chow variety: it is only set-theoretical and it does not corepresent the moduli functor of such sheaves. But the moduli space of perverse rank 0 instantons is still a coarse moduli space for some functor: it is the moduli space of modules over some associative (but non-commutative) algebra. We show that this moduli space contains an irreducible component, whose normalisation is the symmetric power of \mathbb{A}^4 .

The structure of the paper is as follows. In Section 2 we recall a few known results including Nakajima's description of the moduli space of framed torsion-free sheaves on a plane and Frenkel–Jardim's description of the (set-theoretical) moduli space of framed instantons in terms of ADHM data. In Section 3, we introduce perverse instantons and we sketch proof of representability of the stack of framed perverse instantons on \mathbb{P}^3 (in the plane case, this theorem is due to Drinfeld; see [4]). Then, in Section 4, we study the notion of stability of ADHM data in terms of Geometric Invariant Theory (GIT). This is crucial in Section 5, where we describe the Gieseker and Donaldson–Uhlenbeck type compactifications of the moduli space of framed instantons. In Section 6, we study the moduli space of perverse instantons of rank 0 relating them to the moduli space of modules over a certain non-commutative algebra. In particular, we show an example when this moduli space is reducible. In Section 7, we gather several examples and counter-examples to some conjectures, e.g. to the Frenkel–Jardim conjecture on smoothness and irreducibility of moduli space of torsion-free framed instantons or to their conjecture on weak instantons. In Section 8, we study an analogue of the hyper-Kähler structure on the moduli space of perverse instantons and we relate our moduli spaces to moduli spaces of framed modules of Huybrechts and Lehn. In Section 9, we give a very short sketch of deformation theory for stable framed perverse instantons and we study smoothness of moduli spaces of framed locally free instantons.

2. Preliminaries. In this section, we introduce notation and collect a few known results needed in later sections.

2.1. Geometric Invariant Theory. Let G be a reductive group. Let X be an affine k -scheme (possibly non-reduced or reducible) with a left G -action. A character

$\chi : G \rightarrow \mathbb{G}_m$ gives a G -linearisation of the trivial line bundle $L := (X \times \mathbb{A}^1 \rightarrow X)$ via $g \cdot (x, z) = (gx, \chi(g^{-1})z)$. So we can consider the corresponding GIT quotient

$$X^{ss}(L)/G = \text{Proj} \left(\bigoplus_{n \geq 0} H^0(X, L^n)^G \right).$$

It is equal to

$$X //_{\chi} G = \text{Proj} \left(\bigoplus_{n \geq 0} k[X]^{G, \chi^n} \right)$$

and it is projective over $X/G = \text{Spec}(k[X]^G)$ (see [21] for this description). The corresponding map

$$X //_{\chi} G \rightarrow X/G$$

can be identified with the map describing change of polarisation from χ to the trivial character $1 : G \rightarrow \mathbb{G}_m$. The GIT (semi)stable points of the G -action on (X, L) given by χ are called χ -(semi)stable. Note that all points of X are 1-semistable, i.e. GIT semistable for the trivial character 1.

We say that x is χ -polystable if $G \cdot (x, z)$ is closed for $z \neq 0$. In particular, $X //_{\chi} G$ is in bijection with the set of χ -polystable points and a χ -polystable point is χ -stable if and only if its stabiliser in G is trivial.

2.2. Torsion-free sheaves on \mathbb{P}^2 and ADHM data. Let V and W be k -vector spaces of dimensions c and r , respectively. Set

$$\mathbf{B} = \text{Hom}(V, V) \oplus \text{Hom}(V, V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W).$$

An element of \mathbf{B} is written as (B_1, B_2, i, j) .

The map $\mu : \mathbf{B} \rightarrow \text{Hom}(V, V)$ given by

$$\mu(B_1, B_2, i, j) = [B_1, B_2] + ij$$

is called the *moment map*.

We say that $(B_1, B_2, i, j) \in \mathbf{B}$ satisfies the *ADHM equation* if $[B_1, B_2] + ij = 0$, i.e. $(B_1, B_2, i, j) \in \mu^{-1}(0)$. An element of \mathbf{B} satisfying the ADHM equation is called an *ADHM datum*.

DEFINITION 2.1. We say that an ADHM datum is

- (1) *stable*, if for every subspace $S \subsetneq V$ (note that we allow $S = 0$) such that $B_k(S) \subset S$ for $k = 1, 2$ we have $\text{im } i \not\subset S$.
- (2) *costable*, if for every non-zero subspace $S \subset V$ such that $B_k(S) \subset S$ for $k = 1, 2$ we have $S \not\subset \ker j$,
- (3) *regular*, if it is stable and costable.

The group $G = \text{GL}(V)$ acts on \mathbf{B} via

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

If we consider the adjoint action of G on $\text{End}(V)$ then the map μ is G -equivariant. In particular, G acts on $\tilde{\mu}^{-1}(0)$, i.e. on the set of ADHM data satisfying the ADHM equation. Let $\chi : G \rightarrow \mathbb{G}_m$ be the character given by the determinant. We consider the G -action on the trivial line bundle over \mathbf{B} but with a non-trivial linearisation given the character χ .

LEMMA 2.2. (1) All χ -semistable points of $\mu^{-1}(0)$ are χ -stable and they correspond to stable ADHM data.

(2) All χ^{-1} -semistable points of $\mu^{-1}(0)$ are χ^{-1} -stable and they correspond to costable ADHM data.

We have the following well-known theorem (see [28, Theorem 2.1, Remark 2.2 and Lemma 3.25]):

THEOREM 2.3. The moduli space $\mathcal{M}(\mathbb{P}^2; r, c)$ of rank $r > 0$ torsion-free sheaves on \mathbb{P}^2 with $c_2 = c$, framed along a line l_∞ is isomorphic to the GIT quotient $\mu^{-1}(0) //_\chi G$. Moreover, orbits of regular ADHM data are in bijection with locally free sheaves.

DEFINITION 2.4. A complex of locally free sheaves

$$\mathcal{C} = (0 \rightarrow \mathcal{C}^{-1} \xrightarrow{\alpha} \mathcal{C}^0 \xrightarrow{\beta} \mathcal{C}^1 \rightarrow 0)$$

is called a *monad* if α is injective and β is surjective (as maps of sheaves). In this case $\mathcal{H}^0(\mathcal{C}) = \ker \beta / \text{im } \alpha$ is called the *cohomology* of the monad \mathcal{C} .

Now let us briefly recall how to recover a torsion-free sheaf from a stable ADHM datum.

Let $(B_1, B_2, i, j) \in \mathbf{B}$ be a stable ADHM datum. Denote $\tilde{W} = V \oplus V \oplus W$ and fix homogeneous coordinates $[x_0, x_1, x_2]$ on \mathbb{P}^2 . Let us define maps $\alpha : V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^2}$ and $\beta : \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^2} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$ by

$$\alpha = \begin{pmatrix} B_1 x_0 - 1 \otimes x_1 \\ B_2 x_0 - 1 \otimes x_2 \\ j x_0 \end{pmatrix} \tag{1}$$

and

$$\beta = (-B_2 x_0 + 1 \otimes x_2 \quad B_1 x_0 - 1 \otimes x_1 \quad i x_0). \tag{2}$$

Then (B_1, B_2, i, j) gives rise to the complex

$$V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^2}(1).$$

This complex is a monad. Injectivity of α follows from injectivity on the line $x_0 = 0$ and surjectivity of β follows from stability of the ADHM datum (see [28, Lemma 2.7]). We can recover a torsion-free sheaf as the cohomology of this monad.

Let $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^2; r, c)$ be the moduli space of rank r locally free sheaves on \mathbb{P}^2 with $c_2 = c$, framed along a line l_∞ . By Theorem 2.3 $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^2, r, c)$ is isomorphic to the quotient of regular ADHM data by the group G .

Let $\mathcal{M}_0(\mathbb{P}^2; r, c)$ denotes the affine quotient $\mu^{-1}(0)/G$. This space contains the moduli space $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^2; r, c)$ and it can be considered as a partial Donaldson–Uhlenbeck

compactification. We have a natural set-theoretical decomposition

$$\mathcal{M}_0(\mathbb{P}^2; r, c) = \bigsqcup_{0 \leq d \leq c} \mathcal{M}_0^{\text{reg}}(\mathbb{P}^2; r, c - d) \times S^d(\mathbb{A}^2),$$

where \mathbb{A}^2 is considered as the completion of l_∞ in \mathbb{P}^2 . Then the morphism

$$\mathcal{M}(\mathbb{P}^2; r, c) \simeq \mu^{-1}(0) //_{\chi} G \rightarrow \mu^{-1}(0) / G \simeq \mathcal{M}_0(\mathbb{P}^2; r, c)$$

coming from the GIT (see Subsection 2.1) can be identified with the map

$$(E, \Phi) \rightarrow ((E^{**}, \Phi), \text{Supp}(E^{**}/E))$$

(see [28, Exercise 3.53] and [37, Theorem 1]). This morphism is an analogue of the morphism from the Gieseker compactification of the moduli space of (semistable) locally free sheaves on a surface by means of torsion-free sheaves to its Donaldson–Uhlenbeck compactification. In a very special case of rank one this corresponds to the morphism from the Hilbert space to the Chow space (the so called *Hilbert–Chow morphism*).

In the rest of this section, to agree with the standard notation we need to assume that the characteristic of the base field is zero (or it is sufficiently large).

Let us define a symplectic form ω on \mathbf{B} by

$$\omega((B_1, B_2, i, j), (B'_1, B'_2, i', j')) := \text{Tr}(B_1 B'_2 - B_2 B'_1 + ij' - i'j).$$

We will use the same notation for the form induced on the tangent bundle $T\mathbf{B}$. One can easily check that μ is a *momentum map*, i.e.

- (1) μ is G -equivariant, i.e. $\mu(g \cdot x) = \text{Ad}_{g^{-1}}^* \mu(x)$,
- (2) $\langle d\mu_x(v), \xi \rangle = \omega(\xi_x, v)$ for any $x \in \mathbf{B}$, $v \in T_x \mathbf{B}$ and $\xi \in \mathfrak{g}$ (ξ_x denotes the image of ξ under the tangent of the orbit map of x).

In particular, [19, Lemma 3.2] implies that the moduli space of semistable sheaves is smooth (there are many others proofs of this fact: a sheaf-theoretic proof is trivial but we mention the above proof since another argument using ADHM data given in [36, Lemma 3.2] seems a bit too complicated).

2.3. Mathematical instantons on \mathbb{P}^3 .

DEFINITION 2.5. A torsion-free sheaf E on \mathbb{P}^3 is called a *mathematical* (r, c) -instanton, if E has rank r , $c_2 E = c$, $H^1(\mathbb{P}^3, E(-2)) = H^2(\mathbb{P}^3, E(-2)) = 0$ and there exists a line $l \subset \mathbb{P}^3$ such that the restriction of E to l is isomorphic to the trivial sheaf \mathcal{O}_l^r .

Let us fix a line $l_\infty \subset \mathbb{P}^3$. A choice of an isomorphism $\Phi : E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^r$ is called a *framing* of E along l_∞ . A pair (E, Φ) consisting of a mathematical (r, c) -instanton E and its framing $\Phi : E|_{l_\infty} \xrightarrow{\sim} \mathcal{O}_{l_\infty}^r$ is called a *framed* (r, c) -instanton.

In the following we skip adjective “mathematical” and we will refer to mathematical instantons simply as instantons.

The following lemma is well known (for locally free sheaves see [30, Chapter II, 2.2]):

LEMMA 2.6. *If a torsion-free sheaf E on \mathbb{P}^n is trivial on one line then it is slope semistable. In particular, an instanton on \mathbb{P}^3 is slope semistable.*

Proof. The sheaf E is trivial on a line $m \subset \mathbb{P}^n$ if and only if $E|_m$ is torsion-free and $H^1(E|_m(-1)) = 0$. Since these are open conditions it follows that if E is trivial on one line then it is trivial on a general line. Now if $E' \subset E$ then for a general line m we have $E'|_m \subset E|_m \simeq \mathcal{O}_{\mathbb{P}^1}^{\text{rk } E}$, so $\mu(E') = \text{deg}(E'|_m) / \text{rk } E' \leq 0$. □

If E is a rank 2 locally free sheaf with $c_1 E = 0$ on \mathbb{P}^3 then $H^1(\mathbb{P}^3, E(-2))$ and $H^2(\mathbb{P}^3, E(-2))$ are Serre dual to each other. Let us also recall that if k has characteristic 0 then for a rank 2 locally free sheaf E with $c_1 E = 0$ existence of a line l such that the restriction of E to l is trivial is equivalent to $H^0(E(-1)) = 0$ (this follows from the Grauert–Mülich restriction theorem). This leads to a more traditional definition of rank 2 instantons as rank 2 vector bundles on \mathbb{P}^3 with vanishing $H^0(E(-1))$ and $H^1(E(-2))$ (see [30, Chapter II, 4.4]). Usually, one also adds vanishing of $H^0(E)$ which in this case is equivalent to slope stability (if $H^0(E) \neq 0$ then $E \simeq \mathcal{O}_{\mathbb{P}^3}^2$, so this cannot happen if $c \geq 1$).

Note that if E is locally free of rank ≥ 3 then vanishing of $H^2(\mathbb{P}^3, E(-2))$ does not follow from the remaining conditions (see Example 7.4).

We say that a locally free sheaf E is *symplectic*, if it admits a non-degenerate symplectic form (or equivalently, an isomorphism $\varphi : E \rightarrow E^*$ such that $\varphi^* = -\varphi$). It is easy to see that a non-trivial symplectic sheaf has an even rank. Obviously, for a symplectic locally free sheaf, vanishing of $H^2(\mathbb{P}^3, E(-2))$ follows from vanishing of $H^1(\mathbb{P}^3, E(-2))$ (by the Serre duality).

The following fact was known for a very long time:

THEOREM 2.7. (Barth, Atiyah [1, Theorem 2.3]) *Let E be a symplectic (r, c) -instanton. Then E is the cohomology of a monad*

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^c \rightarrow \mathcal{O}_{\mathbb{P}^3}^{2c+r} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^c \rightarrow 0.$$

In the following we will need the following lemma (it should be compared with [9, Proposition 15] dealing with the rank 1 case).

LEMMA 2.8. *There exist framed locally free (r, c) -instantons if and only if either $r = 1$ and $c = 0$ or $r > 1$ and c is an arbitrary non-negative integer. Moreover, if there exist framed locally free (r, c) -instantons then there exist framed locally free (r, c) -instantons F such that $\text{Ext}^2(F, F) = 0$.*

Proof. The case $r = 0$ is clearly not possible. Let us first assume that $r = 1$. Since the only line bundle with trivial determinant is $E = \mathcal{O}_{\mathbb{P}^1}$ we see that $c_2(E) = 0$. So to finish the proof it is sufficient to show existence of locally free $(2, c)$ -instantons. If E is such an instanton then $F = E \oplus \mathcal{O}_{\mathbb{P}^3}^{r-2}$ can be given a structure of framed locally free (r, c) -instanton.

Existence of locally free $(2, c)$ -instantons is well known. For example, we can use Serre’s construction (see [30, Chapter I, Theorem 5.1.1]) to construct the so called *t’Hooft bundles*. More precisely, let L_1, \dots, L_c be a collection of c disjoint lines in \mathbb{P}^3 and let Y denotes their sum (as a subscheme of \mathbb{P}^3). Then by the above mentioned theorem there exists a rank 2 locally free sheaf E that sits in a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow E \rightarrow J_Y \rightarrow 0.$$

One can easily see that E is a $(2, c)$ -instanton. Moreover, it is easy to see that $\text{Ext}^2(F, F) = 0$ as $\text{Ext}^2(E, E) = 0$ and $H^2(E) = H^2(E^*) = 0$ (note that $E^* \simeq E$). \square

2.4. Generalised ADHM data after Frenkel–Jardim. Let X be a smooth projective variety over an algebraically closed field k and let $\mathcal{O}_X(1)$ be a fixed ample line bundle. Let V and W be k -vector spaces of dimensions c and r , respectively. Set

$$\mathbf{B} = \text{Hom}(V, V) \oplus \text{Hom}(V, V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$$

and $\tilde{\mathbf{B}} = \mathbf{B} \otimes H^0(\mathcal{O}_X(1))$. An element of $\tilde{\mathbf{B}}$ is written as $(\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j})$, where \tilde{B}_1 and \tilde{B}_2 are treated as maps $V \rightarrow V \otimes H^0(\mathcal{O}_X(1))$, \tilde{i} as a map $W \rightarrow V \otimes H^0(\mathcal{O}_X(1))$ and \tilde{j} as a map $V \rightarrow W \otimes H^0(\mathcal{O}_X(1))$.

Let us define an analogue of the moment map

$$\tilde{\mu} = \tilde{\mu}_{W,V} : \tilde{\mathbf{B}} \rightarrow \text{End}(V) \otimes H^0(\mathcal{O}_X(2))$$

by the formula

$$\tilde{\mu}(\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j}) = [\tilde{B}_1, \tilde{B}_2] + \tilde{i}\tilde{j}.$$

As before an element of $\tilde{\mu}^{-1}(0)$ is called an *ADHM datum* (or an *ADHM* (r, c) -datum for X if we want to show dependence on r, c and X).

If we fix a point $p \in X$ then for a k -vector space U the evaluation map $\text{ev}_p : H^0(\mathcal{O}_X(1)) \rightarrow \mathcal{O}_X(1)_p \simeq k$ tensored with identity on U gives a map $U \otimes H^0(\mathcal{O}_X(1)) \rightarrow U$ which we also denote by ev_p . For simplicity, we will use the notation $\tilde{B}_1(p) = \text{ev}_p \tilde{B}_1 \in \text{Hom}(V, V)$, etc. For an ADHM datum $x = (\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j})$, $x(p)$ denotes the quadruple $(\tilde{B}_1(p), \tilde{B}_2(p), \tilde{i}(p), \tilde{j}(p))$. Note that for maps to be well defined we need to fix an isomorphism $\mathcal{O}_X(1)_p \simeq k$ at each point $p \in X$. This does not cause any problems as all the notions that we consider are independent of these choices.

DEFINITION 2.9. We say that an ADHM datum $x \in \tilde{\mathbf{B}}$ is

- (1) *FJ-stable* (*FJ-costable*, *FJ-regular*), if $x(p)$ is stable (respectively: costable, regular) for all $p \in X$,
- (2) *FJ-semistable*, if there exists a point $p \in X$ such that $x(p)$ is stable,
- (3) *FJ-semiregular*, if it is FJ-stable and there exists a point $p \in X$ such that $x(p)$ is regular.

Definition 2.9 in case of $X = \mathbb{P}^1$ (not \mathbb{P}^3 !) was introduced by I. Frenkel and M. Jardim in [9], but we slightly change the notation and we call stability, semistability, etc. introduced in [9], FJ-stability, FJ-semistability, etc. The reason for this change will become apparent in later sections. Namely, in [17] Jardim generalised this definition of (semi)stability of ADHM data to all projective spaces and claimed in [17, Proposition 4] that his notion of semistability is equivalent to GIT semistability of ADHM data. We show that this assertion is false.

Let us specialise to the case $X = \mathbb{P}^1$. Let $[x_0, x_1, x_2, x_3]$ be homogeneous coordinates in \mathbb{P}^3 and let us embed X into \mathbb{P}^3 by $[y_0, y_1] \rightarrow [y_0, y_1, 0, 0]$. Then x_0 and x_1 can be considered as elements of $H^0(X, \mathcal{O}_X(1))$.

Let us set $\tilde{W} = V \oplus V \oplus W$. Then any point $x = (\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j}) \in \tilde{\mathbf{B}} = \mathbf{B} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$ gives rise to the following maps of sheaves on \mathbb{P}^3 : map

$\alpha : V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3}$ given by

$$\alpha = \begin{pmatrix} \tilde{B}_1 + 1 \otimes x_2 \\ \tilde{B}_2 + 1 \otimes x_3 \\ \tilde{j} \end{pmatrix} \tag{3}$$

and map $\beta : \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ given by

$$\beta = (-\tilde{B}_2 - 1 \otimes x_3 \quad \tilde{B}_1 + 1 \otimes x_2 \quad \tilde{i}). \tag{4}$$

It follows from easy calculations that $\beta\alpha = 0$ if and only if $(\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j}) \in \tilde{\mu}^{-1}(0)$. So if x is an ADHM datum then we get the complex

$$\mathcal{C}_x^\bullet = (V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \tag{5}$$

considered in degrees $-1, 0, 1$.

