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ABSOLUTE C-EMBEDDING OF SPACES WITH COUNTABLE CHARACTER AND PSEUDOCHARACTER CONDITIONS

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1. Introduction. Absolute C-embeddings have been studied extensively by C. E. Aull. We will use his notation $P = C[Q]$ to mean that a space satisfying property Q is C-embedded in every space having property Q that it is embedded in if (and only if) it has property P. The first result of this type is due to Hewitt [**5**] where he proves that if Q is "Tychonoff" then P is almost compactness. Aull [**2**] proves that if Q is "T_4 and countable pseudocharacter" or "T_4 and first countable" then P is "countably compact". In this paper we show that P is almost compactness if Q is "Tychonoff" and any of countable pseudocharacter, perfect, or first countability. Unfortunately for the last case we require the assumption that $2^{\aleph_0} = \aleph_1$. Finally we show that P is countable compactness if Q is Tychonoff and "closed sets have a countable neighborhood base". In each of the above results C-embedding may be replaced by C^*-embeddings and the results hold if restricted to closed embeddings.

We assume that all hypothesized topological spaces are Tychonoff. A space X is *almost compact* if given disjoint zero sets at least one is compact. This is equivalent to $|\beta X \backslash X| \leqq 1$ where βX is the Stone–Čech compactification of X [**4**, 6J]. The *pseudocharacter* of $x \in X$, denoted $\psi(x, X)$, is the minimum cardinality κ such that $\{x\}$ is a G_κ set in X. X is *perfect* if every closed subset of X is a G_δ. We shall use the notation and terminology of [**4**].

2. Absolute C-embedding of spaces with countable pseudocharacter. We show that P = almost compact when Q is countable pseudocharacter. Hewitt's result shows that an almost compact space is C-embedded in any Tychonoff space in which it is embedded. Hence we only have to show the converse. First we list some lemmas that we will need.

LEMMA 2.1. *A space X is compact if, and only if, every infinite subset of X has a complete accumulation point. That is, for every subset D of X there is a point $x \in \bar{D}$ such that every neighbourhood of x meets D in a set of the same cardinality as D.*

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LEMMA 2.2. *If X is not compact and κ is the minimum cardinality of a subset of X with no complete accumulation point then κ is regular.*

Proof. Suppose that $(x_\alpha)_{\alpha<\kappa} \subseteq X$ has no complete accumulation point in X and

$$\sum (\alpha_j : j < \lambda) = \kappa$$

where each $\alpha_j < \kappa$. By the minimality of κ the sets $(x_\alpha)_{\alpha<\alpha_j}$ for each $j < \lambda$ all have complete accumulation points y_j. Similarly the set $(y_j)_{j<\lambda}$ has a complete accumulation point if $\lambda < \kappa$. But if x is such a point then x is a complete accumulation of the set $(x_\alpha)_{\alpha<\kappa}$. Therefore $\lambda = \kappa$ and κ is regular.

THEOREM 2.3. *Let X be a space which is not almost compact. Then X can be embedded in a space Y such that:* (1) *X is not C^*-embedded in Y,* (2) *for $x \in X$, $\psi(x, X) = \psi(x, Y)$ and* (3) *for $y \in Y \backslash X$, $\psi(y, Y) = \omega$.*

Proof. Since X is not almost compact we can choose disjoint non-compact zero sets Z_1 and Z_2. We will embed X into a space Y so that Z_1 and Z_2 are not completely separated. By Lemmas 2.1 and 2.2 we can choose regular cardinals κ_1 and κ_2 and sets

$$(x_\alpha)_{\alpha<\kappa_1} \subseteq Z_1 \quad (y_\beta)_{\beta<\kappa_2} \subseteq Z_2$$

such that neither set has a complete accumulation point in X. Next we choose $\mathscr{C} \subset \omega^\omega$ such that $f, g \in \mathscr{C}$ implies $\{n: f(n) = g(n)\}$ is finite, and maximal with respect to this property. We note that $\mathscr{C}$ has the following property: (*) if $h \in \omega^\omega$ then there is an $f \in \mathscr{C}$ such that $\{n: f(n) > h(n)\}$ is infinite. This follows easily from the maximality of $\mathscr{C}$. Our space Y consists of

$$X \cup (\kappa_1 \times \omega) \cup (\kappa_2 \times \omega) \cup \mathscr{C}.$$