Let l_∞ be the line in \mathbb{P}^3 given by $x_0 = x_1 = 0$. It is easy to see that after restricting to l_∞ the cohomology of the above complex of sheaves becomes the trivial rank r sheaf on \mathbb{P}^1 . Moreover, we have the following lemma:

LEMMA 2.10. (see [9, Proposition 11]) *Let us fix an ADHM datum $x \in \tilde{\mathbf{B}}$. Then the corresponding complex \mathcal{C}_x^\bullet is a monad if and only if the ADHM datum x is FJ-stable.*

Proof. The map α is always injective as a map of sheaves, so we only need to check when β is surjective. This is exactly the content of [9, Proposition 11]. □

The main theorem of [9] is existence of the following set-theoretical bijections:

THEOREM 2.11. ([9, Main Theorem]) *The above construction of monads from ADHM data on \mathbb{P}^1 gives bijections between the following objects:*

- FJ-stable ADHM data and framed torsion-free instantons;
- FJ-semiregular ADHM data and framed reflexive instantons;
- FJ-regular ADHM data and framed locally free instantons.

3. Perverse instantons on \mathbb{P}^3 . In this section we introduce perverse sheaves and perverse instantons and we show that perverse instantons are perverse sheaves (this fact is non-trivial!). We also sketch proof of an analogue of Drinfeld’s representability theorem in the three-dimensional case.

3.1. Tilting and torsion pairs on \mathbb{P}^3 .

DEFINITION 3.1. Let \mathcal{A} be an abelian category. A *torsion pair* in \mathcal{A} is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of \mathcal{A} such that the following conditions are satisfied:

- (1) for all objects $T \in \text{Ob } \mathcal{T}$ and $F \in \mathcal{F}$ we have $\text{Hom}_{\mathcal{A}}(T, F) = 0$,
- (2) for every object $E \in \text{Ob } \mathcal{A}$ there exist objects $T \in \text{Ob } \mathcal{T}$ and $F \in \text{Ob } \mathcal{F}$ such that the following short exact sequence is exact in \mathcal{A} :

$$0 \rightarrow T \rightarrow E \rightarrow F \rightarrow 0.$$

We will need the following theorem of Happel, Reiten and Smalø:

THEOREM 3.2. (see [13, Proposition I.2.1]) *Assume that \mathcal{A} is the heart of a bounded t -structure on a triangulated category \mathcal{D} and suppose that $(\mathcal{T}, \mathcal{F})$ is a torsion pair in \mathcal{A} . Then the full subcategory*

$$\mathcal{B} = \{E \in \text{Ob } \mathcal{D} : H^i(E) = 0 \text{ for } i \neq 0, -1, H^{-1}(E) \in \text{Ob } \mathcal{F}, \text{ and } H^0(E) \in \text{Ob } \mathcal{T}\}$$

of \mathcal{D} is the heart of a bounded t -structure on \mathcal{D} .

In the situation of the above theorem we say that \mathcal{B} is obtained from \mathcal{A} by *tilting* with respect to the torsion pair $(\mathcal{T}, \mathcal{F})$.

Let E be a coherent sheaf on a noetherian scheme X . The *dimension* $\dim E$ of the sheaf E is by definition the dimension of the support of E . For a d -dimensional sheaf E there exists a unique filtration

$$0 \subset T_0(E) \subset T_1(E) \subset \dots \subset T_d(E)$$

such that $T_i(E)$ is the maximal subsheaf of E of dimension $\leq i$ (see [15, Definition 1.1.4]).

Let \mathcal{A} be an abelian category. Then any object in \mathcal{A} can be viewed as a complex concentrated in degree zero. This yields an equivalence between \mathcal{A} and the full subcategory of the derived category $D(\mathcal{A})$ of \mathcal{A} of complexes K^\bullet with $H^i(K^\bullet) = 0$ for $i \neq 0$.

In the following by $D^b(X)$ we denote the bounded derived category of the abelian category of coherent sheaves on the scheme X . The object of $D^b(X)$ corresponding to a coherent sheaf F is called a *sheaf object* and by abuse of notation it is also denoted by \mathcal{F} .

If \mathcal{C} is a complex of coherent sheaves on X then $\mathcal{H}^p(\mathcal{C})$ denotes its p -th cohomology. We use this notation since we would like to distinguish cohomology $\mathcal{H}^p(\mathcal{F})$ of a sheaf object \mathcal{F} and cohomology of a sheaf $H^p(\mathcal{F}) = H^p(X, \mathcal{F})$.

Let $\mathcal{A} = \text{Coh } X$ be the category of coherent sheaves on a smooth projective 3-fold X . Let \mathcal{T} be the full subcategory of \mathcal{A} whose objects are all coherent sheaves of dimension ≤ 1 . Let \mathcal{F} be the full subcategory of \mathcal{A} whose objects are all coherent sheaves E which do not contain subsheaves of dimension ≤ 1 (i.e. $T_1(E) = 0$). Clearly, $(\mathcal{T}, \mathcal{F})$ form a torsion pair in \mathcal{A} .

DEFINITION 3.3. A complex $\mathcal{C} \in D^b(X)$ is called a *perverse sheaf* if the following conditions are satisfied:

- (1) $\mathcal{H}^i(\mathcal{C}) = 0$ for $i \neq 0, 1$,
- (2) $\mathcal{H}^0(\mathcal{C}) \in \text{Ob } \mathcal{F}$,
- (3) $\mathcal{H}^1(\mathcal{C}) \in \text{Ob } \mathcal{T}$.

DEFINITION 3.4. A *moduli lax functor of perverse sheaves* is the lax functor $\text{Perv}(X) : \text{Sch}/k \rightarrow \text{Group}$ from the category of k -schemes to the category of groupoids, which to a k -scheme S assigns the groupoid that has S -families of perverse sheaves on X as objects and isomorphisms of perverse sheaves as morphisms.

Theorem 3.2 implies that perverse sheaves form an abelian category which is a shift of the tilting of $\text{Coh } X$ with respect to the pair $(\mathcal{T}, \mathcal{F})$. This fact is crucial in proof of the following theorem (cf. [11, VIII 5.1, 1.1, 1.2], [35, Theorem 3.5], [4, Lemma 5.5]):

THEOREM 3.5. *The moduli lax functor of perverse sheaves on a smooth three-dimensional projective variety is a (non-algebraic!) k -stack.*

PROPOSITION 3.6. *Let*

$$\mathcal{C} = (\mathcal{C}^{-1} \xrightarrow{\alpha} \mathcal{C}^0 \xrightarrow{\beta} \mathcal{C}^1)$$

be a complex of locally free sheaves on X and assume that there exists a curve $j : C \hookrightarrow X$ such that $Lj^*\mathcal{C}$ is a sheaf object in $D^b(C)$. Then the object of $D^b(X)$ corresponding to \mathcal{C} is a perverse sheaf.

Proof. Note that $Lj^*\mathcal{C}$ is represented by the complex

$$j^*\mathcal{C}^{-1} \xrightarrow{j^*\alpha} j^*\mathcal{C}^0 \xrightarrow{j^*\beta} j^*\mathcal{C}^1.$$

Since $\mathcal{H}^{-1}(Lj^*\mathcal{C}) = 0$, the restriction of α to C is injective. Since \mathcal{C}^{-1} is torsion-free this implies that α is injective and hence $\mathcal{H}^{-1}(\mathcal{C}) = 0$.

Similarly, by assumption we have $\mathcal{H}^1(Lj^*\mathcal{C}) = 0$ and hence the restriction of β to C is surjective. This implies that β is surjective in codimension 1 (i.e. it is surjective outside of a subset of codimension ≥ 2). Therefore $\mathcal{H}^1(\mathcal{C}) = \text{coker } \beta$ is a sheaf of dimension ≤ 1 , i.e. $\mathcal{H}^1(\mathcal{C})$ is an object of \mathcal{T} .

By definition we have a short exact sequence

$$0 \rightarrow \mathcal{H}^0(\mathcal{C}) \rightarrow E = \text{coker } \alpha \rightarrow \text{im } \beta \rightarrow 0.$$

Therefore to finish the proof it is sufficient to show that $T_1(E) = 0$. To prove this let us consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 0 & \longrightarrow & \ker \gamma & \xrightarrow{\alpha} & \mathcal{C}^0 & \xrightarrow{\gamma} & E/T_1(E) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \mathcal{C}^{-1} & \xrightarrow{\alpha} & \mathcal{C}^0 & \longrightarrow & E \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & \longrightarrow & 0 & \longrightarrow & T_1(E) \\
 & & & & & & \uparrow \\
 & & & & & & 0
 \end{array}$$

Using the snake lemma we get the following short exact sequence

$$0 \rightarrow \mathcal{C}^{-1} \rightarrow \ker \gamma \rightarrow T_1(E) \rightarrow 0.$$

Since $\ker \gamma$ is torsion-free (as a subsheaf of \mathcal{C}^0), \mathcal{C}^{-1} is reflexive and the map $\mathcal{C}^{-1} \rightarrow \ker \gamma$ is an isomorphism outside of the support of $T_1(E)$ (i.e. outside of a subset of codimension ≥ 2), the map $\mathcal{C}^{-1} \rightarrow \ker \gamma$ must be an isomorphism. In particular, $T_1(E) = 0$ and $\mathcal{H}^0(\mathcal{C})$ is an object of \mathcal{F} . □

PROPOSITION 3.7. *Let*

$$\mathcal{C} = (\mathcal{C}^{-1} \xrightarrow{\alpha} \mathcal{C}^0 \xrightarrow{\beta} \mathcal{C}^1)$$

be a complex of locally free sheaves on \mathbb{P}^3 and assume that there exists a curve $j : C \hookrightarrow \mathbb{P}^3$ such that $Lj^*\mathcal{C}$ is a locally free sheaf object in $D^b(C)$. Then $\mathcal{H}^0(\mathcal{C})$ is torsion-free.

Proof. Let Z be the set of points $p \in \mathbb{P}^3$ such that $\alpha(p) = \alpha \otimes k(p) : \mathcal{C}^{-1} \otimes k(p) \rightarrow \mathcal{C}^0 \otimes k(p)$ is not injective. It is a closed subset of \mathbb{P}^3 (in the Zariski topology). Since $\mathcal{H}^0(Lj^*\mathcal{C})$ is locally free, it is easy to see that Z does not intersect C (if $Z \cap C \neq \emptyset$ then the cokernel of $j^*\alpha$ would contain torsion that would also be contained in $\mathcal{H}^0(Lj^*\mathcal{C})$). But since we are on \mathbb{P}^3 this implies that Z has dimension at most 1. But the support of $T_2(\text{coker } \alpha)$ is contained in Z and $T_2(\text{coker } \alpha)$ is pure of dimension 2 (by the previous proposition). Therefore $\text{coker } \alpha$ is torsion-free, which implies that $\mathcal{H}^0(\mathcal{C})$ is also torsion-free. \square

3.2. Definition and basic properties of perverse instantons. Let us denote by j the embedding of a line l into \mathbb{P}^3 . The pull back j^* induces the left derived functor $Lj^* : D^b(\mathbb{P}^3) \rightarrow D^b(l)$.

DEFINITION 3.8. A rank r perverse instanton is an object \mathcal{C} of the derived category $D^b(\mathbb{P}^3)$ satisfying the following conditions:

- (1) $H^p(\mathbb{P}^3, \mathcal{C} \otimes \mathcal{O}_{\mathbb{P}^3}(q)) = 0$ if either $p = 0, 1$ and $p + q < 0$ or $p = 2, 3$ and $p + q \geq 0$,
- (2) $\mathcal{H}^p(\mathcal{C}) = 0$ for $p \neq 0, 1$,
- (3) there exists a line $j : l \hookrightarrow \mathbb{P}^3$ such that $Lj^*\mathcal{C}$ is isomorphic to the sheaf object $\mathcal{O}_l^{\oplus r}$.

Let us fix a line $j : l_\infty \hookrightarrow \mathbb{P}^3$ and choose coordinates $[x_0, x_1, x_2, x_3]$ in \mathbb{P}^3 so that l_∞ is given by $x_0 = x_1 = 0$. A framing Φ along l_∞ of a perverse instanton \mathcal{C} is an isomorphism $\Phi : Lj^*\mathcal{C} \rightarrow \mathcal{O}_{l_\infty}^{\oplus r}$. A framed perverse instanton is a pair (\mathcal{C}, Φ) consisting of a perverse instanton \mathcal{C} and its framing Φ .

Any instanton is a perverse instanton. By the Riemann–Roch theorem for any perverse instanton \mathcal{C} there exists $c \geq 0$ such that $\text{ch}(\mathcal{C}) = r - c[H]^2$. We also have a distinguished triangle $\mathcal{H}^0(\mathcal{C}) \rightarrow \mathcal{C} \rightarrow \mathcal{H}^1(\mathcal{C})[-1] \rightarrow \mathcal{H}^0[1]$. However, it is not a priori clear if a perverse instanton is a perverse sheaf in the sense of Definition 3.3. We will prove that this is indeed the case in Corollary 3.16.

As before there is a natural $G = \text{GL}(r)$ -action on $\tilde{\mathbf{B}}$ which induces a G -action on the set of ADHM data. More precisely, let us recall that the group $G = \text{GL}(V)$ acts on \mathbf{B} via

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$$

and it induces the action on $\tilde{\mathbf{B}}$. If we consider the adjoint action of G on $\text{End}(V)$ then the map $\tilde{\mu}$ is G -equivariant. In particular, G acts on $\tilde{\mu}^{-1}(0)$, i.e. on the set of ADHM data.

The main motivation for introducing perverse instantons is the following theorem:

THEOREM 3.9. *There exists a bijection between isomorphism classes of perverse instantons (\mathcal{C}, Φ) with $\text{ch}(\mathcal{C}) = r - c[H]^2$ framed along l_∞ and $\text{GL}(c)$ -orbits of ADHM (r, c) -data for \mathbb{P}^1 .*

The bijection in the theorem is the same as in Section 2.4. Namely, if $x = (\tilde{B}_1, \tilde{B}_2, \tilde{\iota}, \tilde{j})$ is an (r, c) -complex ADHM datum then we can associate to it the complex

$$C_x^\bullet = (V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \tag{6}$$

where α and β are defined as in Section 2.4. This complex is a perverse instanton and it comes with an obvious framing along l_∞ .

In the following we sketch proof of a stronger version of the above theorem showing that isomorphism already holds at the level of stacks. To do so, first we need to generalise the above definition to families of perverse instantons.

Let S be a (locally noetherian) k -scheme. We set $j_S = j \times \text{Id}_S : S \times_k l_\infty \rightarrow \mathbb{P}_S^3 = S \times_k \mathbb{P}^3$.

DEFINITION 3.10. An S -family of framed perverse (r, c) -instantons is an object $C \in \text{Ob } D^b(\mathbb{P}_S^3)$ together with an isomorphism $\Phi : Lj_S^* C \rightarrow \mathcal{O}_{l_\infty \times_S}^r$ such that for every geometric point $s : \text{Spec } K \rightarrow S$, the derived pull-back $(Lj_s^* C, Ls^* \Phi)$ is a framed perverse (r, c) -instanton on \mathbb{P}_K^3 .

A morphism $\varphi : (C_1, \Phi_1) \rightarrow (C_2, \Phi_2)$ of S -families of framed perverse instantons is a morphism $\varphi : C_1 \rightarrow C_2$ in $D^b(\mathbb{P}_S^3)$ such that $\Phi_1 = \Phi_2 \circ Lj_S^* \varphi$.

DEFINITION 3.11. A moduli lax functor of framed perverse (r, c) -instantons is the lax functor $\text{Perv}_r^c(\mathbb{P}^3, l_\infty) : \text{Sch}/k \rightarrow \text{Group}$ from the category of k -schemes to the category of groupoids, which to a k -scheme S assigns the groupoid that has S -families of framed perverse (r, c) -instantons as objects and isomorphisms of framed perverse instantons as morphisms.

In order for the definition to make geometric sense we have to note that the moduli lax functor is a stack, i.e. it defines a sheaf of categories in the faithfully flat topology:

LEMMA 3.12. *The moduli lax functor of framed perverse (r, c) -instantons on \mathbb{P}^3 is a k -stack of finite type.*

Let G be an algebraic group acting on a scheme X . Then we can form a quotient stack $[X/G]$ which to any scheme S assigns the groupoid whose objects are pairs (P, φ) consisting of a principal G -bundle P on S and a G -equivariant morphism $\varphi : P \rightarrow X$. A morphism in this groupoid is an isomorphism $h : (P_1, \varphi_1) \rightarrow (P_2, \varphi_2)$ of pairs, i.e. such an isomorphism $h : P_1 \rightarrow P_2$ of principal G -bundles that $\varphi_1 = \varphi_2 \circ h$.

In the three-dimensional case we have the following analogue of Drinfeld’s theorem on representability of the stack of framed perverse sheaves on \mathbb{P}^2 (see [4, Theorem 5.7]):

THEOREM 3.13. *The moduli stack $\text{Perv}_r^c(\mathbb{P}^3, l_\infty)$ is isomorphic to the quotient stack $[\tilde{\mu}^{-1}(0)/\text{GL}(V)]$.*

Proof of the above theorem is analogous to proof of [4, Theorem 5.7] and it follows from the following two lemmas.

LEMMA 3.14. *Let C be a perverse instanton on \mathbb{P}^3 . Then*

$$H^q(\mathbb{P}^3, C(-1) \otimes \Omega_{\mathbb{P}^3}^{-p}(-p)) = 0$$

for $q \neq 1$ and for $q = 1, p \leq -3$ or $p > 0$.

This lemma and its proof are analogous to [28, Lemma 2.4] and [9, Proposition 26].

LEMMA 3.15. *An S -family of perverse instantons $\mathcal{C} \in D^b(\mathbb{P}_S^3)$ is canonically isomorphic to the complex of sheaves*

$$\mathcal{O}_{\mathbb{P}_S^3}(-1) \otimes R^1(p_1)_*(\mathcal{C} \otimes \Omega^2(1)) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}_S^3} \otimes R^1(p_1)_*(\mathcal{C} \otimes \Omega^1) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}_S^3}(1) \otimes R^1(p_1)_*(\mathcal{C}(-1))$$

in degrees $-1, 0, 1$ coming from Beilinson’s construction. Moreover, α is injective (as a map of sheaves), the sheaves $R^1(p_1)_(\mathcal{C}(-1) \otimes \Omega^p(p))$ are locally free for $p = 0, 1, 2$ and we have a canonical isomorphism*

$$R^1(p_1)_*(\mathcal{C} \otimes \Omega^2(1)) \simeq R^1(p_1)_*(\mathcal{C}(-1)).$$

The above lemma follows from the previous lemma by standard arguments using Beilinson’s construction (i.e. proof of existence of Beilinson’s spectral sequence) in families.

COROLLARY 3.16. *Let \mathcal{C} be a perverse instanton on \mathbb{P}^3 . Then $\mathcal{H}^0(\mathcal{C})$ is torsion-free and $\mathcal{H}^1(\mathcal{C})$ is of dimension ≤ 1 .*

Proof. The assertion follows immediately from Proposition 3.7 and Lemma 3.15 applied for S being a point. □

3.3. Analysis of singularities of perverse instantons.

DEFINITION 3.17. Let E be a coherent sheaf on a smooth variety X . Then the set of points where the sheaf E is not locally free is called the *singular locus* of E and it is denoted by $S(E)$.

It is easy to see that the singular locus of an arbitrary coherent sheaf on X is a closed subset of X (in the Zariski topology). Here we study the singular locus of perverse instantons on \mathbb{P}^3 .

From the proof of [9, Proposition 10] it follows that in case of complex ADHM data if $\text{coker } \alpha$ is not reflexive then it is non-locally free along a certain (possibly non-reduced or reducible) curve of degree c^2 (not $2c!$) that does not intersect l_∞ . If $\text{coker } \alpha$ is reflexive then it is non-locally free only in a finite number of points.

We have two short exact sequences:

$$0 \rightarrow \mathcal{H}^0(\mathcal{C}) \rightarrow \text{coker } \alpha \rightarrow \text{im } \beta \rightarrow 0$$

and

$$0 \rightarrow \text{im } \beta \rightarrow \mathcal{C}^1 \rightarrow \mathcal{H}^1(\mathcal{C}) = \text{coker } \beta \rightarrow 0.$$

It follows that $\text{im } \beta$ is torsion-free and it is non-locally free exactly along the support of $\mathcal{H}^1(\mathcal{C})$ (which is at most one-dimensional).

Obviously, $\mathcal{H}^0(\mathcal{C})$ can be non-locally free only at the points of $S(\text{coker } \alpha)$ or at the points of $S(\text{im } \beta)$. Moreover, the one-dimensional components of $S(\mathcal{H}^0(\mathcal{C}))$ are contained in $S(\text{coker } \alpha)$. This follows from the fact that the kernel of a map from a locally free sheaf to a torsion-free sheaf is reflexive.

4. GIT approach to perverse instantons. In this section we consider ADHM data for an arbitrary manifold X . We have a natural $G = \text{GL}(V)$ -action on \mathbf{B} which induces a G -action on the set of ADHM data $\tilde{\mu}^{-1}(0)$. Let $\chi : G \rightarrow \mathbb{G}_m$ be the character given by the determinant. We can consider the G -action on $\mathbf{B} \times \mathbb{A}^1$ with respect to this character (i.e. a non-trivial G -linearisation of the trivial line bundle on \mathbf{B}).

The main aim of this section is to study different notions of stability obtained via Geometric Invariant Theory when taking quotients $\tilde{\mu}^{-1}(0) //_{\chi} G$ and $\tilde{\mu}^{-1}(0)/G$.

This section is just a careful rewriting of [37, Section 2] but we give a bit more details for the convenience of the reader.

DEFINITION 4.1. We say that an ADHM datum is

- (1) *stable*, if for every subspace $S \subsetneq V$ (note that we allow $S = 0$) such that $\tilde{B}_k(S) \subset S \otimes H^0(\mathcal{O}_X(1))$ for $k = 1, 2$ we have $\text{im } \tilde{i} \not\subset S \otimes H^0(\mathcal{O}_X(1))$.
- (2) *costable*, if for every non-zero subspace $S \subset V$ such that $\tilde{B}_k(S) \subset S \otimes H^0(\mathcal{O}_X(1))$ for $k = 1, 2$ we have $S \not\subset \ker \tilde{j}$,
- (2) *regular*, if it is stable and costable.

We say that $(\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j})$ satisfies the *ADHM equation* if $[\tilde{B}_1, \tilde{B}_2] + \tilde{i}\tilde{j} = 0$.

The following lemma generalises [28, Lemma 3.25]. Its proof is similar to the proof given in [28].

LEMMA 4.2. *Let x be an ADHM datum. Then x is stable if and only if $G \cdot (x, z)$ is closed for some (or, equivalently, all) $z \neq 0$.*

Proof. Let sections $\{s_l\}$ form a basis of $H^0(\mathcal{O}_X(1))$. Then \tilde{B}_k and \tilde{i} can be written as

$$\tilde{B}_k = \sum B_{lk} \otimes s_l, \quad \tilde{i} = \sum i_l \otimes s_l.$$

Assume that $G \cdot (x, z)$ is closed for $z \neq 0$. Suppose that there exists $S \subsetneq V$ such that $\tilde{B}_k(S) \subset S \otimes H^0(\mathcal{O}_X(1))$ for $k = 1, 2$ and $\text{im } \tilde{i} \subset S \otimes H^0(\mathcal{O}_X(1))$. Let us fix a subspace $S^\perp \subset V$ such that $V = S \oplus S^\perp$. Then we have

$$B_{kl} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad i_l = \begin{pmatrix} * \\ 0 \end{pmatrix}.$$

If we set $g(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix}$ then we have

$$g(t)B_{kl}g(t^{-1}) = \begin{pmatrix} * & t* \\ 0 & * \end{pmatrix}, \quad g(t)i_l = i_l.$$

Therefore there exists limit $\lim_{t \rightarrow 0} g(t)x$ in \mathbf{B} . On the other hand, when $t \rightarrow 0$ then

$$g(t)(x, z) = (g(t)x, \det(g(t))^{-1}z) = (g(t)x, t^{\dim S^\perp}z)$$

has a limit $(\lim_{t \rightarrow 0} g(t)x, 0)$ which does not belong to $G \cdot (x, z)$. Contradiction shows that x has to be stable.

Now suppose that x is stable and $G \cdot (x, z)$ is not closed. By the Hilbert–Mumford criterion there exists a 1-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$ such that the limit $\lim_{t \rightarrow 0} \lambda(t)(x, z)$ exists and it belongs to $\overline{G \cdot (x, z)} \setminus G \cdot (x, z)$. Let $V(m)$ consist of

vectors $v \in V$ such that $\lambda(t) \cdot v = t^m v$ for every $t \in \mathbb{G}_m$. Then we have a decomposition $V = \bigoplus_m V(m)$ and we can choose a basis of V such that

$$\lambda(t) = \begin{pmatrix} t^{a_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & t^{a_c} \end{pmatrix}$$

where $a_1 \geq \dots \geq a_c$. Existence of $\lim_{t \rightarrow 0} \lambda(t) \tilde{B}_k \lambda(t^{-1})$ implies that the limits $\lim_{t \rightarrow 0} \lambda(t) \tilde{B}_{lk} \lambda(t^{-1})$ exist for every l . Let $B_{lk} = (b_{ij})$. Then $(\lambda(t) \tilde{B}_{lk} \lambda(t^{-1}))_{ij} = t^{a_i - a_j} b_{ij}$. This shows that $b_{ij} = 0$ if $a_i < a_j$. Therefore $\tilde{B}_k(V(m)) \subset (\bigoplus_{l \geq m} V(l)) \otimes H^0(\mathcal{O}_X(1))$. Similarly, one can show that $\text{im } \tilde{\iota} \subset (\bigoplus_{m \geq 0} V(m)) \otimes H^0(\mathcal{O}_X(1))$. Let us set $S = \bigoplus_{m \geq 0} V(m)$. Then $\tilde{B}_k(S) \subset S \otimes H^0(\mathcal{O}_X(1))$ and $\text{im } \tilde{\iota} \subset S \otimes H^0(\mathcal{O}_X(1))$, so from the stability condition it follows that $S = V$. Therefore $\det \lambda(t) = t^N$ for $N \geq 0$. If $N = 0$ then $V(0) = V$ and $\lambda \equiv \text{Id}$. This is impossible because $\lim_{t \rightarrow 0} \lambda(t)(x, z) \notin G \cdot (x, z)$. If $N > 0$ then $\lambda(t)(x, z) = (\lambda(t)x, \det(\lambda(t))^{-1}z) = (\lambda(t)x, t^{-N}z)$ which diverges as $t \rightarrow 0$. This gives a contradiction. \square

PROPOSITION 4.3. *The following conditions are equivalent:*

- (1) x is stable,
- (2) x is χ -stable,
- (3) x is χ -semistable.

Similar assertion holds if we replace stable with costable and χ with χ^{-1} .

Proof. By Lemma 4.2 x is stable if and only if x is χ -polystable. So to prove the proposition it is sufficient to prove that if x is stable then its stabiliser in G is trivial. Assume that $g \in G$ acts trivially on x and consider $S = \ker(g - \text{Id})$. Then $\text{im } \tilde{\iota} \subset S \otimes H^0(\mathcal{O}_X(1))$ and $\tilde{B}_k(S) \subset S \otimes H^0(\mathcal{O}_X(1))$ so $S = V$ and $g = \text{Id}$. If x is χ -semistable let y be a χ -polystable ADHM datum such that (y, w) is in the closure of $G \cdot (x, z)$. Since $\overline{G \cdot (x, z)}$ is disjoint from the zero-section ([21, Lemma 2.2]) we know that $w \neq 0$. Then y is χ -stable and in particular it has a trivial stabiliser in G . Therefore the orbit of (y, w) has the maximal dimension. But the set $\overline{G \cdot (x, z)} \setminus G \cdot (x, z)$ is composed from the orbits of smaller dimension than the dimension of $G \cdot (x, z)$. Therefore $G \cdot (x, z) = G \cdot (y, w)$ and x is also χ -stable. \square

LEMMA 4.4. *Let x be an ADHM datum. Then x is 1-stable (i.e. stable for the trivial character) if and only if it is regular.*

Proof. Let us recall that x is 1-stable if and only if the stabiliser of x in G is trivial and the orbit $G \cdot x$ is closed. Then for any character and any $z \neq 0$ the orbit $G \cdot (x, z)$ is closed as well. In particular, x is both χ -stable and χ^{-1} -stable, which by Proposition 4.3 gives implication \Rightarrow .

Proof of the other implication is similar to the proof of Lemma 4.2. Suppose that x stable and costable and $G \cdot x$ is not closed. There exists a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists and belongs to $\overline{G \cdot x} \setminus G \cdot x$. Let $V = \bigoplus_m V(m)$ be the weight decomposition with respect to λ . As in proof of Lemma 4.2 existence of the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ implies that

$$\tilde{B}_k(V(m)) \subset \left(\bigoplus_{l \geq m} V(l) \right) \otimes H^0(\mathcal{O}_X(1)),$$

$$\text{im } \tilde{\iota} \subset \left(\bigoplus_{m \geq 0} V(m) \right) \otimes H^0(\mathcal{O}_X(1))$$

and

$$\bigoplus_{m > 0} V(m) \subset \ker \tilde{j}.$$

The stability condition implies that $V = \bigoplus_{m \geq 0} V(m)$ and costability gives $\bigoplus_{m \geq 1} V(m) = \{0\}$. So $V = V(0)$ which contradicts our assumption that the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ does not belong to $G \cdot x$. The stabiliser of x in G is trivial because x is also χ -stable. □

LEMMA 4.5. *Let $x \in \tilde{\mu}_{\tilde{W}, V}^{-1}(0)$. Then x is 1-polystable if and only if there exist subspaces $V_1, V_2 \subset V$ and quadruples $x_1 \in \tilde{\mu}_{\tilde{W}, V_1}^{-1}(0)^{\text{s.c}}$ and $x_2 \in \tilde{\mu}_{\{0\}, V_2}^{-1}(0)$ such that $V = V_1 \oplus V_2, x = x_1 \oplus x_2$ and $\text{GL}(V_2) \cdot x_2$ is closed. Moreover, such splitting is unique.*

Proof. Let us remind that x is 1-polystable if and only if $\text{GL}(V) \cdot x$ is closed.

Assume first that $x = (\tilde{B}_1, \tilde{B}_2, \tilde{\iota}, \tilde{j})$ has a closed orbit and define V_1 as the intersection of all subspaces $S \subset V$ such that $\tilde{B}_k(S) \subset S \otimes H^0(\mathcal{O}_X(1))$ for $k = 1, 2$ and $\text{im } \tilde{\iota} \subset S \otimes H^0(\mathcal{O}_X(1))$. Choose V_2 such that $V = V_1 \oplus V_2$. Let $\{s_l\}$ be a basis of $H^0(\mathcal{O}_X(1))$. Then $\tilde{B}_k, \tilde{\iota}$ and \tilde{j} can be written as

$$\tilde{B}_k = \sum B_{lk} \otimes s_l, \quad \tilde{\iota} = \sum i_l \otimes s_l, \quad \tilde{j} = \sum j_l \otimes s_l$$

where

$$B_{kl} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad i_l = \begin{pmatrix} * \\ 0 \end{pmatrix}, \quad j_l = (* *).$$

If $\lambda(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix}$ then we have

$$\lambda(t)B_{kl}\lambda(t^{-1}) = \begin{pmatrix} * & t* \\ 0 & * \end{pmatrix}, \quad \lambda(t)i_l = i_l, \quad j_l\lambda(t^{-1}) = (* t*).$$

Hence there exists $x' = (\tilde{B}'_1, \tilde{B}'_2, \tilde{\iota}', \tilde{j}')$ $= \lim_{t \rightarrow 0} \lambda(t) \cdot x$ which has the following properties:

- $\tilde{B}'_k(V_a) \subset V_a \otimes H^0(\mathcal{O}_X(1))$ for $k, a = 1, 2$,
- $\tilde{B}'_{k|V_1} = \tilde{B}_{k|V_1}$ for $k = 1, 2$,
- $\tilde{\iota}' = \tilde{\iota}$,
- $\tilde{j}'_{|V_2} = 0$,
- $\tilde{j}'_{|V_1} = \tilde{j}_{|V_1}$.

Since the orbit of x is closed, we have $x' \in \text{GL}(V) \cdot x$. There exists $g \in \text{GL}(V)$ such that $x' = g \cdot x$. So if we find V'_1, V'_2 and x'_1, x'_2 satisfying conditions in the lemma for x' , then $g \cdot V'_1, g \cdot V'_2$ and $g^{-1} \cdot x'_1, g^{-1} \cdot x'_2$ satisfy it for x .

Let V'_1 be the intersection of all subspaces $S \subset V$ such that $\tilde{B}'_k(S) \subset S \otimes H^0(\mathcal{O}_X(1))$ for $k = 1, 2$ and $\text{im } \tilde{\iota}' \subset S \otimes H^0(\mathcal{O}_X(1))$. Properties of x' show that V_1 is one of such subspaces so $V'_1 \subset V_1$. On the other hand $g \cdot V'_1$ destabilises x so $V_1 \subset g^{-1} \cdot V'_1$ and by the dimension count we obtain $V'_1 = V_1$. Let us set $V'_2 = V_2$ and

$x'_1 = (\tilde{B}'_{1|V_1}, \tilde{B}'_{2|V_1}, \tilde{i}', \tilde{j}'_{|V_1}), x'_2 = (\tilde{B}'_{1|V_2}, \tilde{B}'_{2|V_2}, 0, \tilde{j}'_{|V_2}) = (\tilde{B}'_{1|V_2}, \tilde{B}'_{2|V_2}, 0, 0)$. It is clear that $x'_1 \in \tilde{\mu}^{-1}_{W, V_1}(0)$ and $x'_2 \in \tilde{\mu}^{-1}_{\{0\}, V_2}(0)$ and $x' = x'_1 \oplus x'_2$. Since V'_1 is minimal destabilising space for x' , we also know that x'_1 is stable.

Now assume that $x''_1 \in \tilde{\mu}^{-1}_{W, V_1}(0)$ is in the closure of the $GL(V_1)$ -orbit of x'_1 and $x''_2 \in \tilde{\mu}^{-1}_{\{0\}, V_2}(0)$ is in the closure of the $GL(V_2)$ -orbit of x'_2 . Then $x''_1 \oplus x''_2$ is in the closure of the $GL(V)$ -orbit of $x' = x'_1 \oplus x'_2$. This orbit is closed by the assumption so we can find $g \in GL(V)$ such that $g \cdot x' = x''_1 \oplus x''_2$.

We can write

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$x'_1 = \left(\sum_l B'_{11l} \otimes s_l, \sum_l B'_{21l} \otimes s_l, \sum_l i'_{1l} \otimes s_l, \sum_l j'_{1l} \otimes s_l \right)$$

$$x'_2 = \left(\sum_l B'_{12l} \otimes s_l, \sum_l B'_{22l} \otimes s_l, 0, 0 \right)$$

$$x''_1 = \left(\sum_l B''_{11l} \otimes s_l, \sum_l B''_{21l} \otimes s_l, \sum_l i''_{1l} \otimes s_l, \sum_l j''_{1l} \otimes s_l \right)$$

$$x''_2 = \left(\sum_l B''_{12l} \otimes s_l, \sum_l B''_{22l} \otimes s_l, 0, 0 \right)$$

$$x' = \left(\begin{pmatrix} B'_{11l} & 0 \\ 0 & B'_{12l} \end{pmatrix} \otimes s_l, \begin{pmatrix} B'_{21l} & 0 \\ 0 & B'_{22l} \end{pmatrix} \otimes s_l, \begin{pmatrix} i'_{1l} \\ 0 \end{pmatrix} \otimes s_l, \begin{pmatrix} j'_{1l} & 0 \end{pmatrix} \otimes s_l \right)$$

$$x''_1 \oplus x''_2 = \left(\begin{pmatrix} B''_{11l} & 0 \\ 0 & B''_{12l} \end{pmatrix} \otimes s_l, \begin{pmatrix} B''_{21l} & 0 \\ 0 & B''_{22l} \end{pmatrix} \otimes s_l, \begin{pmatrix} i''_{1l} \\ 0 \end{pmatrix} \otimes s_l, \begin{pmatrix} j''_{1l} & 0 \end{pmatrix} \otimes s_l \right)$$

The equality $g \cdot x' = x''_1 \oplus x''_2$ gives us for each l and $k = 1, 2$ the following equalities:

$$\begin{aligned} \begin{pmatrix} g_{11}B'_{k1l} & g_{12}B'_{k2l} \\ g_{21}B'_{k1l} & g_{22}B'_{k2l} \end{pmatrix} &= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} B'_{k1l} & 0 \\ 0 & B'_{k2l} \end{pmatrix} \\ &= \begin{pmatrix} B''_{k1l} & 0 \\ 0 & B''_{k2l} \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} B''_{k1l}g_{11} & B''_{k1l}g_{12} \\ B''_{k2l}g_{21} & B''_{k2l}g_{22} \end{pmatrix} \\ \begin{pmatrix} g_{11}i'_{1l} \\ g_{21}i'_{1l} \end{pmatrix} &= \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} i'_{1l} \\ 0 \end{pmatrix} = \begin{pmatrix} i''_{1l} \\ 0 \end{pmatrix} \\ \begin{pmatrix} j''_{1l}g_{11} & j''_{1l}g_{12} \end{pmatrix} &= \begin{pmatrix} j''_{1l} & 0 \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} j'_{1l} & 0 \end{pmatrix} \end{aligned}$$

Let $S = \ker g_{21} \subset V_1$. The equalities above show that for every l and $k = 1, 2$ we have

$$B'_{kl}(S) \subset S \quad \text{im } i'_{1l} \subset S,$$

which shows that S is a destabilising space for x'_1 . Since x_1 is stable $S = V_1$ and $g_{21} = 0$. Therefore g_{11} and g_{22} are isomorphisms. Hence

$$g_{11} \cdot x'_1 = x''_2 \quad \text{and} \quad g_{22} \cdot x'_2 = x''_2$$

which shows that the orbits $\text{GL}(V_1) \cdot x'_1$ and $\text{GL}(V_2) \cdot x'_2$ are closed. By the same argument as in Proposition 4.3 we show that x_1 has a trivial stabiliser in $\text{GL}(V_1)$. Hence x_1 is 1-stable and by Lemma 4.5 it is costable.

REMARK 4.6. Note that the decomposition $V = V_1 \oplus V_2$ is unique since V_1 (respectively V_2) is the smallest (respectively the biggest) subspace of V such that

$$\tilde{B}_k(V_1) \subset V_1 \otimes H^0(\mathcal{O}_X(1)) \text{ for } k = 1, 2 \text{ and } \text{im } \tilde{i} \subset V_1 \otimes H^0(\mathcal{O}_X(1))$$

$$\text{(respectively } \tilde{B}_k(V_2) \subset V_2 \otimes H^0(\mathcal{O}_X(1)) \text{ for } k = 1, 2 \text{ and } V_2 \subset \ker \tilde{j}\text{)}.$$

Obviously, the splitting $x = x_1 \oplus x_2$ is also unique.

Now let us prove the opposite implication \Leftarrow . Fix $x = (\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j})$ admitting a splitting $x = x_1 \oplus x_2$ as in the statement of the lemma. Let Y be the unique closed orbit contained in the closure of $\text{GL}(V)$ -orbit of x . By [28, Theorem 3.6] there exists $x^0 \in Y$ and $\lambda : \mathbb{G}_m \rightarrow \text{GL}(V)$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x = x^0$. The implication proved above shows that there exists a unique splitting $x^0 = x^0_1 \oplus x^0_2$ and $V = V^0_1 \oplus V^0_2$ as in the lemma. Put

$$\bar{V}^0_k = \lim_{t \rightarrow 0} \lambda(t) V_k \text{ and } \bar{x}^0_k = \lim_{t \rightarrow 0} \lambda(t) \cdot x_k \text{ for } k = 1, 2.$$

The first limit exists because subspaces in V of fixed dimension are parameterised by Grassmanians which are projective. The remaining limits are restrictions of x^0 to \bar{V}^0_1 and \bar{V}^0_2 , respectively.

REMARK 4.7. Let us note that

$$\lambda(s) \lim_{t \rightarrow 0} \lambda(t)(V_k) = \lim_{t \rightarrow 0} \lambda(ts)(V_k) = \bar{V}^0_k.$$

Hence for any $s \in \mathbb{G}_m$ and $k = 1, 2$ we have $\lambda(s)(\bar{V}^0_k) = \bar{V}^0_k$.