$\kappa_1 \times \omega$ and $\kappa_2 \times \omega$ are open and discrete. For $x \in X$, x has a neighbourhood $U \subseteq X$ which meets $\{x_\alpha: \alpha < \kappa_1\}$ in a set of cardinality less than κ_1 and meets $\{y_\beta: \beta < \kappa_2\}$ in a set of cardinality less than κ_2. Define a neighbourhood of x in Y to be

$$U \cup [\bigcup \{\{\alpha\} \times [n_\alpha, \omega) : n_\alpha \in \omega, x_\alpha \in U\}] \cup [\bigcup \{\{\beta\} \times [n_\beta, \omega) : n_\beta \in \omega, y_\beta \in U\}].$$

In other words $\{x_\alpha\} \cup \{\alpha\} \times \omega$ is the one point compactification of $\{\alpha\} \times \omega$ and similarly for y_β. Finally we describe neighbourhoods of the elements of $\mathscr{C}$. For each $f \in \mathscr{C}$ we will define $f_1 : \kappa_1 \to \omega$ and $f_2 : \kappa_2 \to \omega$ and neighbourhoods of f will be subsets of the graph of f_1 and f_2. First assume κ_1 and κ_2 are uncountable, and let $\{U_n\}_{n\in\omega}$ be a shrinking family of subsets of κ_1 such that $|U_n \backslash U_{n+1}| = \kappa_1$ for each n and $\bigcap_n U_n = \emptyset$. Similarly let $\{V_n\}_{n\in\omega}$ be subsets of κ_2 with the analogous properties. If

κ_1 or κ_2 is ω then let U_n or V_n be $\{n, n+1, n+2, \ldots,\}$. Now we define f_1. If $\kappa_1 = \omega$ then $f_1 = f$; otherwise let $f(\alpha) = f(n)$ if $\alpha \in U_n \backslash U_{n+1}$ and similarly define $f_2(\beta) = f(m)$ if $\beta \in V_m \backslash V_{m+1}$. A neighbourhood base for f is

$$\{W_{\alpha,\beta,n,k}: \alpha < \kappa_1, \beta < \kappa_2, n < \omega, k < \omega\}$$

where

$$W_{\alpha,\beta,n,k} = \{f\} \cup [(U_n \cap [\alpha, \kappa_1)) \times \omega \cap \text{graph of } f_1] \cup [(V_k \cap [\beta, \kappa_2)) \times \omega \cap \text{graph of } f_2]$$

(i.e., tails of the graphs of f_1 intersected with U_n, together with tails of the graph of f_2 intersected with V_k). The neighbourhoods of $f \in \mathscr{C}$ are clopen, for if $g \in \mathscr{C}$ and $g \neq f$ then $\{n: g(n) = f(n)\}$ is finite. So if N is such that $n > N$ implies $g(n) \neq f(n)$ then

$$\{U_N \times \omega \cap \text{graph of } f_1\} \cap \{U_N \times \omega \cap \text{graph of } g_1\} = \emptyset;$$

for if $\alpha \in U_N$ then $\alpha \in U_k \backslash U_{k+1}$ for $k \geqq N$ and

$$f_1(\alpha) = f(k) \neq g(k) = g(\alpha);$$

but for

$$(\alpha, j) \in \{U_N \times \omega \cap \text{graph of } f_1\} \cap \{U_N \times \omega \cap \text{graph of } g_1\}$$

we must have

$$f_1(\alpha) = j = g_1(\alpha).$$

Similarly

$$\{V_N \times \omega \cap \text{graph of } f_2\} \cap \{V_N \times \omega \cap \text{graph of } g_2\} = \emptyset.$$

It is easily checked that no other elements of Y are in the closure. We now check that Y has the desired properties.

Claim 1. Z_1 and Z_2 are not completely separated in Y.

Proof. Let A and B be closed neighbourhoods of Z_1 and Z_2 respectively. We will show that $A \cap B \cap \mathscr{C} \neq \emptyset$.

Since A is a neighbourhood of $(x_\alpha):\alpha < \kappa_1$, for each $\alpha < \kappa_1$, there is an integer $h_1(\alpha)$ such that

$$\{\alpha\} \times [h_1(\alpha), \omega) \subseteq A.$$

Let $A_n = h_1^{\leftarrow}([0, n])$. The A_n's are increasing subsets of κ_1 and

$$\bigcup \{A_n : n < \omega\} = \kappa_1.$$

Case 1. $\kappa_1 = \omega$. This means $h_1 \in \omega^\omega$ so there exists $f \in \mathscr{C}$ such that $\{n: f(n) > h_1(n)\}$ is infinite. This means every neighbourhood of f meets A because a neighbourhood of f contains, for some $m \in \omega$, $\{(k, f_1(k)):$

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$k \geqq m\}$. Now choose $n \geqq m$ such that $f(n) > h_1(n)$, which we can do since $\{n: f(n) > h_1(n)\}$ is infinite. This means that $(n, f(n)) \in \kappa_1 \times \omega$ is in A because

$$\{n\} \times (h_1(n), \omega) \subseteq A.$$

Thus $f \in A$ since A is closed. Note that any $f \in \mathscr{C}$ such that $\{n: f(n) > h_1(n)\}$ is infinite is in A.