First, let us suppose that $\lambda(t)(V_1) = V_1$ for all t . Then $\bar{V}^0_1 = V_1$ and for $k, l = 1, 2$ we have

$$\begin{aligned} (\lim_{t \rightarrow 0} \lambda(t) \cdot \tilde{B}_l)(\bar{V}^0_k) &= (\lim_{t \rightarrow 0} \lambda(t) \cdot \tilde{B}_l \cdot \lambda(t^{-1}))(\lim_{s \rightarrow 0} \lambda(s) V_k) = \lim_{t \rightarrow 0} \lambda(t) \cdot \tilde{B}_l(V_k) \subset \\ &\subset \lim_{t \rightarrow 0} \lambda(t)(V_k \otimes H^0(\mathcal{O}_X(1))) = \bar{V}^0_k \otimes H^0(\mathcal{O}_X(1)) \end{aligned}$$

$$\text{im}(\lim_{t \rightarrow 0} \lambda(t) \tilde{i}) = \lim_{t \rightarrow 0} \lambda(t) \text{im } \tilde{i} \subset \lim_{t \rightarrow 0} \lambda(t) V_1 \otimes H^0(\mathcal{O}_X(1)) = \bar{V}^0_1 \otimes H^0(\mathcal{O}_X(1))$$

and

$$\ker \lim_{t \rightarrow 0} \tilde{j}\lambda(t^{-1}) = \lim_{t \rightarrow 0} \ker \tilde{j}\lambda(t^{-1}) \supset \lim_{t \rightarrow 0} \lambda(t)V_2 = \bar{V}_2^0.$$

Therefore by the characterisation of V_1^0 and V_2^0 given in Remark 4.6 we have $V_1^0 \subset \bar{V}_1^0$ and $\bar{V}_2^0 \subset V_2^0$. Since $\lambda(t)$ preserves V_1 for all t we see that $\bar{x}_1^0 \in \overline{\text{GL}(V_1)} \cdot x_1 = \text{GL}(V_1) \cdot x_1$. This shows that \bar{x}_1^0 is stable and again using Remark 4.6 we obtain equality $V_1^0 = \bar{V}_1^0$. Then the dimension count shows that $V_2^0 = \bar{V}_2^0$ and in particular $V = \bar{V}_1^0 \oplus \bar{V}_2^0$.

Consider the unipotent group

$$U_\lambda = \{u \in \text{GL}(V) \mid \lim_{t \rightarrow 0} (\lambda(t)u\lambda(t)^{-1}) = 1\}.$$

Observe that we can replace λ by $\lambda' = u\lambda u^{-1}$ for $u \in U_\lambda$. Indeed, one can easily prove that $\lim_{t \rightarrow 0} \lambda'(t)x = ux^0$ and ux^0 represents the same orbit as x^0 .

We will show that there exists $u \in U_\lambda$ such that

$$u(\bar{V}_k^0) = V_k \text{ for } k = 1, 2. \tag{7}$$

Set $m = \dim \bar{V}_1^0$ and $n = \dim V$. By Remark 4.7 one can choose a basis $\mathcal{A} = (\alpha_i)_{i=1}^n$ of V such that $(\alpha_i)_{i=1}^m$ is a basis of \bar{V}_1^0 , $(\alpha_i)_{i=m+1}^n$ is a basis of \bar{V}_2^0 and $\lambda(t)\alpha_i = t^{a_i}\alpha_i$ for some $a_i \in \mathbb{Z}$ and all $t \in \mathbb{G}_m$. Let (u_{ij}) be the matrix of $u \in \text{GL}(V)$. Then

$$(u_{ij}) \in U_\lambda \iff u_{ii} = 1 \text{ and } u_{ij} = 0 \text{ for } i \neq j \text{ such that } a_i \leq a_j. \tag{8}$$

Let $\mathcal{B} = (\beta_j)_{j=m+1}^n$ be a basis of V_2 such that

$$\beta_j = \alpha_j + \sum_{i=1}^m c_{ij}\alpha_i \text{ for } j = (m + 1), \dots, n.$$

Such a basis exists because $\dim V_2 = \dim \bar{V}_2^0$ and $V_2 \cap \bar{V}_2^0 = \{0\}$. Let $b_j = \min\{a_i : i = j \text{ or } c_{ij} \neq 0\}$. Then

$$\begin{aligned} \lim_{t \rightarrow 0} \lambda(t) \text{span}(\beta_j) &= \lim_{t \rightarrow 0} \text{span} \left(t^{a_j}\alpha_j + \sum_{i=1}^m c_{ij}t^{a_i}\alpha_i \right) \\ &= \text{span} \left(\lim_{t \rightarrow 0} \left(t^{a_j-b_j}\alpha_j + \sum_{i=1}^m c_{ij}t^{a_i-b_j}\alpha_i \right) \right). \end{aligned}$$

The vector $\lim_{t \rightarrow 0} (t^{a_i-b_j}\alpha_j + \sum_{i=1}^m c_{ij}t^{a_i-b_j}\alpha_i)$ exists by the definition of b_j . Moreover, since $\lim_{t \rightarrow 0} \lambda(t)V_2 = \bar{V}_2^0$, it must be contained in \bar{V}_2^0 . Therefore $b_j = a_j$ and $c_{ij} = 0$ when $a_i \leq a_j$. Now, with respect to the basis \mathcal{A} , we define $u \in \text{GL}(V)$ by matrix (u_{ij}) with the following coefficients:

$$u_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c_{ij} & \text{for } i \leq m \text{ and } j \geq m + 1, \\ 0 & \text{in the remaining cases.} \end{cases}$$

By (8) such u belongs to U_λ and satisfies (7).

Using Remark 4.7 we get for $k = 1, 2$

$$\lambda'(t)(V_k) = u\lambda(t)u^{-1}(V_k) = u\lambda(t)(\overline{V}_k^0) = u(\overline{V}_k^0) = V_k.$$

We can therefore assume that $\lambda = \lambda_1 \times \lambda_2$, where $\lambda_a, a = 1, 2$, is a one-parameter subgroup of $\text{GL}(V_a)$. Then, since x_1, x_2 have closed orbits, we get

$$x^0 = \lim_{t \rightarrow 0} (\lambda_1(t) \cdot x_1 \oplus \lambda_2(t) \cdot x_2) \in \text{GL}(V_1) \cdot x_1 \oplus \text{GL}(V_2) \cdot x_2 \subseteq \text{GL}(V) \cdot x,$$

which proves that the orbit of x is also closed.

In general, by similar arguments as above there exists an element $u \in U_\lambda$ such that $u(\overline{V}_1^0) = V_1$. Consider the one-parameter subgroup $\lambda' = u\lambda u^{-1}$. Then we have $\lambda'(t)(V_1) = V_1$ for all t and $\lim_{t \rightarrow 0} (\lambda'(t) \cdot x) = ux^0$. Thus, the previous part of the proof implies that $ux^0 \in \text{GL}(V) \cdot x$ and it proves that the orbit of x is closed. □

REMARK 4.8. Note that up to now we have never assumed that either r or c is positive. In fact, $r = 0$ is very interesting due to Lemma 4.5. This lemma shows that at least set-theoretically one can reduce the study of 1-polystable ADHM data to regular ADHM data and 1-polystable ADHM data in the rank 0 case. We explain the geometric meaning of this fact in the next section.

Note that in case of rank 0 there are no stable ADHM data so $\tilde{\mu}^{-1}(0) //_\chi G = \emptyset$. But the quotient $\tilde{\mu}^{-1}(0)/G$ is still a highly non-trivial scheme.

5. Gieseker and Donaldson–Uhlenbeck partial compactifications of instantons. In this section we consider ADHM data for $X = \mathbb{P}^1$, which by Theorem 3.9 correspond to perverse instantons on \mathbb{P}^3 . In this case we obtain a similar picture as that known from framed torsion-free sheaves on \mathbb{P}^2 (see 2.2).

DEFINITION 5.1. A perverse instanton \mathcal{C} is called *stable (costable, regular)* if it comes from some stable (respectively: costable, regular) ADHM datum.

Let us recall that we have a natural action of $G = \text{GL}(V)$ on the set $\tilde{\mu}^{-1}(0)$ of ADHM data. This action induces an action on the open subset $\tilde{\mu}^{-1}(0)^s$ of stable ADHM data, which by Lemma 4.3 corresponds to χ -stable points for the character $\chi : G \rightarrow \mathbb{G}_m$ given by the determinant. The proof of Lemma 4.3 shows that $\mu^{-1}(0)^s \rightarrow \mu^{-1}(0)^s/G$ is a principal G -bundle in the étale topology (In fact, in positive characteristic we also need to check scheme-theoretical stabilisers. Then the assertion follows from a version of Luna’s slice theorem. We leave the details to the reader.)

Let $\overline{\mathcal{M}}(\mathbb{P}^3; r, c) : \text{Sch}/k \rightarrow \text{Sets}$ be the functor which to a scheme S assigns the set of isomorphism classes of S -families of stable framed perverse (r, c) -instantons. Theorem 3.13 and the above remarks imply that this functor is representable:

THEOREM 5.2. *The quotient $\mathcal{M}(\mathbb{P}^3; r, c) := \tilde{\mu}^{-1}(0) //_\chi G$ is a fine moduli scheme for the functor $\overline{\mathcal{M}}(\mathbb{P}^3; r, c)$. In particular, there is a bijection between G -orbits of stable ADHM data and isomorphism classes of stable framed perverse instantons.*

Since every FJ-stable ADHM datum is stable we get as a corollary the following theorem generalising the main theorem of [9]:

THEOREM 5.3. *Let $\tilde{\mu}^{-1}(0)^{FJ}$ be the set of FJ-stable ADHM data. Then the GIT quotient $\mathcal{M}^J(\mathbb{P}^3; r, c) := \tilde{\mu}^{-1}(0)^{FJ} //_\chi G$ represents the moduli functor of rank r instantons*

on \mathbb{P}^3 with $c_2 = c$, framed along a line l_∞ . In particular, there is a bijection between G -orbits of FJ-stable ADHM data and isomorphism classes of framed (r, c) -instantons. Moreover, orbits of FJ-regular ADHM data are in bijection with isomorphism classes of locally free instantons.

Let $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^3; r, c)$ be the moduli space of regular framed perverse (r, c) -instantons. By Theorem 5.2 $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^3; r, c)$ is isomorphic to the quotient of regular ADHM data by the group G . The space $\mathcal{M}(\mathbb{P}^3; r, c)$ contains the moduli space $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^3; r, c)$ as an open subset and it can be considered as its partial Gieseker compactification. Note also that FJ-semiregular ADHM data are regular, so $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^3; r, c)$ contains the moduli space of framed reflexive (r, c) -instantons as an open subset.

Let $\mathcal{M}_0(\mathbb{P}^3; r, c)$ denote the quotient $\tilde{\mu}^{-1}(0)/G$. This is an affine scheme and it contains the moduli space $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^3; r, c)$ as an open subset. It can be considered as its partial Donaldson–Uhlenbeck compactification.

PROPOSITION 5.4. *For every stable rank $r > 0$ perverse instanton \mathcal{C} on \mathbb{P}^3 there exists a regular rank r perverse instanton \mathcal{C}' and a rank 0 perverse instanton \mathcal{C}'' such that we have a distinguished triangle*

$$\mathcal{C}'' \rightarrow \mathcal{C} \rightarrow \mathcal{C}' \rightarrow \mathcal{C}''[1].$$

Proof. Fix a stable ADHM datum $x = (\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j}) \in \tilde{\mu}^{-1}(0)^s$ corresponding to a perverse instanton \mathcal{C} (see Theorem 3.9). Then by [28, Theorem 3.6] there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow \text{GL}(V)$ such that $x^0 = \lim_{t \rightarrow 0} (\lambda(t) \cdot x)$ exists and it is contained in the unique closed orbit in $\overline{\text{GL}(V)} \cdot x$. Let us set $x^0 = (\tilde{B}_1^0, \tilde{B}_2^0, \tilde{i}^0, \tilde{j}^0)$ and fix a splitting $x^0 = x_1^0 \oplus x_2^0$, $V = V_1 \oplus V_2$ as in Lemma 4.5.

As before we can consider the weight decomposition

$$V = \bigoplus_{m \in \mathbb{Z}} V(m), \text{ where } V(m) = \{v \in V \mid \lambda(t) \cdot v = t^m v\}.$$

Since x is stable we have $V = \bigoplus_{m \geq 0} V(m)$. Let i^0 be the composition of i and the natural projection $p_1 : V_1 \oplus V_2 \rightarrow V_1$. We claim that

$$V_1 = V(0), \quad x_1^0 = (\tilde{B}_{1|V_1}^0, \tilde{B}_{2|V_1}^0, \tilde{i}^0, \tilde{j}^0),$$

$$V_2 = \bigoplus_{m \geq 1} V(m), \quad x_2^0 = (\tilde{B}_{1|V_2}^0, \tilde{B}_{2|V_2}^0, 0, 0).$$

By Remark 4.6 it is enough to show that $V(0)$ is the smallest destabilising subspace for x^0 and $\bigoplus_{m \geq 1} V(m)$ is the biggest subspace “decostabilising” x^0 . It is easy to see that $V(0)$ indeed destabilises x^0 . If there was a proper subspace $S \subset V(0)$ with the same property then $S \oplus \bigoplus_{m \geq 1} V(m)$ would destabilise x . A similar argument applies to $\bigoplus_{m \geq 1} V(m)$.

Varagnolo and Vasserot in [37, proof of Theorem 1] claimed that $x_2^0 = (\tilde{B}_{1|V_2}, \tilde{B}_{2|V_2}, 0, 0)$. In our case this equality does not hold. Let us set $x_2 = (\tilde{B}_{1|V_2}, \tilde{B}_{2|V_2}, 0, 0)$. Since $\tilde{j}_{V_2} = 0$ one can easily see that x_2 satisfies the ADHM equation and $x_2 \in \tilde{\mu}_{0, V_2}^{-1}(0)$.

Although x and $x_1^0 \oplus x_2$ in general are not equal, we still have the following exact triple of complexes:

$$0 \rightarrow \mathcal{C}_{x_2}^\bullet \rightarrow \mathcal{C}_x^\bullet \rightarrow \mathcal{C}_{x_1^0}^\bullet \rightarrow 0. \tag{9}$$

This triple gives rise to the required distinguished triangle. □

As a corollary to the above proposition we can describe the morphism from Gieseker to Donaldson–Uhlenbeck partial compactifications of $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^3; r, c)$. Namely, we have a natural set-theoretical decomposition

$$\mathcal{M}_0(\mathbb{P}^3; r, c) = \bigsqcup_{0 \leq d \leq c} \mathcal{M}_0^{\text{reg}}(\mathbb{P}^3; r, c - d) \times \mathcal{M}_0(\mathbb{P}^3; 0, d).$$

Then the natural morphism

$$\mathcal{M}(\mathbb{P}^3; r, c) \simeq \tilde{\mu}^{-1}(0) //_{\chi} G \rightarrow \tilde{\mu}^{-1}(0) / G \simeq \mathcal{M}_0(\mathbb{P}^3; r, c)$$

coming from the GIT (see Subsection 2.1) can be identified with the map

$$(\mathcal{C}, \Phi) \rightarrow ((\mathcal{C}', \Phi'), \mathcal{C}''),$$

where \mathcal{C}' and \mathcal{C}'' are as in Proposition 5.4 and Φ' is induced on \mathcal{C}' via Φ . This morphism is analogous to the one described in Subsection 2.2.

PROPOSITION 5.5. *For every framed rank $r > 0$ instanton E on \mathbb{P}^3 there exists a unique regular rank r instanton E' containing E . Moreover, the inclusion map $E \rightarrow E'$ is uniquely determined and we have a short exact sequence*

$$0 \rightarrow E \rightarrow E' \rightarrow E'' \rightarrow 0,$$

where E'' is a rank 0 instanton (see Definition 6.1).

Proof. Let us consider the short exact sequence from the proof of previous proposition. Since $\mathcal{C}_{x_2}^\bullet$ is a rank 0 perverse instanton, we have $\mathcal{H}^0(\mathcal{C}_{x_2}^\bullet) = 0$. Thus we obtain the following long exact sequence of cohomology groups

$$0 \rightarrow \mathcal{H}^0(\mathcal{C}_x^\bullet) \rightarrow \mathcal{H}^0(\mathcal{C}_{x_1^0}^\bullet) \rightarrow \mathcal{H}^1(\mathcal{C}_{x_2}^\bullet) \rightarrow \mathcal{H}^1(\mathcal{C}_x^\bullet) \rightarrow \mathcal{H}^1(\mathcal{C}_{x_1^0}^\bullet) \rightarrow 0.$$

By Lemma 2.10 x is FJ-stable if and only if $\mathcal{H}^1(\mathcal{C}_x^\bullet) = 0$. In particular, if x is FJ-stable then $\mathcal{H}^1(\mathcal{C}_x^\bullet) = 0$. This implies that $\mathcal{H}^1(\mathcal{C}_{x_1^0}^\bullet) = 0$ and hence x_1^0 is also FJ-stable. Therefore we can set $E' = \mathcal{H}^0(\mathcal{C}_{x_1^0}^\bullet)$ and $E'' = \mathcal{H}^1(\mathcal{C}_{x_2}^\bullet)$. Let us set $c' = \dim V_1$. Our choice of x_1^0 and x_2 shows that E' is a torsion-free (r, c') -instanton corresponding to a costable ADHM datum, and E'' is the first cohomology of a perverse $(0, c - c')$ -instanton (let us recall that such instantons have no other non-trivial cohomology). □

The above proposition allows us to describe the morphism from the moduli space $\mathcal{M}^f(\mathbb{P}^3; r, c)$ of framed instantons to the Donaldson–Uhlenbeck partial compactification of $\mathcal{M}_0^{\text{reg}}(\mathbb{P}^3; r, c)$.

6. Perverse instantons of rank 0. In this section we describe the moduli space $\mathcal{M}_0(\mathbb{P}^3; 0, c)$ of perverse instantons of rank 0. Let us recall that this “moduli space” does not corepresent any functor and in particular, as for Chow varieties, we do not have any deformation theory. But we can still show that closed points of this moduli space can be interpreted as certain one-dimensional sheaves on \mathbb{P}^3 . Then we relate the moduli space to modules over a certain non-commutative algebra and we show that already $\mathcal{M}_0(\mathbb{P}^3; 0, 2)$ is reducible.

6.1. Rank 0 instantons.

DEFINITION 6.1. A rank 0 instanton E on \mathbb{P}^3 is a pure sheaf of dimension 1 such that $H^0(\mathbb{P}^3, E(-2)) = 0$ and $H^1(\mathbb{P}^3, E(-2)) = 0$.

The above definition is motivated by the following lemma:

LEMMA 6.2. If \mathcal{C} is a rank 0 perverse instanton then $\mathcal{C}[1]$ is a sheaf object whose underlying sheaf is a rank 0 instanton. On the other hand, if E is a rank 0 instanton then the object $E[-1]$ in $D^b(\mathbb{P}^3)$ is a rank 0 perverse instanton.

Proof. If \mathcal{C} is a rank 0 perverse instanton then only $E = \mathcal{H}^1(\mathcal{C})$ is non-zero and hence $\mathcal{C}[1]$ is a sheaf object. Clearly, it has dimension ≤ 1 , since there exists a line l such that the support of E does not intersect l . Since

$$H^p(\mathbb{P}^3, \mathcal{C} \otimes \mathcal{O}_{\mathbb{P}^3}(q)) = H^{p-1}(\mathbb{P}^3, E(q))$$

we see the required vanishing of cohomology. To prove that E is pure of dimension 1 note that the torsion in E would give a section of $H^0(\mathbb{P}^3, E(-2))$. This proves the first part of the lemma.

Now assume that E is a rank 0 instanton and set $\mathcal{C} = E[-1]$. Conditions 2 and 3 from Definition 3.8 are trivially satisfied for \mathcal{C} . To check the condition 1 it is sufficient to prove that $H^0(\mathbb{P}^3, E(q)) = 0$ for $q \leq -2$ and $H^1(\mathbb{P}^3, E(-2)) = 0$ for $q \geq -2$. By [15, Lemma 1.1.12] there exists an $E(m)$ -regular section of $\mathcal{O}_{\mathbb{P}^3}(1)$ and it gives rise to the sequence

$$0 \rightarrow E(m-1) \rightarrow E(m) \rightarrow E' \rightarrow 0$$

in which E' is some sheaf of dimension 0. Using such sequences and the definition of rank 0 instanton it is easy to check the required vanishing of cohomology groups. \square

By definition closed points of $\mathcal{M}_0(\mathbb{P}^3; 0, d)$ correspond to closed $\text{GL}(c)$ -orbits of ADHM $(0, c)$ -data for \mathbb{P}^1 . By Theorem 3.9 and the above lemma there exists a bijection between isomorphism classes of rank 0 instantons E whose scheme-theoretical support is a curve of degree c not intersecting l_∞ and $\text{GL}(c)$ -orbits of ADHM $(0, c)$ -data for \mathbb{P}^1 . So $\mathcal{M}_0(\mathbb{P}^3; 0, d)$ can be thought of as the moduli space of some pure sheaves of dimension 1. Note however that this moduli space is only set-theoretical and it is not a coarse moduli space.

In the characteristic zero case $\tilde{\mu}^{-1}(0)/G$ is a subscheme of the quotient \tilde{B}/G , which is a normal variety. Moreover, the coordinate ring for the variety \tilde{B}/G can be described using the First Fundamental Theorem for Matrices (see [22, 2.5, Theorem]). More precisely, if $\text{char} k = 0$ then

$$k[\tilde{B}/G] = k[\tilde{B}]^G = k[\text{Tr}_{i_1 \dots i_m} : 1 \leq i_1, \dots, i_m \leq 4, m \leq c^2],$$

where $\text{Tr}_{i_1 \dots i_m} : \tilde{B} \simeq \text{End}(V)^4 \rightarrow k$ is the *generalised trace* defined by

$$(A_1, A_2, A_3, A_4) \rightarrow \text{Tr}(A_{i_1} A_{i_2} \dots A_{i_m}).$$

This in principle allows us to find $\tilde{\mu}^{-1}(0)/G$ as the image of $\tilde{\mu}^{-1}(0)$ in \tilde{B}/G . In practice, computer assisted computations using this interpretation almost never work due to complexity of the problem.

6.2. Schemes of modules over an associative ring. In this subsection we recall a construction of the moduli space of d -dimensional modules over an associative ring. It is mostly a folklore, but note that our moduli space is not the same as the one constructed by King in [21]. We are interested in the moduli space that was introduced by Procesi in [31] (see also [26] for a more functorial approach) but it is non-interesting from the point of view of finite-dimensional (as k -vector spaces) algebras. In our treatment we restrict to the simplest case although the constructions act in much more general set-up.

Let k be an algebraically closed field and let R be a finitely generated associative k -algebra with unit. Let us fix a positive integer d . Let Mod_R^d denote the *scheme of d -dimensional R -module structures*. By definition it is the affine (algebraic) k -scheme representing the functor from commutative k -algebras (with unit) to the category of sets sending a k -algebra A to

$$\begin{aligned} \text{Mod}_R^d(A) &= \{\text{left } R \otimes_k A\text{-module structures on } A^d\} \\ &= \{A\text{-algebra maps } R \otimes_k A \rightarrow \text{Mat}_{d \times d}(A)\}, \end{aligned}$$

where $\text{Mat}_{d \times d}(A)$ denotes the set of $d \times d$ -matrices with values in A .

Let us choose a surjective homomorphism $\pi : k\langle x_1, \dots, x_n \rangle \rightarrow R$ from the free associative algebra with unit. Then the above functor is naturally equivalent to the functor sending A to the set of n -tuples (M_1, \dots, M_n) of $d \times d$ -matrices with coefficients in A such that $f(M_1, \dots, M_n) = 0$ for all $f \in \ker \pi$. In particular, the k -points of Mod_R^d correspond to R -module structures on k^d (i.e. to d -dimensional R -modules with a choice of a k -basis).

We have a natural $\text{GL}(d)$ -action on Mod_R^d which corresponds to a change of bases (it gives the conjugation action on the set of matrices). By the GIT, there exists a uniform good quotient $Q_R^d = \text{Mod}_R^d / \text{GL}_d$.

Let us recall that if S is a k -scheme then a *family of d -dimensional R -modules parameterised by S* (or simply an *S -family of R -modules*) is a locally free coherent \mathcal{O}_S -module \mathcal{F} together with a k -algebra homomorphism $R \rightarrow \text{End } \mathcal{F}$.

PROPOSITION 6.3. *The quotient Q_R^d corepresents the moduli functor $Q_R^d : \text{Sch}/k \rightarrow \text{Sets}$ given by*

$$S \rightarrow \{\text{Isomorphism classes of } S\text{-families of } R\text{-modules}\}.$$

We call it the moduli space of d -dimensional R -modules.

Proof of this proposition is completely standard and we leave it to the reader (cf. [15, Lemma 4.1.2] and [21, Proposition 5.2])

The quotient Q_R^d parameterises closed GL_d -orbits in Mod_R^d . An orbit of a k -point is closed if and only if it corresponds to a semisimple representation of R . Therefore

the k -points of Q_R^d correspond to isomorphism classes of d -dimensional semisimple R -modules. Equivalently, Q_R^d parameterises S-equivalence classes of d -dimensional R -modules, where two modules are S-equivalent if the graded objects associated to their Jordan–Hölder filtrations are isomorphic.

Note that if R is commutative then Q_R^d is the moduli space of zero-dimensional coherent sheaves of length d on $X = \text{Spec } R$. Usually, the moduli spaces on non-projective varieties do not make sense but in case of zero-dimensional sheaves we can take any completion of X to a projective scheme \bar{X} and consider the open subscheme of the moduli space of zero-dimensional coherent sheaves of length d on \bar{X} , which parameterises sheaves with support contained in X .

We will need the following proposition:

PROPOSITION 6.4. *Let R be a commutative k -algebra and let $X = \text{Spec } R$. Then we have a canonical morphism $f : S^d X \rightarrow Q_R^d$ from the d -th symmetric power of X , which is a bijection on the sets of closed points. If k is a field of characteristic zero then f is an isomorphism.*

Proof. Let us consider the morphism $Q_R^1 \times \dots \times Q_R^1 \rightarrow Q_R^d$ from the d copies of Q_R^1 , given by taking a direct sum. Clearly, $Q_R^1 = \text{Spec } R$ and the morphism factors through $S^d \mathbb{A}^n$ as it is invariant with respect to the natural action of symmetric group exchanging components of the product. The induced morphism $S^d \mathbb{A}^n \rightarrow Q_R^d$ is an isomorphism on the level of closed k -points since a simple module over a commutative algebra is one-dimensional (e.g. by Schur’s lemma).

The second part follows from [15, Example 4.3.6] (note that the proof works also if the characteristic is sufficiently high) and the interpretation of Q_R^d that we gave above. □

REMARK 6.5. We note in the next subsection that the scheme of pairs of commuting $d \times d$ -matrices is irreducible. But already the scheme of triples of commuting $d \times d$ -matrices (i.e. Mod_R^d for $R = k[x_1, x_2, x_3]$) is reducible for $d \geq 30$ (see [12, Proposition 3.1]). Still the above proposition says that its quotient Q_R^d is irreducible if R is commutative and $\text{Spec } R$ is irreducible.

EXAMPLE 6.6. Let us consider $M_{k[x_1, x_2]}^2$. Let us set

$$B_1 = \begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix} \begin{pmatrix} y_3 & y_2(y_3 - y_4) \\ 0 & y_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y_1 & 1 \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} 1 & 0 \\ y_1 & 1 \end{pmatrix} \begin{pmatrix} y_5 & y_2(y_5 - y_6) \\ 0 & y_6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y_1 & 1 \end{pmatrix}.$$

One can easily check that the condition $[B_1, B_2] = 0$ is satisfied. Therefore we can define the map $\psi : \mathbb{A}^6 \rightarrow M_{k[x_1, x_2]}^2$ by sending (y_1, \dots, y_6) to (B_1, B_2) . By the previous remark $M_{k[x_1, x_2]}^2$ is irreducible and one can check that the above defined map is dominant and generically finite.

Let us recall that we have the map $\eta : \mathbb{A}^4 = \mathbb{A}^2 \times \mathbb{A}^2 \rightarrow S^2 \mathbb{A}^2 \rightarrow Q_{k[x_1, x_2]}^2$. One can easily see that the image of a point $(y_1, \dots, y_6) \in \mathbb{A}^6$ in $Q_{k[x_1, x_2]}^2$ coincides with the

image under η of the quadruple $(y_3, y_4, y_5, y_6) \in \mathbb{A}^4$ consisting of pairs of eigenvalues of matrices $(B_1, B_2) = \psi(y_1, \dots, y_6)$.

6.3. Moduli interpretation for instantons of rank 0. Let us first consider the ADHM data for a point for $r = 0$ and some positive $c > 0$. The moment map

$$\mu : \mathbf{B} = \text{End}(V) \oplus \text{End}(V) \rightarrow \text{End}(V)$$

is in this case given by $(B_1, B_2) \rightarrow [B_1, B_2]$, where as usual V is a k -vector space of dimension c . In this case $\mu^{-1}(0)$ is known as the *variety of commuting matrices*. It is known to be irreducible by classical results of Gerstenhaber [10] and Motzkin and Taussky [27]. This implies that the quotient $\mu^{-1}(0)/\text{GL}(V)$ is also irreducible. In fact, one can see from the definition that $\mu^{-1}(0)/\text{GL}(V)$ is isomorphic to the scheme $Q_{k[x_1, x_2]}^c$ of equivalence classes of c -dimensional $k[x_1, x_2]$ -modules. Therefore by Proposition 6.4 the points of $\mu^{-1}(0)/\text{GL}(V)$ are in bijection with the points of c -th symmetric power $S^c(\mathbb{A}^2)$ of \mathbb{A}^2 (this should be compared with [28, Proposition 2.10] which gives a different bijection). In characteristic zero we get that $\mu^{-1}(0)/\text{GL}(V)$ is isomorphic to $S^c\mathbb{A}^2$.

Now let us consider ADHM data for \mathbb{P}^1 . Again it follows from the definitions that the quotient $\mathcal{M}_0(\mathbb{P}^3; 0, c) = \tilde{\mu}^{-1}(0)/\text{GL}(V)$ is isomorphic to the scheme Q_R^c of equivalence classes of c -dimensional R -modules for a non-commutative k -algebra

$$R = k\langle y_1, y_2, z_1, z_2 \rangle / (y_1y_2 - y_2y_1, z_1z_2 - z_2z_1, y_1z_2 - z_2y_1 + y_2z_1 - z_1y_2).$$

Let us define a two-sided ideal in R by

$$I = (y_1z_2 - z_2y_1, y_1z_1 - z_1y_1, y_2z_2 - z_2y_2).$$

It is easy to see that $R/I \simeq k[y_1, y_2, z_1, z_2]$ so we have a surjection $R \rightarrow R' = k[y_1, y_2, z_1, z_2]$. This induces a closed embedding of affine schemes

$$\text{Mod}_{R'}^c \subset \text{Mod}_R^c$$

(see [26, Proposition 1.2]). Therefore we get a morphism

$$Q_{R'}^c \rightarrow Q_R^c,$$

which is a set-theoretical injection of quotients. If k has characteristic zero then this morphism is a closed embedding.

By Proposition 6.4 we get the following induced affine map

$$\varphi : S^c\mathbb{A}^4 \rightarrow Q_R^c \simeq \mathcal{M}_0(\mathbb{P}^3; 0, c),$$

which is a set-theoretical injection.

Geometric interpretation of the map φ is the following. Note that \mathbb{A}^4 parameterises the lines in \mathbb{P}^3 that do not intersect l_∞ . Then for a point in $S^c\mathbb{A}^4$, the image corresponds to the rank 0 instanton $E = \mathcal{O}_{l_1}(1) \oplus \dots \oplus \mathcal{O}_{l_c}(1)$, where l_1, \dots, l_c are the lines not intersecting l_∞ . If all these lines are disjoint then the corresponding rank 0 instanton E gives a point in the Hilbert scheme of curves of degree c and one can check that the corresponding component has dimension $4c$. This suggest the following proposition:

PROPOSITION 6.7. *The image of φ is an irreducible component of $\mathcal{M}_0(\mathbb{P}^3; 0, c)$ of dimension $4c$.*

Proof. Let l_1, \dots, l_c be disjoint lines not intersecting l_∞ . Let $E = \mathcal{O}_{l_1}(1) \oplus \dots \oplus \mathcal{O}_{l_c}(1)$ be the corresponding rank 0 instanton E and let $x \in \tilde{\mu}^{-1}(0) = \text{Mod}_R^c$ be an ADHM datum corresponding to E . Let X denote the R -module corresponding to x .

By Theorem 9.1 there exists a surjective map $T_x \text{Mod}_R \rightarrow \text{Ext}_{\mathbb{P}^3}^1(E, E)$ whose kernel is the tangent space of the orbit $\mathcal{O}(x)$ of x at x . The support of E does not intersect l_∞ so we do not need to tensor by J_{l_∞} . Note that the theorem (and its proof) still works in our case but in the formulation given above: $d\varphi_e$ is not injective. Let us also note that this fact, together with Voigt’s theorem, shows that $\text{Ext}_R^1(X, X) \simeq \text{Ext}_{\mathbb{P}^3}^1(E, E)$.

LEMMA 6.8. *Let l be any line in \mathbb{P}^3 . Then $\dim \text{Ext}_{\mathbb{P}^3}^1(\mathcal{O}_l, \mathcal{O}_l) = 4$.*

Proof. Using the short exact sequence

$$0 \rightarrow J_l \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_l \rightarrow 0$$

we see that $\text{Ext}_{\mathbb{P}^3}^1(\mathcal{O}_l, \mathcal{O}_l) \simeq \text{Hom}(J_l, \mathcal{O}_l)$. To compute this last group we can assume that l is given by equations $x_2 = x_3 = 0$. Then we have a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{(x_2, x_3)} \mathcal{O}_{\mathbb{P}^3}(-1)^2 \rightarrow J_l \rightarrow 0$$

which gives an exact sequence

$$0 \rightarrow \text{Hom}(J_l, \mathcal{O}_l) \rightarrow \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1)^2, \mathcal{O}_l) \xrightarrow{f} \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(-2), \mathcal{O}_l).$$

Since f is the zero map, we see that $\text{Hom}(J_l, \mathcal{O}_l) \simeq \text{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1)^2, \mathcal{O}_l) \simeq H^0(\mathcal{O}_l(1))^{\oplus 2}$ is four-dimensional. □

The above lemma implies that $\text{Ext}_{\mathbb{P}^3}^1(E, E)$ is $4c$ -dimensional. Since $\text{Hom}_R(X, X)$ is c -dimensional, the orbit $\mathcal{O}(X)$ is of dimension $c^2 - c$. But then the dimension of Mod_R^c at X is at most $c^2 - c + 4c = c^2 + 3c$. Since the pre-image of the closed subscheme $\varphi(S^c \mathbb{A}^4)$ in Mod_R^c is of dimension at least $4c + (c^2 - c)$ (as all the fibers of the restricted map contain closed orbits of dimension at least $c^2 - c$), we see that $\varphi(S^c \mathbb{A}^4)$ is an irreducible component of $\mathcal{M}_0(\mathbb{P}^3; 0, c)$. □

REMARK 6.9. One can easily see that $\dim \text{Ext}_{\mathbb{P}^3}^2(\mathcal{O}_l, \mathcal{O}_l) = 3$. Therefore the instanton E from the above proof has $\dim \text{Ext}_{\mathbb{P}^3}^2(E, E) = 3c$ so it is potentially obstructed (cf. Theorem 9.1). On the other hand, the above proof shows that the corresponding point in $\mathcal{M}_0(\mathbb{P}^3; 0, c)$ is smooth.

The following example shows that $\mathcal{M}_0(\mathbb{P}^3; 0, c)$ need not be irreducible (unlike in the case of ADHM data for a point). But it is still possible that it is a connected locally complete intersection of dimension $4c$.

EXAMPLE 6.10. Let us consider ADHM data on \mathbb{P}^1 for $r = 0$ and $c = 2$ in the characteristic zero case. In this case one can compute that $\tilde{\mu}^{-1}(0)$ has two irreducible and reduced components: X_1 of dimension 11 and X_2 of dimension 10 intersecting along an irreducible and reduced scheme of dimension 9 (to see this fact we first performed a computer assisted computation in Singular). We can explicitly describe these two components as follows.

Let V_1 and V_2 denote varieties of pairs of commuting 2×2 matrices (see Example 6.6). Let us note that $\tilde{\mu}^{-1}(0)$ is a subvariety in $V_1 \times V_2$ given by equation $[B_{11}, B_{22}] + [B_{12}, B_{21}] = 0$, where $(B_{11}, B_{21}) \in V_1$ and $(B_{12}, B_{22}) \in V_2$ are pairs of 2×2 matrices.

Let us set

$$B_{1k} = \begin{pmatrix} 1 & 0 \\ y_{1k} & 1 \end{pmatrix} \begin{pmatrix} y_{3k} & y_{2k}(y_{3k} - y_{4k}) \\ 0 & y_{4k} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y_{1k} & 1 \end{pmatrix},$$

and

$$B_{2k} = \begin{pmatrix} 1 & 0 \\ y_{1k} & 1 \end{pmatrix} \begin{pmatrix} y_{5k} & y_{2k}(y_{5k} - y_{6k}) \\ 0 & y_{6k} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y_{1k} & 1 \end{pmatrix}.$$

As in Example 6.6 the condition $[B_{1k}, B_{2k}] = 0$ is satisfied for both $k = 1$ and $k = 2$.

Thus we can define a map ψ from \mathbb{A}^{12} to the product $V_1 \times V_2$ by sending (y_{ij}) to $(B_{11}, B_{21}, B_{12}, B_{22})$ defined above. Computations in Singular show that $\psi^{-1}(\tilde{\mu}^{-1}(0))$ has three irreducible components Y_1, Y_2, Y_3 given by the following ideals:

$$I_1 = ((y_{31} - y_{41})(y_{52} - y_{62}) - (y_{51} - y_{61})(y_{32} - y_{42})),$$

$$I_2 = (y_{11} - y_{12}, y_{21} - y_{22})$$

and

$$I_3 = (y_{21}y_{12} - y_{21}y_{11} + 1, y_{21} + y_{22}).$$

Further computations show that $\psi(Y_1)$ is eleven-dimensional and $\psi(Y_2)$ and $\psi(Y_3)$ are equal and ten-dimensional. This shows that the restriction $\psi|_{Y_1}$ is a generically finite morphism from Y_1 to the eleven-dimensional component X_1 of $\mu^{-1}(0)$. Similarly, $\psi|_{Y_2}$ and $\psi|_{Y_3}$ are generically finite morphisms from Y_2 and Y_3 to the ten-dimensional component X_2 .

We have dominant morphisms $Y_1 \rightarrow X_1/\text{GL}(2)$ and $Y_2 \rightarrow X_2/\text{GL}(2)$. For a quadruple of matrices $(B_{11}, B_{21}, B_{12}, B_{22}) \in X_2$ obtained as the image of a point $(y_{ij}) \in Y_2$, the isotropy group of $\text{GL}(2)$ contains matrices of the form

$$\begin{pmatrix} 1 & 0 \\ y_{11} & 1 \end{pmatrix} \begin{pmatrix} t_1 & y_{21}(t_1 - t_2) \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y_{11} & 1 \end{pmatrix}$$

for arbitrary $t_1, t_2 \in \mathbb{G}_m$. One can see that Y_2 is mapped dominantly onto the image of $S^2\mathbb{A}^4$ in $\tilde{\mu}^{-1}(0)/\text{GL}(2)$ and therefore for a generic quadruple $(B_{11}, B_{21}, B_{12}, B_{22}) \in X_2$ the isotropy group is two-dimensional and it is equal to the above described group (one can also compute this isotropy group explicitly for all such quadruples).

One can also check that the isotropy group of a generic point in X_1 is one-dimensional (so the corresponding R -module is simple) and therefore $\tilde{\mu}^{-1}(0)/\text{GL}(2)$ is pure of dimension 8 with irreducible components given by $X_1/\text{GL}(2)$ and $X_2/\text{GL}(2)$.

This also proves that the injection $\varphi : S^2\mathbb{A}^4 \rightarrow \tilde{\mu}^{-1}(0)/\text{GL}(2)$ maps $S^2\mathbb{A}^4$ onto an irreducible component of the quotient $\tilde{\mu}^{-1}(0)/\text{GL}(2)$.

Now we need to check that the components $X_1/\text{GL}(2)$ and $X_2/\text{GL}(2)$ do not coincide. For this we need the following lemma:

LEMMA 6.11. *Let G be a linear algebraic group acting on a reducible variety X with two irreducible components X_1 and X_2 . Then $X_1/G \cap X_2/G = (X_1 \cap X_2)/G$.*

Proof. First, observe that since G is irreducible, the closure of an orbit of a point $x \in X$ is contained in the same irreducible component as x . The intersection $X_1 \cap X_2$ is closed and G -invariant so $(X_1 \cap X_2)/G$ can be regarded as subvariety in $X_1/G \cap X_2/G$.