Case 2. $\kappa_1 > \omega$. Note that κ_1 is regular by Lemma 2.2. Recall that

$$A_n = h_1^{\leftarrow}([0, n]).$$

For each n choose $h(n) \in \omega$ such that

$$|U_n \backslash U_{n+1} \cap A_{h(n)}| = \kappa_1.$$

Now choose $f \in \mathscr{C}$ such that

$$|\{n: f(n) > h(n)\}| = \omega.$$

An arbitrary neighbourhood of f contains a set of the form

$$(U_m \cap [\delta, \kappa_1)) \times \omega \cap \text{graph of } f_1.$$

Let $n > m$ be such that $f(n) > h(n)$ and choose

$$\alpha \in [\delta, \kappa_1) \cap U_n \backslash U_{n+1} \cap A_{h(n)}.$$

Since $\alpha > \delta$ and $\alpha \in U_m \supseteq U_n \backslash U_{n+1}$ we have that $(\alpha, f_1(\alpha))$ is in the above neighbourhood of f and since $\alpha \in A_{h(n)}$, $h_1(\alpha) \leqq h(n)$. Also $h(n) < f(n)$ and $f(n) = f_1(\alpha)$ because $\alpha \in U_n \backslash U_{n+1}$. Therefore $(\alpha, f_1(\alpha))$ is in A because

$$\{\alpha\} \times [h_1(\alpha), \omega) \subseteq A$$

and we conclude that $f \in A$. Again we note that any $f \in \mathscr{C}$ such that

$$|\{n: f(n) > h(n)\}| = \omega$$

is in A.

Now we consider B. For each $\beta < \kappa_2$ we choose $g_1(\beta) \in \omega$ such that $\{\beta\} \times [g_1(\beta), \omega) \subseteq B$. We let

$$B_n = g_1^{\leftarrow}([0, n]).$$

If $\kappa_2 = \omega$ then let $g = g_1$. Otherwise for each n choose $g(n)$ so that

$$|V_n \backslash V_{n+1} \cap B_{g(n)}| = \kappa_2.$$

Now we can show that if $f \in C$ such that $|\{n: f(n) > g(n)\}| = \omega$ then $f \in B$. To complete the proof that Z_1 and Z_2 are not completely separated we observe that there exists $f \in \mathscr{C}$ such that

$$|\{n: f(n) > h(n) + g(n)\}| = \omega.$$

Hence $f \in A \cap B$.

Claim 2. Y is completely regular. It suffices to show complete regularity for points of X since all points of $Y \backslash X$ are either isolated or have a base of clopen neighbourhoods which miss X. Let $x \in X$ and (without loss of generality) not in the closure of $(y_\beta)_{\beta<\kappa_1}$. Since $(x_\alpha)_{\alpha<\kappa_2}$ has no complete accumulation point, we can choose a cozero set neighbourhood W of x which misses $(y_\beta)_{\beta<\kappa_2}$ and

$$|W \cap (x_\alpha)_{\alpha<\kappa_1}| < \kappa_1.$$

Let $\gamma < \kappa_1$ be such that if $\alpha > \gamma$ then $x_\alpha \notin W$. Let $h \in C^*(X)$ such that $h(x) = 1$ and $h(X \backslash W) = 0$. Define $h' \in C^*(Y)$ by

$$h'(f) = 0 \text{ for } f \in \mathscr{C} \text{ and}$$
$$h'((\beta, n)) = 0 \text{ for } \beta \in \kappa_2 \text{ and}$$
$$h'((\alpha, n)) = h(x_\alpha) \text{ for } \alpha \in \kappa_1.$$

It is easy to check that h' is continuous. Conditions 2 and 3 of the theorem are immediate by the construction.