Let us take $y \in X_1/G \cap X_2/G \subset X/G$. By assumption it is the image of a closed orbit of a point $x \in X$. We claim that $x \in X_1 \cap X_2$. Note that y is the image of some points $x_1 \in X_1, x_2 \in X_2$. Since each $\overline{Gx_i} \subset X_i$ for $i = 1, 2$ contains a unique closed orbit, it must be the orbit of x and it is contained in $X_1 \cap X_2$. Therefore y lies in $(X_1 \cap X_2)/G$. □

The image of a quadruple $(B_{11}, B_{21}, B_{12}, B_{22}) \in X_2$ in the image of $S^2\mathbb{A}^4$ in $\tilde{\mu}^{-1}(0)/GL(2)$ is given by quadruples of pairs of eigenvalues of matrices B_{ij} . But for a quadruple of 2×2 matrices $(B_{11}, B_{21}, B_{12}, B_{22}) \in X_1 \cap X_2$ obtained as the image of $(y_{ij}) \in Y_1 \cap Y_2$ we have the equation

$$(y_{31} - y_{41})(y_{52} - y_{62}) = (y_{51} - y_{61})(y_{32} - y_{42})$$

for the eigenvalues. Therefore the image of $Y_1 \cap Y_2$ in $\tilde{\mu}^{-1}(0)/GL(2)$ has dimension 7. Together with the above lemma this proves the following corollary:

COROLLARY 6.12. *$\mathcal{M}_0(\mathbb{P}^3; 0, 2)$ has two eight-dimensional irreducible components intersecting along a seven-dimensional variety.*

7. Examples and counter-examples. In this section we consider generalised ADHM data in the case $X = \mathbb{P}^1$. We provide a few examples showing, e.g. a relation between our notion of stability and that of Frenkel and Jardim. We also show a few counter-examples to some expectations of Frenkel and Jardim.

In this section we keep notation from Section 2.4.

7.1. Relation between GIT semistability and FJ-semistability. The following lemma follows immediately from definitions:

LEMMA 7.1. *Let us fix an ADHM datum $x \in \tilde{\mathbf{B}} = \mathbf{B} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))$. If x is FJ-semistable then it is also stable.*

Jardim in [17, Proposition 4] claims that the opposite implication also holds but the following example shows that this assertion is false.

EXAMPLE 7.2. We consider ADHM data in case $r = 1$ and $c = 2$. Let us fix coordinate systems in V and W and consider an element $x = (\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j}) \in \tilde{\mathbf{B}}$ given by

$$\tilde{B}_1 = \begin{bmatrix} x_0 & x_0 \\ x_1 & x_1 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} x_0 & -x_0 \\ x_1 & -x_1 \end{bmatrix}, \quad \tilde{i} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \quad \text{and} \quad \tilde{j} = \begin{bmatrix} -2x_1 & 2x_0 \end{bmatrix}.$$

It is easy to see that $\tilde{\mu}(x) = 0$. Hence x is an ADHM datum.

We claim that this ADHM datum is stable and costable. To prove that consider a vector subspace $S \subset V$ such that $\text{im } \tilde{i} \subset S \otimes H^0(\mathcal{O}_X(1))$. We claim that S must be two-dimensional. Otherwise, there exist constants $a, b \in k$ such that every element in

$S \otimes H^0(\mathcal{O}_X(1))$ can be written as $\begin{bmatrix} af(x_0, x_1) \\ bf(x_0, x_1) \end{bmatrix}$ for some linear polynomial f in x_0 and x_1 . But $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \in \text{im } \tilde{\gamma}$ cannot be written in this way. Therefore $S = V$, which proves that the ADHM datum x is stable. Since $\ker \tilde{j} = 0$, the ADHM datum x is also costable.

Now fix a point $p = [a : b] \in \mathbb{P}^1$ and consider the subspace $S \subset V = k^2$ spanned by vector $s = \begin{bmatrix} a \\ b \end{bmatrix}$. Then

$$(\tilde{B}_1(p))(s) = (a + b) \cdot s, \quad (\tilde{B}_2(p))(s) = (a - b) \cdot s, \quad (\tilde{\gamma}(p))(1) = s \quad \text{and} \quad (\tilde{j}(p))(s) = 0.$$

Therefore $(x(p))(S) \subset S$ and x restricted to any point of \mathbb{P}^1 is neither stable nor costable. In particular, the ADHM datum x is regular but not FJ-semistable.

Let us focus on the example given above and study cohomology groups of the complex C_x^\bullet corresponding to the ADHM datum x .

First, let us describe the locus of points $p = [x_0, x_1, x_2, x_3] \in \mathbb{P}^3$ where the map $\alpha(p)$ is not injective. It is equivalent to describing the locus

$$\text{rk} \begin{pmatrix} x_0 + x_2 & x_0 \\ x_1 & x_1 + x_2 \\ x_0 + x_3 & -x_0 \\ x_1 & -x_1 + x_3 \\ -2x_1 & 2x_0 \end{pmatrix} \leq 1.$$

Easy computations show that this set is an intersection of two planes:

$$\begin{cases} x_0 + x_1 + x_2 = 0 \\ x_0 - x_1 + x_3 = 0 \end{cases}.$$

Similarly, the locus of points $p \in \mathbb{P}^3$ where

$$\beta(p) = \begin{pmatrix} -x_0 - x_3 & x_0 & x_0 + x_2 & x_0 & x_0 \\ -x_1 & x_1 - x_3 & x_1 & x_1 + x_2 & x_1 \end{pmatrix}$$

is not surjective is the line given by equations $x_2 = x_3 = 0$.

Using this one can see that $\mathcal{H}^1(C_x^\bullet)$ is a pure sheaf of dimension 1 and $\mathcal{H}^0(C_x^\bullet)$ is a torsion-free sheaf whose reflexivisation is locally free.

7.2. Relation to Diaconescu’s approach to ADHM data. We will use notation from [7, Section 2] (see also [34, 2.9.2]). Let us set $\mathcal{X} = (X, M_1 = \mathcal{O}_X(-1), M_2 = \mathcal{O}_X(-1), E_\infty = W \otimes \mathcal{O}_X(-1))$ and consider an ADHM sheaf $\mathcal{E} = (E = V \otimes \mathcal{O}_X, \Phi_1, \Phi_2, \varphi, \psi)$ for this data.

DEFINITION 7.3. We say that \mathcal{E} is *stable* if for every subspace $S \subsetneq V$ (possibly $S = 0$) such that $\Phi_k(S \otimes \mathcal{O}_X(-1)) \subset S \otimes \mathcal{O}_X$ for $k = 1, 2$ we have $\text{im } \Psi \not\subset S \otimes \mathcal{O}_X$.

The above stability notion is similar to Diaconescu’s stability [7, Definition 2.2] but with stability condition only for subsheaves E' of the form $S \otimes \mathcal{O}_X$ for some $0 \subsetneq S \subsetneq V$.

Let $\tilde{B}_k : V \rightarrow V \otimes H^0(\mathcal{O}_X(1))$ be induced by Φ_k and let $\tilde{\gamma} : W \rightarrow V \otimes H^0(\mathcal{O}_X(1))$ and $\tilde{j} : V \rightarrow W \otimes H^0(\mathcal{O}_X(1))$ be induced by ψ and φ , respectively.

Then giving \mathcal{E} is equivalent to giving a point $x = (\tilde{B}_1, \tilde{B}_2, \tilde{\iota}, \tilde{j}) \in \tilde{\mathbf{B}}$ such that $\tilde{\mu}(x) = 0$. Moreover, \mathcal{E} is stable in the above sense if and only if x is stable.

7.3. Counter-example to the Frenkel–Jardim conjecture. In [9] Frenkel and Jardim conjectured that the moduli space $\mathcal{M}^f(\mathbb{P}^3; r, c)$ of framed instantons is smooth and irreducible. Here we show that this conjecture is false.

Let us consider the map $\varphi : \mathcal{O}_{\mathbb{P}^3}^4 \rightarrow \mathcal{O}_{\mathbb{P}^3}^3(1)$ given by

$$\varphi = \begin{pmatrix} x_2 & x_3 & 0 & 0 \\ 0 & x_2 & x_3 & 0 \\ 0 & 0 & x_2 & x_3 \end{pmatrix}.$$

Now let us consider the sheaf E defined by the short exact sequence

$$0 \rightarrow E \rightarrow \mathcal{O}_{\mathbb{P}^3}^4 \xrightarrow{\tilde{\varphi}} \mathcal{O}_m(1)^3 \rightarrow 0,$$

where m is the line $x_0 = x_1 = 0$ and $\tilde{\varphi}$ is the composition the natural restriction map $\mathcal{O}_{\mathbb{P}^3}(1)^3 \rightarrow \mathcal{O}_m^3(1)$ with φ . It is easy to see that E is a $(4, 3)$ -instanton, trivial on the line $l_\infty := (x_2 = x_3 = 0)$. From the defining sequence we have an exact sequence

$$\text{Ext}^2(\mathcal{O}_{\mathbb{P}^3}^4, E) \rightarrow \text{Ext}^2(E, E) \rightarrow \text{Ext}^3(\mathcal{O}_m(1)^3, E) \rightarrow \text{Ext}^3(\mathcal{O}_{\mathbb{P}^3}^4, E).$$

Then $\text{Ext}^l(\mathcal{O}_{\mathbb{P}^3}^4, E) = H^l(E)^4 = 0$ for $l = 2, 3$ and $\text{Ext}^3(\mathcal{O}_m(1)^3, E)$ is Serre dual to $\text{Hom}(E, \mathcal{O}_m(-3)^3)$. But after restricting to m we have

$$E|_m \twoheadrightarrow \mathcal{O}_m(-3) \rightarrow \mathcal{O}_m^4 \rightarrow \mathcal{O}_m(1)^3 \rightarrow 0,$$

and it is easy to see that $\text{Hom}(E, \mathcal{O}_m(-3))$ is one-dimensional. In particular, $\dim \text{Ext}^2(E, E) = 3$.

On the other hand, by Lemma 2.8 there exists a locally free $(4, 3)$ -instanton F such that $\text{Ext}^2(F, F) = 0$. It corresponds to a smooth point of an irreducible component of expected dimension (see Theorem 9.1). Therefore the point corresponding to E in the moduli space of framed instantons is either singular or lives in a component of unexpected dimension (in which case the moduli space would not be irreducible).

Another way of looking at this example is defining an ADHM datum for \mathbb{P}^1 , for $r = 4, c = 3$. We define an ADHM datum $x = (\tilde{B}_1, \tilde{B}_2, \tilde{\iota}, \tilde{j})$ by setting $\tilde{B}_1 = 0, \tilde{B}_2 = 0, \tilde{j} = 0$ and

$$\tilde{\iota} = \begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & x_0 & x_1 & 0 \\ 0 & 0 & x_0 & x_1 \end{pmatrix}.$$

It is easy to see that these matrices satisfy the ADHM equations and define an FJ-stable ADHM datum (in fact, $\tilde{\iota}_p$ is surjective for every $p \in \mathbb{P}^1$). The corresponding torsion-free framed $(4, 3)$ -instanton E can be described by the above sequence. In terms of ADHM data we proved that the moment map μ is not submersion at x (see Theorem 9.1) but there exist ADHM data at which μ is a submersion.

More generally, one can easily see that if $c < r < 3c/2$ then $M(l_\infty; r, c)$ is either singular or reducible. Indeed, one can find an FJ-stable complex ADHM data $x \in \mathbb{B}$ for which only \tilde{i} is non-zero. Then the rank of $d\mu_x$ is at most $2cr < 3c^2$, so $d\mu_x$ is not surjective. On the other hand, by Lemma 2.8 there exists an irreducible component of expected dimension which proves our claim.

7.4. Weak instantons.

DEFINITION 7.4. A *weakly instanton sheaf* (or a *weak instanton*) is a torsion-free sheaf E on \mathbb{P}^3 such that

- $c_1(E) = 0$,
- $H^0(E(-1)) = H^1(E(-2)) = H^3(E(-3)) = 0$.

Weak instantons were introduced by Frenkel and Jardim (see [9, 2.4]) to deal with FJ-semistable data in the rank 1 case.

We say that a torsion-free sheaf on \mathbb{P}^3 has *trivial splitting type* if there exists a line such that the restriction of this sheaf to a line is a trivial sheaf. In this case the restriction to a general line is also a trivial sheaf.

LEMMA 7.5. *Let E be a locally free sheaf on \mathbb{P}^3 of trivial splitting type. Then $H^0(E(-1)) = H^3(E(-3)) = 0$. In particular, if $H^1(E(-2)) = 0$ then E is a weak instanton.*

Proof. If E is of trivial splitting type then both $E(-1)$ and $E^*(-1)$ have no sections. Since $H^3(E(-3))$ is Serre dual to $H^0(E^*(-1))$ this shows the first part. The second one follows from the first one by noting that for a sheaf of trivial splitting type we have $c_1(E) = 0$. □

DEFINITION 7.6. We say that a perverse instanton \mathcal{C} is *mini-perverse* if $\mathcal{H}^0(\mathcal{C})$ is torsion-free and $\mathcal{H}^1(\mathcal{C})$ is a sheaf of finite length.

Obviously, any instanton is mini-perverse, but the opposite implication does not hold.

LEMMA 7.7. *An ADHM datum $x \in \tilde{\mathbf{B}}$ is FJ-semistable if and only if the corresponding perverse instanton \mathcal{C}_x^\bullet is mini-perverse.*

Proof. The “if” implication is a content of [9, Proposition 17]. To prove the converse note that the restriction of a mini-perverse instanton \mathcal{C}_A corresponding to $A \in \tilde{\mathbf{B}}$ to a general hyperplane containing l_∞ gives a locally free sheaf on \mathbb{P}^2 . But this shows that for a general point $x \in \mathbb{P}^1$ the ADHM datum $A(x)$ (corresponding to this restriction) is regular. □

Note that in Example 7.2 the constructed perverse instanton is not mini-perverse. So the above lemma gives another proof that this perverse instanton is not FJ-semistable.

LEMMA 7.8. *If \mathcal{C} is a mini-perverse instanton then $\mathcal{H}^0(\mathcal{C})$ is a weak instanton of trivial splitting type.*

Proof. Let us set $E = \mathcal{H}^0(\mathcal{C})$ and $T = \mathcal{H}^1(\mathcal{C})$. If \mathcal{C} is a perverse instanton then we have the distinguished triangle

$$E \rightarrow \mathcal{C} \rightarrow T[-1] \rightarrow E[1].$$

The long cohomology exact sequence for this triangle gives exactness of the following sequence:

$$0 = H^0(\mathcal{C}(-2)) \rightarrow H^1(T(-2)) \rightarrow H^1(E(-2)) \rightarrow H^1(\mathcal{C}(-2)) = 0.$$

Since T has dimension zero we see that $H^1(E(-2)) = 0$ and so E is a weak instanton. The fact that it is of trivial splitting type follows from the fact that $\mathcal{H}^0(\mathcal{C})$ is trivial on l_∞ . □

LEMMA 7.9. *A zero-dimensional coherent sheaf E on a smooth variety X has homological dimension equal to the dimension of X .*

Proof. By the Auslander–Buchsbaum theorem it is sufficient to prove that $E_x = E \otimes \mathcal{O}_{X,x}$ has depth zero. Assume that it has depth at least 1. Then there exists an element $y \in m_x \subset \mathcal{O}_{X,x}$ such that multiplication by y defines an injective homomorphism $\varphi_y : E_x \rightarrow E_x$. Note that φ_y is an isomorphism since E_x is zero-dimensional and $H^0(\varphi_y)$ is an isomorphism as it is a linear injection of k -vector spaces of the same dimension. But this implies that $m_x E_x = E_x$ which contradicts Nakayama’s lemma. □

LEMMA 7.10. *If a locally free sheaf E appears as $\mathcal{H}^0(\mathcal{C})$ for some mini-perverse instanton \mathcal{C} then $\mathcal{H}^1(\mathcal{C}) = 0$. In particular, E is an instanton.*

Proof. By Lemma 3.15 \mathcal{C} is isomorphic in $D^b(\mathbb{P}^3)$ to the complex

$$(0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^c \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{2c+r} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)^c \rightarrow 0).$$

Set $T = \mathcal{H}^1(\mathcal{C})$. We have a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^c \simeq \text{im } \alpha \rightarrow \ker \beta \rightarrow E \rightarrow 0$$

which, together with our assumption on E , implies that $\ker \beta$ is locally free. On the other hand, we have an exact sequence

$$0 \rightarrow \ker \beta \rightarrow \mathcal{O}_{\mathbb{P}^3}^{2c+r} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)^c \rightarrow T \rightarrow 0,$$

which implies that the homological dimension of T is at most two.

But if $T \neq 0$ then Lemma 7.9 implies that the homological dimension of T is equal to 3, a contradiction. □

EXAMPLE 7.11. In [9, 2.4] Frenkel and Jardim ask if every weak instanton of trivial splitting type come from some FJ-semistable ADHM datum. In view of Lemma 7.7 this would imply that such an instanton is of the form $\mathcal{H}^0(\mathcal{C})$ for some mini-perverse instanton \mathcal{C} . Here we give a negative answer to this question. Note that if the answer were positive then by Lemma 7.10 every locally free weak instanton of trivial splitting type would be an instanton. So it is sufficient to show a weakly instanton sheaf which is locally free of trivial splitting type but which is not an instanton.

We use [5, Example 1.6] to show a rank 3 locally free sheaf E on \mathbb{P}^3 which is trivial on a general line and has vanishing $H^1(E(-2))$ and it does not appear as $\mathcal{H}^0(\mathcal{C})$ for some mini-perverse instanton (there are no such sheaves in the rank 2 case). This gives a negative answer to the question posed in [9, 2.4].

Let $q \geq 1$ and $c_2 \geq 2q$ be integers. Let Z_1 and Z_2 be plane curves of degree $c_2 - q$ and q contained in different planes. Assume that they intersect in $0 \leq s \leq q$ simple

points and set $Z = Z_1 \cup Z_2$. Then there exists a rank 3 vector bundle E which sits in a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}^2 \rightarrow E \rightarrow I_Z \rightarrow 0.$$

Then E is trivial along any line disjoint with Z .

Using the short exact sequence

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_Z \rightarrow 0$$

we see that $H^1(I_Z(-2)) = 0$. Therefore $H^1(E(-2)) = 0$. Obviously, $H^0(E(-1)) = 0$. Since E is locally free, the Serre duality implies that $H^3(E(-3))$ is dual to $H^0(E^*(-1))$. But E is slope semistable and hence $E^*(-1)$ has no sections. Thus E is a weak instanton of trivial splitting type.

On the other hand, $H^2(E(-2))$ has dimension $\chi(E(-2))$ as all the other cohomology of $E(-2)$ vanish. But the Riemann–Roch theorem implies that $\chi(E(-2)) = \frac{1}{2}c_3 = s + q^2 + \frac{1}{2}c_2(c_2 - 2q + 1)$, so E is not an instanton.

7.5. Perverse instantons of charge 1. In [9] the moduli spaces of framed torsion-free instantons with $c = 1$ and $r \geq 2$ were described quite explicitly. Let us recall that such instantons come from FJ-stable ADHM datum. We can generalise this description to the case of stable ADHM datum. For $c = 1$ general ADHM datum consists of complex numbers B_{lk} and i_k, j_k which can be regarded as vectors in W . The ADHM equation reduce to

$$\tilde{i}\tilde{j} = 0. \tag{10}$$

Stability is equivalent to $\tilde{i} \neq 0$ and costability to $\tilde{j} \neq 0$. The group $\text{GL}(V)$ is just \mathbb{G}_m and $t \in \mathbb{G}_m$ acts trivially on \tilde{B}_k , it acts on i_k by multiplication by t and on j_k by multiplication by t^{-1} . The moduli of perverse instantons for fixed $r \geq 1$ and $c = 1$ is isomorphic to $\mathbb{A}^4 \times \mathcal{B}(r)$ where $\mathcal{B}(r)$ is the set of solutions of equation (10) modulo the action of \mathbb{G}_m . Note however, that there exist stable ADHM data also for $r = 1$ whereas there are no FJ-stable ones (see [9, Propositions 4 and 15]).

PROPOSITION 7.12. *For $r \geq 2$ $\mathcal{B}(r)$ is a quasi projective variety of dimension $4(r - 1)$ and $\mathcal{B}(1) \simeq \mathbb{P}^1$.*

Proof. We follow the proof of Proposition 7 in [9]. Let

$$i_1 = (x_1, \dots, x_r), \quad i_2 = (y_1, \dots, y_r),$$

$$j_1 = \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}, \quad j_2 = \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix}.$$

Then equation (10) reduces to

$$\sum_{k=1}^r x_k z_k = \sum_{k=1}^r y_k w_k = \sum_{k=1}^r x_k w_k + y_k z_k = 0. \tag{11}$$

Such an ADHM datum is stable if and only if i_1 or i_2 is not a zero vector. One can also easily show that FJ-stability is equivalent to the vectors i_1 and i_2 being linearly independent. $\mathcal{B}(r)$ is the complete intersection of the three quadrics (11) in the open subset of the $(4r - 1)$ -dimensional weighted projective space

$$X = \mathbb{P}(\underbrace{1, \dots, 1}_{2r}, \underbrace{-1, \dots, -1}_{2r}).$$

This shows that $\mathcal{B}(r)$ is quasi-projective.

REMARK 7.13. A point in the complete intersection of the quadrics (11) in X corresponds to an ADHM datum which is either stable or costable.

Let us consider the map

$$\mu : \mathbb{A}^{4r} \rightarrow \mathbb{A}^3$$

given by

$$\begin{aligned} \mu(x_1, \dots, x_r, y_1, \dots, y_r, z_1, \dots, z_r, w_1, \dots, w_r) \\ = \left(\sum_{k=1}^r x_k z_k, \sum_{k=1}^r y_k w_k, \sum_{k=1}^r x_k w_k + y_k z_k \right). \end{aligned}$$

The derivative of μ is given by

$$D\mu = \begin{pmatrix} z_1 & \dots & z_r & 0 & \dots & 0 & x_1 & \dots & x_r & 0 & \dots & 0 \\ 0 & \dots & 0 & w_1 & \dots & w_r & 0 & \dots & 0 & y_1 & \dots & y_r \\ w_1 & \dots & w_r & z_1 & \dots & z_r & y_1 & \dots & y_r & x_1 & \dots & x_r \end{pmatrix}$$

Frankel and Jardim claimed that for $r \geq 2$ the matrix $D\mu$ has maximal rank 3 if and only if (x_1, \dots, x_r) and (y_1, \dots, y_r) are linearly independent. However, only the implication " \Leftarrow " is true and their result on non-singularity at points corresponding to FJ-stable ADHM data remains correct. It also follows that $\dim \mathcal{B}(r) = 4r - 4$. In characteristic different from 2, setting $x_1 = z_2 = w_2 = 1, y_1 = 2$ and all other coefficients equal 0 gives an example of stable ADHM datum which is not FJ-stable but it corresponds to a non-singular point in the moduli space of perverse instantons. On the other hand, if i_1 and i_2 are linearly dependent and $j_1 = j_2 = 0$ then $D\mu$ has clearly rank 2. This shows a stable ADHM datum which is neither costable nor FJ-stable but it gives a singular point.

In the case $r = 1$, equations (11) reduce to $\tilde{i} = 0$ or $\tilde{j} = 0$. Stability is equivalent to $\tilde{i} \neq 0$ so $D\mu$ has rank 2 for all stable ADHM datum. Clearly, we have $\mathcal{B}(1) \simeq \mathbb{P}^1$. \square

8. A general study of ADHM data for \mathbb{P}^1 . In this section we introduce a hypersymplectic reduction which is a holomorphic analogue of a hyper-Kähler structure. We also relate the moduli space of framed instantons to the moduli space of framed modules of Huybrechts and Lehn. The relation is not as straightforward as in the surface case since many framed instantons are not Gieseker δ -semistable framed modules on \mathbb{P}^3 for all parameters δ . The relation shows existence of the moduli space of framed instantons without Theorem 5.2.

8.1. Hypersymplectic reduction. Let X be a smooth quasi-projective k -variety. As an analogue of a hyper-Kähler structure we introduce the following:

DEFINITION 8.1. We say that X has a *hypersymplectic structure* if there exist a non-degenerate symmetric form g on TX and maps of vector bundles $I, J, K : TX \rightarrow TX$ such that

- (1) $g(Iv, Iw) = g(Jv, Jw) = g(Kv, Kw) = g(v, w)$,
- (2) $I^2 = J^2 = K^2 = IJK = -1$.

If we have a hypersymplectic manifold then we can define non-degenerate symplectic forms $\omega_1, \omega_2, \omega_3$ on X by $\omega_1(v, w) = g(Iv, w)$, $\omega_2(v, w) = g(Jv, w)$ and $\omega_3(v, w) = g(Kv, w)$. Assume that there exists a reductive k -group G acting on X and preserving g, I, J, K . As an analogue of a hyper-Kähler moment map we have the following:

DEFINITION 8.2. A map $\mu = (\mu_1, \mu_2, \mu_3) : X \rightarrow k^3 \otimes \mathfrak{g}^*$ is called a *hypersymplectic moment map* if it satisfies the following properties:

- (1) μ_l is G -equivariant for $l = 1, 2, 3$,
- (2) $\langle d\mu_{l,x}(v), \xi \rangle = \omega_l(\xi_x, v)$ for $l = 1, 2, 3$ and for any $x \in X, v \in T_xX$ and $\xi \in \mathfrak{g}$.

Let V_x be the image of the tangent map (at the unit) to the orbit map $\varphi_x : G \rightarrow X$ sending g to $g \cdot x$.

PROPOSITION 8.3. *Let us take a point $x \in X$. Then the following conditions are equivalent:*

- (1) $d\mu_x$ is surjective.
- (2) $d\varphi_x$ is an injection and $S = IV_x + JV_x + KV_x$ is a direct sum.
- (3) The map $\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} \rightarrow T_xX$ given by $(\xi_1, \xi_2, \xi_3) \rightarrow (I\xi_{1,x}, J\xi_{2,x}, K\xi_{3,x})$ is injective.

Proof. Since

$$\langle d\mu_x(v), \xi \rangle = (g(I\xi_x, v), g(J\xi_x, v), g(K\xi_x, v)),$$

the kernel of $d\mu_x$ is equal to the orthogonal complement S^\perp of S (with respect to g). Since g is non-degenerate we have

$$\dim S + \dim S^\perp = \dim X.$$

Hence $d\mu_x$ is surjective if and only if $\dim S = 3 \dim \mathfrak{g}$. This is clearly equivalent to saying that $d\varphi_x$ is injective (i.e. $\dim V_x = \dim \mathfrak{g}$) and $IV_x + JV_x + KV_x$ is a direct sum. Equivalence with the last condition is clear. □

PROPOSITION 8.4. *Let $\eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{g}^*$ satisfy $\text{Ad}_g^*(\eta_i) = \eta_i$ for all $g \in G$. If $x \in \mu^{-1}(\eta)$ and $g|_{V_x}$ is non-degenerate then $d\mu_x$ is surjective.*

Proof. By assumption μ sends a G -orbit of x into a point. Hence $d\mu_x(V_x) = 0$. This immediately implies that V_x, IV_x, JV_x, KV_x are orthogonal to each other (with respect to g). But then the assertion follows from the above proposition. Indeed, if there exists (ξ_1, ξ_2, ξ_3) such that $I\xi_{1,x} + J\xi_{2,x} + K\xi_{3,x} = 0$ then $g(\xi_{1,x}, \zeta_x) = g(I\xi_{1,x} + J\xi_{2,x} + K\xi_{3,x}, I\zeta_x) = 0$ for any $\zeta \in \mathfrak{g}$. Therefore $\xi_{1,x} = 0$ and similarly $\xi_{2,x} = \xi_{3,x} = 0$. □

Note that it can easily happen that the form g restricted to $V_x + S$ is zero and $d\mu_x(S) = 0$ although $d\mu_x$ is surjective (this happens, e.g. in Example 7.3).

8.2. Hypersymplectic moment map for ADHM data on \mathbb{P}^1 . In this subsection we assume that the characteristic of the base field is zero.

Let us fix a basis x_0, x_1 of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. Then a point $x \in \tilde{\mathbf{B}} = \mathbf{B} \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ can be thought of as a matrix

$$x = \begin{pmatrix} B_{11} & B_{12} & i_1 & j_1 \\ B_{21} & B_{22} & i_2 & j_2 \end{pmatrix},$$

where $(B_{l1}, B_{l2}, i_l, j_l) \in \mathbf{B}$ for $l = 1, 2$ is written as in case of the usual ADHM data. Using this notation we define a symmetric form g on $T\tilde{\mathbf{B}}$ by

$$g(x, x') = \text{Tr}(B_{11}B'_{22} + B_{22}B'_{11} - B_{21}B'_{12} - B_{12}B'_{21} + i_1j'_2 + i'_1j_2 - i_2j'_1 - i'_2j_1).$$

Let us choose a standard quaternion basis:

$$I = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Then I, J, K can be thought of as operators acting on $T\tilde{\mathbf{B}}$. Let us write $\tilde{\mu} : \tilde{\mathbf{B}} \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \text{End } V \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$ as the sum $\mu_1x_0^2 + \mu_2x_0x_1 + \mu_3x_1^2$ in which $\mu_l : \tilde{\mathbf{B}} \rightarrow \text{End } V$ for $l = 1, 2, 3$ are the corresponding components. Let us set $\tilde{\mu}_1(x) = \sqrt{-1}\mu_2(x)$, $\tilde{\mu}_2(x) = \mu_1(x) + \mu_3(x)$ and $\tilde{\mu}_3(x) = \sqrt{-1}(-\mu_1(x) + \mu_3(x))$.

By a straightforward computation we get the following proposition:

PROPOSITION 8.5. *(g, I, J, K) define a hypersymplectic structure on $\tilde{\mathbf{B}}$. Moreover, $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3) : \tilde{\mathbf{B}} \rightarrow k^3 \otimes \mathfrak{g}^*$ is a hypersymplectic moment map.*

This, together with Proposition 8.3, implies the following corollary which can be used for checking smoothness of the moduli space of framed perverse instantons:

COROLLARY 8.6. *Let $x \in \tilde{\mathbf{B}}$ be a stable ADHM datum. Then $d\tilde{\mu}_x$ is surjective if and only if there exist no $(\xi_1, \xi_2, \xi_3) \in \mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g} - \{(0, 0, 0)\}$ such that*

$$\xi_{1,x} + I\xi_{2,x} + J\xi_{3,x} = 0.$$

8.3. Relation to moduli spaces of framed modules. Let X be a smooth n -dimensional projective variety defined over an algebraically closed field k . Let us fix an ample line bundle $\mathcal{O}_X(1)$ and a coherent sheaf F on X . Let us also fix a polynomial $\delta \in \mathbb{Q}[t]$ of degree $\leq (n - 1)$. When writing δ as

$$\delta(m) = \delta_1 \frac{m^{n-1}}{(n-1)!} + \delta_2 \frac{m^{n-2}}{(n-2)!} + \dots + \delta_n,$$

we will assume that the first non-zero coefficient is positive.

Let us recall a few definitions from [14]. A *framed module* is a pair (E, α) , where E is a coherent sheaf and $\alpha : E \rightarrow F$ is a homomorphism. Let us set $\epsilon(\alpha) = 0$ if $\alpha = 0$ and

$\epsilon(\alpha) = 1$ if $\alpha \neq 0$. Then we define the *Hilbert polynomial* of (E, α) as $P(E, \alpha) = P(E) - \epsilon(\alpha) \cdot \delta$. If E has positive rank then we also define the *slope* of (E, α) as $\mu(E, \alpha) = (\text{deg}(E) - \epsilon(\alpha)\delta_1) \cdot \text{rk } E$.

DEFINITION 8.7. A framed module (E, α) is called *Gieseker δ -(semi)stable* if for all framed submodules $(E', \alpha') \subset (E, \alpha)$ we have $\text{rk } E \cdot P(E', \alpha') \leq \text{rk } E' \cdot P(E, \alpha)$.

If E is torsion-free than we say that (E, α) is *slope δ_1 -(semi)stable* if for all framed submodules $(E', \alpha') \subset (E, \alpha)$ of rank $0 < \text{rk } E' < \text{rk } E$ we have $\mu(E', \alpha') \leq \mu(E, \alpha)$.

Let us assume that F is a torsion-free sheaf on a divisor $D \subset X$. In the following we identify F with its push forward to X .

LEMMA 8.8. *Let E be a slope semistable torsion-free sheaf on X and let $E|_D \simeq F$ be a framing. Then the corresponding framed module (E, α) , where $\alpha : E \rightarrow E|_D \simeq F$, is slope δ_1 -stable for any small positive constant δ_1 . In particular, (E, α) is Gieseker δ -stable for all polynomials δ of degree $n - 1$ with a small positive leading coefficient.*

Proof. Note that $\ker \alpha = E(-D)$. Let $E' \subset E$ be a subsheaf of rank $r' < r = \text{rk } E$. If $E' \subset \ker \alpha$ then

$$\mu(E', \alpha') = \mu(E') \leq \mu(E) - Dc_1(\mathcal{O}_X(1))^{n-1} < \mu(E) - \delta_1 = \mu(E, \alpha).$$

If $E' \not\subset \ker \alpha$ then

$$\mu(E', \alpha') = \mu(E') - \frac{\delta_1}{r'} \leq \mu(E) - \frac{\delta_1}{r'} < \mu(E) - \frac{\delta_1}{r} = \mu(E, \alpha),$$

which proves the lemma. □

Now [14, Theorem 0.1], together with appropriate modifications in positive characteristic (see [23] for the details) imply the following corollary:

COROLLARY 8.9. *There exists a quasi-projective scheme $M(X; D, F, P)$ which represents the moduli functor $\mathcal{M}(X; D, F, P) : \text{Sch}/k \rightarrow \text{Sets}$, which to a k -scheme of finite type S associates the set of isomorphism classes of S -flat families of pairs $(E, E|_D \simeq F)$, where E is a slope semistable torsion-free sheaf on X with fixed Hilbert polynomial P . It can be constructed as an open subscheme of the projective moduli scheme of Gieseker δ -stable framed modules $M_\delta^s(X; D, F, P) = M_\delta^{ss}(X; D, F, P)$ for any polynomial δ of degree $n - 1$ with a small positive leading coefficient.*

Let X be a surface and let F be a semistable locally free sheaf on a smooth irreducible curve $D \subset X$. Assume that D is numerically proportional to the polarisation $c_1(\mathcal{O}_X(1))$. Then any torsion-free sheaf E on X for which there exists a framing $E|_D \simeq F$ is automatically slope semistable. So in this case we have a quasi-projective moduli space for torsion-free sheaves with framing without any need to introduce the stability condition.

This in particular implies that the moduli spaces of torsion-free sheaves E on \mathbb{P}^2 with fixed rank r , second Chern class and framing $E \simeq \mathcal{O}_{l_\infty}^r$ at the fixed line l_∞ can be considered as an open subscheme of the moduli space of framed modules of [14] and it is a fine moduli space for the corresponding moduli functor (cf. [28, Remark 2.2]).

However, the situation becomes more subtle if we want to consider moduli spaces of (r, c) -instantons E on \mathbb{P}^3 with framing $E \simeq \mathcal{O}_{l_\infty}^r$ at the fixed line $l_\infty \subset \mathbb{P}^3$:

PROPOSITION 8.10. *Let \tilde{E} be an $(r - 1, c)$ -instanton on \mathbb{P}^3 and let $E|_{l_\infty} \simeq \mathcal{O}_{l_\infty}^r$ be a framing of $E = \tilde{E} \oplus \mathcal{O}_{\mathbb{P}^3}$. If $c = c_2(E) > r(r - 1)$ then E is an (r, c) -instanton but the corresponding framed module (E, α) is not Gieseker δ -semistable for any positive polynomial δ .*

Proof. Assume (E, α) is Gieseker δ -semistable for some positive polynomial δ . Then the stability condition for $E' = I_{l_\infty} E \subset E$ gives $\frac{P(E')}{r} \leq \frac{P(E) - \delta}{r}$, i.e. $\delta \leq P(E) - P(E') = rP(\mathcal{O}_{l_\infty})$. Hence

$$\delta(m) \leq r(m + 1)$$

for large m . On the other hand, we have $P(\mathcal{O}_{\mathbb{P}^3}) - \delta \leq \frac{P(E) - \delta}{r}$, which translates into

$$\delta(m) \geq \frac{c(m + 2)}{r - 1}.$$

Hence $c \leq r(r - 1)$. □

Below we show that the moduli space of framed instantons on \mathbb{P}^3 can be constructed as an open subscheme of the moduli space of framed modules but on a different variety. Before giving a precise formulation of this result let us introduce some notation.

Let $\Lambda \simeq \mathbb{P}^1$ be the pencil of hyperplanes passing through $l_\infty = \{x_0 = x_1 = 0\}$ in \mathbb{P}^3 . The coordinates of this \mathbb{P}^1 are denoted by y_0, y_1 . Let $X = \{(H, x) : x \in H\} \subset \mathbb{P}^1 \times \mathbb{P}^3$ be the incidence variety. It is defined by the equation $y_1 x_0 = y_0 x_1$. Let p and q denote the corresponding projections of X onto Λ and \mathbb{P}^3 . We will write $\mathcal{O}_X(a, b)$ for $p^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes q^* \mathcal{O}_{\mathbb{P}^3}(b)$. The projection $q : X \rightarrow \mathbb{P}^3$ is the blow up of \mathbb{P}^3 along the line l_∞ . The exceptional divisor of q will be denoted by D . It is easy to see that $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^3}(-1, 1)$. Note that X is equal to the projectivisation of $N = \mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on \mathbb{P}^1 . The relative $\mathcal{O}_{\mathbb{P}(N)}(1)$ for this projectivisation is equal to $q^* \mathcal{O}_{\mathbb{P}^3}(1)$. We will denote this line bundle by $\mathcal{O}_X(1)$.

THEOREM 8.11. *There exists a quasi-projective scheme $\mathcal{M}^f(\mathbb{P}^3; r, c)$ which represents the moduli functor $\tilde{\mathcal{M}}^f(\mathbb{P}^3; r, c) : \text{Sch}/k \rightarrow \text{Sets}$ given by*

$$S \rightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of } S\text{-flat families} \\ \text{of framed } (r, c)\text{-instantons } E \text{ on } \mathbb{P}^3. \end{array} \right\}$$

It is isomorphic to $M(X; D, \mathcal{O}_D^r, P)$ for a suitably chosen Hilbert polynomial P and an arbitrary polarisation.

Proof. Let E be an instanton on \mathbb{P}^3 .

LEMMA 8.12. *q^*E is slope \tilde{H} -semistable for any ample line bundle \tilde{H} on X .*

Proof. Let us set $\xi = c_1(\mathcal{O}_X(1))$. It is easy to see that q^*E is slope ξ -semistable as otherwise the push forward of the destabilising subsheaf would destabilise $E = q_*(q^*E)$ (see Lemma 2.6).

Moreover, the restriction of q^*E to a general fibre of p is isomorphic to the restriction of E to a hyperplane in \mathbb{P}^3 containing l , which is clearly semistable. So q^*E is slope $f\xi$ -semistable (i.e. slope in the semistability condition is computed as $c_1 \cdot f\xi / \text{rk}$).

The nef cone of X is generated by divisors ξ and $f = p^*c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. So we can write $\tilde{H} = a\xi + bf$ for some positive numbers a and b . Then $\tilde{H}^2 = a^2\xi^2 + 2abf\xi$, so slope \tilde{H} -semistability of q^*E follows from the above. \square

In the proof we also need a generalisation of Ishimura’s generalisation [16, Theorem 1] of Schwarzenberger’s theorem. For a moment let us switch to a different notation:

Let X and $Y \subset X$ be smooth varieties and let S be an arbitrary noetherian k -scheme. Let $\pi : \tilde{X} \rightarrow X$ be the blow up of X along Y . Let E be the exceptional divisor and let $\tilde{\pi} = \pi|_E : E \rightarrow Y$. Let us set $\pi_S = \pi \times \text{Id}_S : \tilde{X} \times S \rightarrow X \times S$ etc.

THEOREM 8.13. (cf. [16, Theorem 1]) *Let \mathcal{F} be a coherent sheaf on $\tilde{X} \times S$ such that $\mathcal{F}|_{E \times S} \simeq \tilde{\pi}^*\mathcal{G}$ for some locally free sheaf \mathcal{G} on $Y \times S$. Then the coherent sheaf $\mathcal{E} = \pi_{S*}\mathcal{F}$ is locally free in an open neighborhood of $Y \times S$ and the natural map $\pi_S^*\mathcal{E} \rightarrow \mathcal{F}$ is an isomorphism.*

Proof. The theorem can be proven in exactly the same way as [16, Theorem 1] using the fact that cohomology commutes with flat base extension. \square

Coming back to the proof of the theorem we will show that the functor $\mathcal{M}(l; r, c)$ is represented by the quasi-projective moduli scheme $M(X; D, \mathcal{O}_D^r, P)$ (for a suitably chosen P and an arbitrary fixed polarisation).

First, let us note that by Lemma 8.12 there exists a natural transformation of functors

$$\Phi : \tilde{\mathcal{M}}^f(\mathbb{P}^3; r, c) \rightarrow \mathcal{M}(X; D, \mathcal{O}_D^r, P)$$

given by sending a flat S -family $(E_S, E|_{l \times S} \simeq \mathcal{O}_{l \times S}^r)$ of framed (r, c) -instantons to the family $(q_S^*E_S, q_S^*E_S|_{D \times S} \simeq \mathcal{O}_{D \times S}^r)$. To show the above claim it is sufficient to prove that the transformation Φ is an isomorphism of functors. First, note that

$$q_{S*}q_S^*E_S \simeq E_S \otimes q_{S*}\mathcal{O}_{X \times S} \simeq E_S,$$

where the first isomorphism comes from the projection formula (note that E_S is locally free around $l \times S$) and the second isomorphism follows since push-forward commutes with flat base extension. Similarly, we have

$$R^1q_{S*}(q_S^*E_S(-D \times S)) \simeq E_S \otimes R^1q_{S*}\mathcal{O}_{X \times S}(-D \times S) = 0,$$

so $q_{S*}(q_S^*E_S|_{D \times S}) \simeq E_S|_{l \times S}$ and the push-forward of $q_S^*E_S|_{D \times S} \simeq \mathcal{O}_{D \times S}^r$ gives an isomorphism $E_S|_{l \times S} \simeq \mathcal{O}_{l \times S}^r$.

Hence Theorem 8.13 implies that the natural transformation

$$\Psi : \mathcal{M}(X; D, \mathcal{O}_D^r, P) \rightarrow \tilde{\mathcal{M}}^f(\mathbb{P}^3; r, c)$$

given by sending a flat S -family $(F_S, F|_{D \times S} \simeq \mathcal{O}_{D \times S}^r)$ to the family $(q_{S*}F_S, (q_{S*}F_S)|_{l \times S} \simeq \mathcal{O}_{l \times S}^r)$ is inverse to Φ . \square

9. Deformation theory and smoothness of moduli spaces of instantons. In this section we give a very quick review of deformation theory for framed perverse instantons. We sketch only a quite simple fact from deformation theory used a few

times throughout the paper without going into long technical results showing, e.g. virtual smoothness of the moduli space of stable perverse instantons.

Then we show that if E_1 and E_2 are locally free instantons then $\text{Ext}^2(E_1, E_2)$ vanishes for low ranks and second Chern classes. This implies that the moduli space of locally free instantons embeds as a Lagrangian submanifold into the moduli space of sheaves on a quartic. It also proves that the moduli space of framed locally free (r, c) -instantons is smooth for low values of r and c .

9.1. Deformation theory for framed perverse instantons. Let (\mathcal{C}, Φ) be a stable framed perverse instanton corresponding to an ADHM datum $x \in \tilde{\mathbf{B}}$. Let $\varphi : G \rightarrow \mathbb{B}$ be the orbit map sending g to gx .

THEOREM 9.1. *Let us consider the complex K*

$$0 \rightarrow K^0 = \mathfrak{g} \xrightarrow{d\varphi_x} K^1 = T_x \tilde{\mathbf{B}} \xrightarrow{d\tilde{\mu}_x} K^2 = T_0(\text{End}(V) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2))) \rightarrow 0$$

Then $H^i(K) = 0$ for $i \neq 1, 2$, $H^1(K) = \text{Ext}^1(\mathcal{C}, J_{l_\infty} \otimes \mathcal{C})$ and $H^2(K) = \text{Ext}^2(\mathcal{C}, \mathcal{C})$. In particular, if $\text{Ext}^2(\mathcal{C}, \mathcal{C}) = 0$ then the moduli space $\mathcal{M}(\mathbb{P}^3; r, c)$ is smooth of dimension $4cr$ at $[(\mathcal{C}, \Phi)]$.

Proof. We know that \mathcal{C} is quasi-isomorphic to the following complex

$$0 \rightarrow \mathcal{C}^{-1} := V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{C}^0 := \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{C}^1 := V \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

where $\dim V = c$ and $\dim \tilde{W} = r + 2c$ (more precisely $\tilde{W} = V \oplus V \oplus W$) and α, β are defined by the ADHM datum x as in 2.4. Let us consider the complex $\mathcal{D} = \text{Hom}^\bullet(\mathcal{C}, J_{l_\infty} \otimes \mathcal{C})$. Then we see that

$$\text{Ext}^i(\mathcal{C}, J_{l_\infty} \otimes \mathcal{C}) = \mathbb{H}^i(\mathbb{P}^3, \mathcal{D}),$$

where $\mathbb{H}^i(X, \mathcal{D})$ denotes the i th hypercohomology group of the complex \mathcal{D} . Let us consider a spectral sequence

$$H^l(\mathbb{P}^3, \mathcal{D}^s) \Rightarrow \mathbb{H}^{s+l}(\mathbb{P}^3, \mathcal{D}).$$

Using this spectral sequence we see that we have a complex

$$0 \rightarrow L^0 = H^2(\mathbb{P}^3, \mathcal{D}^{-2}) \xrightarrow{d_L^0} L^1 = H^0(\mathbb{P}^3, \mathcal{D}^1) \xrightarrow{d_L^1} L^2 = H^0(\mathbb{P}^3, \mathcal{D}^2) \rightarrow 0$$

such that $H^1(L) = \mathbb{H}^1(\mathbb{P}^3, \mathcal{D})$ and $H^2(L) = \mathbb{H}^2(\mathbb{P}^3, \mathcal{D})$. We have $L^0 = \text{Hom}(V, V) \otimes H^2(\mathbb{P}^3, J_{l_\infty}(-2))$, $L^1 = (\text{Hom}(\tilde{W}, V) \oplus \text{Hom}(V, \tilde{W})) \otimes H^0(\mathbb{P}^3, J_{l_\infty}(1))$ and $L^2 = \text{Hom}(V, V) \otimes H^0(\mathbb{P}^3, J_{l_\infty}(2))$. Note that $H^2(\mathbb{P}^3, J_{l_\infty}(-2)) \simeq k$ but $H^0(\mathbb{P}^3, J_{l_\infty}(2)) \simeq k^7$, so this is not yet the complex we were looking for. However, if we write down everything in coordinates we see that d_L^1 is an isomorphism on $\text{Hom}(V, V) \otimes k^4$ and after splitting off the corresponding factors from L^1 and L^2 we get exactly complex K . Obviously, we need to write down everything in coordinates to check that the obtained maps are essentially the same. We leave the details to the reader. Now the theorem follows from the following lemma:

LEMMA 9.2. *Let \mathcal{C} be a framed perverse (r, c) -instanton on \mathbb{P}^3 . Then*

$$\text{Ext}^2(\mathcal{C}, \mathcal{C}) = \text{Ext}^2(\mathcal{C}, J_{l_\infty} \otimes \mathcal{C}).$$

Proof. We have a distinguished triangle

$$J_{l_\infty} \otimes \mathcal{C} \rightarrow \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{O}_{l_\infty} \rightarrow J_{l_\infty} \otimes \mathcal{C}[1].$$

This triangle gives

$$\text{Ext}^1(\mathcal{C}, \mathcal{C} \otimes \mathcal{O}_{l_\infty}) \rightarrow \text{Ext}^2(\mathcal{C}, J_{l_\infty} \otimes \mathcal{C}) \rightarrow \text{Ext}^2(\mathcal{C}, \mathcal{C}) \rightarrow \text{Ext}^2(\mathcal{C}, \mathcal{C} \otimes \mathcal{O}_{l_\infty}).$$

But $\text{Ext}^l(\mathcal{C}, \mathcal{C} \otimes \mathcal{O}_{l_\infty}) = h^l(\mathcal{O}_{l_\infty}^{\otimes 2}) = 0$ for $l = 1, 2$, so we get the required equality. □

This finishes proof of Theorem 9.1. □

REMARK 9.3. Let (\mathcal{C}, Φ) be a stable framed perverse instanton. Then by a standard computation one can see that the tangent space to $\mathcal{M}(\mathbb{P}^3; r, c)$ at the point corresponding to (\mathcal{C}, Φ) is isomorphic to $\text{Ext}^1(\mathcal{C}, J_{l_\infty} \otimes \mathcal{C})$. Moreover, one can show that there exists an appropriate obstruction theory with values in $\text{Ext}^2(\mathcal{C}, \mathcal{C})$ (cf. [15, 2.A.5]).

9.2. Smoothness of the moduli space of framed locally free instantons.

LEMMA 9.4. *Let E be a locally free instanton of rank $r = 2$ or $r = 3$. Then for any plane $\Pi \subset \mathbb{P}^3$ the restriction E_Π is slope semistable.*

Proof. Let us note that we have a long exact cohomology sequences:

$$0 = H^0(E(-1)) \rightarrow H^0(E_\Pi(-1)) \rightarrow H^1(E(-2)) = 0$$

and

$$0 = H^0(E^*(-1)) \rightarrow H^0(E_\Pi^*(-1)) \rightarrow H^1(E^*(-2)) \simeq (H^2(E(-2)))^* = 0,$$

where the isomorphism in the second sequence comes from the Serre duality. This implies that $E_\Pi(-1)$ and $E_\Pi^*(-1)$ have no sections which in ranks 2 and 3 implies semistability of E_Π . □

LEMMA 9.5. *Let E_i be a locally free (r_i, c_i) -instanton on \mathbb{P}^3 , where $i = 1, 2$. Then $\text{Ext}^2(E_1, E_2(-2))$ has dimension at most c_1c_2 .*

Proof. Our assumption implies that E_i is the cohomology of the following monad \mathcal{C}_i^\bullet

$$0 \rightarrow V_i \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{d_{c_i}^{-1}} \tilde{W}_i \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{d_{c_i}^0} V_i \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0,$$

where $\dim V_i = c_i$ and $\dim \tilde{W}_i = 2c_i + r_i$. Let us consider the complex $\mathcal{C}^\bullet = \text{Hom}^\bullet(\mathcal{C}_1^\bullet, \mathcal{C}_2^\bullet)$ defined by

$$\mathcal{C}^i := \bigoplus_k \mathcal{H}om(\mathcal{C}_1^k, \mathcal{C}_2^{k+i})$$

with $d(f) := dc_2 \circ f - (-1)^{\deg f} \circ dc_1$. Since \mathcal{C}_i^\bullet are complexes of locally free sheaves we see that

$$\text{Ext}^p(E_1, E_2(-2)) = \mathbb{H}^p(\mathbb{P}^3, \mathcal{C}^\bullet \otimes \mathcal{O}_{\mathbb{P}^3}(-2)),$$

where \mathbb{H}^p denotes the p th hypercohomology group. But then the spectral sequence

$$H^t(\mathbb{P}^3, \mathcal{C}^s \otimes \mathcal{O}_{\mathbb{P}^3}(-2)) \Rightarrow \mathbb{H}^{s+t}(\mathbb{P}^3, \mathcal{C}^\bullet \otimes \mathcal{O}_{\mathbb{P}^3}(-2))$$

gives an exact sequence

$$0 \rightarrow \text{Ext}^1(E_1, E_2(-2)) \rightarrow \text{Hom}(V_1, V_2) \rightarrow \text{Hom}(V_1, V_2) \rightarrow \text{Ext}^2(E_1, E_2(-2)) \rightarrow 0.$$

Clearly, this implies the required inequality. □

The proof of the following theorem uses the method of proof of [25, Théorème 1].

THEOREM 9.6. *Let E_i be a locally free (r_i, c_i) -instanton on \mathbb{P}^3 , where $i = 1, 2$. If $r_1, r_2 \leq 3$ and $c_1 c_2 \leq 6$ then $\text{Ext}^2(E_1, E_2) = 0$.*

Proof. Let $Z = \{(x, \Pi) \in \mathbb{P}^3 \times (\mathbb{P}^3)^* : x \in \Pi\}$ be the incidence variety of planes containing a point in \mathbb{P}^3 . Let p_1, p_2 denote projections from $\mathbb{P}^3 \times (\mathbb{P}^3)^*$ onto \mathbb{P}^3 and $(\mathbb{P}^3)^*$, respectively, and let us set $q_1 = p_1|_Z$ and $q_2 = p_2|_Z$. On $\mathbb{P}^3 \times (\mathbb{P}^3)^*$ we have a short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3 \times (\mathbb{P}^3)^*}(-1, -1) \rightarrow \mathcal{O}_{\mathbb{P}^3 \times (\mathbb{P}^3)^*} \rightarrow \mathcal{O}_Z \rightarrow 0.$$

Let us tensor this sequence with $p_1^* \mathcal{H}om(E_1, E_2(i))$ and push it down by p_2 . Then we get an exact sequence

$$\text{Ext}^2(E_1, E_2(i-1)) \otimes \mathcal{O}_{(\mathbb{P}^3)^*}(-1) \xrightarrow{\varphi_i} \text{Ext}^2(E_1, E_2(i)) \otimes \mathcal{O}_{(\mathbb{P}^3)^*} \rightarrow R^2 q_{2*} q_1^* \mathcal{H}om(E_1, E_2(i)).$$

But for any plane $\Pi \subset \mathbb{P}^3$ the group $\text{Ext}^2((E_1)_\Pi, (E_2)_\Pi(i))$ is Serre dual to $\text{Hom}((E_2)_\Pi, (E_1)_\Pi(-i-3))$. By Lemma 9.4 both $(E_1)_\Pi$ and $(E_2)_\Pi$ are semistable of the same slope so if $i > -3$ then $\text{Hom}((E_2)_\Pi, (E_1)_\Pi(-i-3)) = 0$. This implies that $R^2 q_{2*} q_1^* \mathcal{H}om(E_1, E_2(i)) = 0$ for $i > -3$ and hence for such i we have a short exact sequence

$$0 \rightarrow F_i = \ker \varphi_i \rightarrow \text{Ext}^2(E_1, E_2(i-1)) \otimes \mathcal{O}_{(\mathbb{P}^3)^*}(-1) \xrightarrow{\varphi_i} \text{Ext}^2(E_1, E_2(i)) \otimes \mathcal{O}_{(\mathbb{P}^3)^*} \rightarrow 0.$$

Now F_i is a vector bundle (again only for $i > -3$). Let s_i denotes its rank. If $s_i < 3$ and $\text{Ext}^2(E_1, E_2(i)) \neq 0$ then $c_{s_i+1}(F_i)$ is non-zero which contradicts the fact that F_i is locally free. Therefore if $\text{Ext}^2(E_1, E_2(i)) \neq 0$ for some $i \geq -2$ then

$$s_i = \dim \text{Ext}^2(E_1, E_2(i-1)) - \dim \text{Ext}^2(E_1, E_2(i)) \geq 3.$$

Applying this inequality for $i = 0$ and $i = -1$ we see that if $\text{Ext}^2(E_1, E_2) \neq 0$ then $\text{Ext}^2(E_1, E_2(-2))$ has dimension at least 7. By Lemma 9.5 this contradicts our assumption on $c_1 c_2$. □

COROLLARY 9.7. *Let $r \leq 3$ and $c \leq 2$. Then the moduli space of framed locally free (r, c) -instantons is smooth of dimension $4cr$.*

Proof. Let E be a locally free (r, c) -instanton. Then $\text{Ext}^2(E, E) = 0$ and by Theorem 9.1 the tangent space to the moduli space is isomorphic to $\text{Ext}^1(E, J_{l_\infty} E)$, so the dimension is equal to $4cr$. \square

Let $M_{\mathbb{P}^3}(r, c)$ denotes the moduli space of Gieseker stable locally free (r, c) -instantons on \mathbb{P}^3 . In case of rank $r = 2$ or 3 Gieseker stability of instanton E is equivalent to $h^0(E) = h^0(E^*) = 0$.

Let $S \subset \mathbb{P}^3$ be any smooth quartic with $\text{Pic } S = \mathbb{Z}$. Let $M_S(r, 4c)$ denotes the moduli space of slope stable vector bundles on S with rank r and Chern classes $c_1 = 0$ and $c_2 = c \cdot h^2|_S = 4c$ (h stands for the class of a hyperplane in \mathbb{P}^3).

Using an idea of A. Tyurin (see [2, Section 9]) one can show the following theorem:

THEOREM 9.8. *Let $r \leq 3$ and $c \leq 2$. Then $M_{\mathbb{P}^3}(r, c)$ is smooth and the restriction $r : M_{\mathbb{P}^3}(r, c) \rightarrow M_S(r, 4c)$ is a morphism which induces an isomorphism of $M_{\mathbb{P}^3}(r, c)$ onto a Lagrangian submanifold of $M_S(r, 4c)$.*

Proof. Smoothness of $M_{\mathbb{P}^3}(r, c)$ follows directly from Theorem 9.6. To prove that the restriction map $r : M_{\mathbb{P}^3}(r, c) \rightarrow M_S(r, c)$ is a morphism we need the following lemma:

LEMMA 9.9. *Let E be a locally free instanton of rank $r = 2$ or $r = 3$. Assume that $h^0(E) = h^0(E^*) = 0$. Then E is slope stable and for any smooth quartic $S \subset \mathbb{P}^3$ with $\text{Pic } S = \mathbb{Z}$, the restriction E_S is slope stable.*

Proof. The (saturated) destabilising subsheaf of E has either rank 1 and then it gives a section of E or it has rank 2 and then E has rank 3 and the determinant of the destabilising subsheaf gives a section of $\wedge^2 E \simeq E^*$. This proves the first assertion. By the same argument to show the second assertion it is sufficient to prove that $h^0(E_S) = h^0(E_S^*) = 0$. But this follows from sequences:

$$0 = H^0(E) \rightarrow H^0(E_S) \rightarrow H^1(E(-4)) = 0$$

and

$$0 = H^0(E^*) \rightarrow H^0(E_S^*) \rightarrow H^1(E^*(-4)) \simeq (H^2(E))^* = 0.$$

\square

Let E be a Gieseker stable locally free (r, c) -instanton. By Lemma 9.9 and Theorem 9.6 we know that the restriction of E to S is slope stable and $\text{Ext}^2(E, E) = 0$. This implies that r is an immersion at the point $[E]$ (see [2, 9.1]). Therefore we only need to show that r is an injection.

To prove that let us take two Gieseker stable locally free (r, c) -instantons E_1 and E_2 . Then by Theorem 9.6 we have an exact sequence

$$\text{Hom}(E_1, E_2) \rightarrow \text{Hom}((E_1)_S, (E_2)_S) \rightarrow \text{Ext}^1(E_1, E_2(-S)) \simeq (\text{Ext}^2(E_2, E_1))^* = 0.$$

This shows that we can lift any isomorphism $(E_1)_S \rightarrow (E_2)_S$ to an isomorphism of E_1 and E_2 and hence r is injective. \square

REMARK 9.10. It is very tempting to conjecture that Theorem 9.6 holds for all pairs of locally free instantons (maybe with some additional assumptions concerning stability of these bundles). This would imply a well known conjecture on smoothness

of the moduli space of locally free instantons. Even then, an analogue of Theorem 9.8 does not immediately follow. But if one restricted to the open subset of bundles for which all exterior powers remain instantons then one could embed it into $M_S(r, 4c)$ as a Lagrangian submanifold.

However, it seems that all these conjectures are just a wishful thinking similar to the original conjecture on smoothness of the moduli space of locally free instantons: there are very few known results and all the methods work only for instantons of low charge.

EXAMPLE 9.11. For $r = 2$ and $c = 1$ the moduli space $M_{\mathbb{P}^3}(2, 1)$ parameterizes only null-correlation bundles and it is known that $M_{\mathbb{P}^3}(2, 1) \simeq \mathbb{P}^5 \setminus \text{Gr}(2, 4)$, where $\text{Gr}(2, 4)$ is the Grassmannian of planes in \mathbb{A}^4 (see [30, Chapter II, Theorem 4.3.4]). By the above theorem this is a Lagrangian submanifold of the moduli space $M_S(2, 4)$. Over complex numbers $M_S(2, 4)$ is known to have a smooth compactification to a holomorphic symplectic variety (see [29]). Note that Lagrangian fibrations $M_S(2, 4) \rightarrow \mathbb{P}^5$ for some K3 surfaces S were constructed by Beauville in [2, Proposition 9.4]. It is possible that the Lagrangian submanifold $M_{\mathbb{P}^3}(2, 1)$ extends to a section of some Lagrangian fibration (possibly after deforming the compactification) providing another example when this is possible (see [33] for the proof that some Lagrangian fibrations can be deformed to Lagrangian fibrations with a section in case of four-dimensional varieties).

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