COROLLARY 2.4. *C[countable pseudocharacter] = almost compact.*

3. Perfect spaces. The purpose of this section is to show that if X is a perfect space then the space Y constructed in 2.3 is also perfect. So let X be perfect and Y be as in Theorem 2.3. Let F be a closed subset of Y. We must show that F is a G_δ set in Y. Let

$$F_1 = F \cap X,\ F_2 = F \cap \mathscr{C}, \text{ and } F_3 = F \backslash (X \cup \mathscr{C}).$$

Since X is perfect, let $F_1 = \bigcap_{n\in\omega} 0_n$, where 0_n is open in X. Let

$$0_n' = 0_n \cup \bigcup \{\{\alpha\} \times [n, \omega) : x_\alpha \in 0_n \text{ and } \alpha < \kappa_1\} \cup \\ \cup \{\{\beta\} \times [n, \omega) : y_\beta \in 0_n \text{ and } \beta < \kappa_2\}.$$

Define W_n to be

$$F_2 \cup [(U_n \times \omega) \cap \bigcup \{\text{graph of } f_1 : f \in F_2\}] \\ \cup [(V_n \times \omega) \cap \bigcup \{\text{graph of } f_2 : f \in F_2\}].$$

W_n is open and contains F_2. Now we let

$$M_n = 0_n' \cup W_n \cup F_3.$$

Then M_n is open and $\bigcap_{n<\omega} M_n = F$ which shows that Y is perfect. For if $y \in Y \backslash F$ then if $y \in X$ there is an n such that $y \notin 0_n$ hence $y \notin M_n$. If $y \in Y \backslash X$ then either $y \in \mathscr{C}$, in which case $y \notin M_n$ for any n, or (without loss of generality) $y = (\alpha, m)$ for some $\alpha < \kappa_1$, $m \in \omega$. In this case, let n be such that $n > m$ and $\alpha \notin U_n$. Therefore $y \notin 0_n'$ since to 0_n we added

$\{\alpha\} \times [n, \omega)$ and $m < n$. Also $y \notin W_n$ since

$$W_n \cap \kappa_1 \times \omega \subset U_n \times \omega.$$

And $y \notin F_3$ hence $y \notin M_n$. We have proved

THEOREM 3.1 *C[perfect] = almost compact.*

4. First countable spaces.

Definition 4.1. X is (κ, λ)-*compact* if every subset A of cardinality κ has a λ-accumulation point x (i.e., for any neighbourhood U of x, $|U \cap A| \geqq \lambda$). X is κ-bounded if every subset of cardinality κ has compact closure.

LEMMA 4.2. [CH] *If X is a first countable, countably compact, $(\aleph_1, \aleph_1)$-compact space then X is $\aleph_1$-bounded.*

Proof. Suppose not. Let $D \subset X$ such that $|D| = \aleph_1$ and $\bar{D}$ is not compact. Since we are assuming [CH] and that X is first countable,

$$|\bar{D}| = |D^\omega| = \aleph_1.$$

Hence if $\bar{D}$ is not compact there is a subset S of cardinality less than or equal to $\aleph_1$ with no complete accumulation point. This contradicts the assumptions that X is countably compact and $(\aleph_1, \aleph_1)$-compact. Hence X is $\aleph_1$-bounded.

LEMMA 4.3. [CH]. *If X is first countable and $\aleph_1$-bounded then X is compact and has cardinality $\aleph_1$.*

Proof. Suppose $|X| > \aleph_1$. Let $D \subset X$ such that $|D| = \aleph_2$. $\bar{D}$ is first countable and has cardinality $\aleph_2 = c^+$ and is therefore not compact [**1**]. Hence without loss of generality we can assume that D has no complete accumulation point. Therefore every point of $\bar{D}$ has a neighbourhood which meets D in a set of cardinality less than or equal to $\aleph_1$. Hence, by Lemma 4.2, $\bar{D}$ is locally compact. Inductively construct compact sets C_α for $\alpha < \omega_1$, such that

(i) $C_\alpha \subset \operatorname{int}_{\bar{D}} C_{\alpha+1}$
(ii) $\exists\, x_{\alpha+1} \in C_{\alpha+1} \backslash C_\alpha$ and
(iii) $|C_\alpha| \leqq \aleph_1$.

Choose $x_0 \in \bar{D}$ and let $C_0 = \{x_0\}$. Let $\alpha \in \omega_1$, and suppose we have defined C_γ for $\gamma < \alpha$ so that (i)–(iii) are satisfied. Since

$$\left|\operatorname{cl}_{\bar{D}} \bigcup_{\gamma<\alpha} C_\gamma\right| \leqq \aleph_1$$

we can choose

$$x_\alpha \in \bar{D} \backslash \operatorname{cl}_{\bar{D}} \bigcup_{\gamma<\alpha} C_\gamma.$$

$\{x_\alpha\} \cup \mathrm{cl}_D \bigcup_{\gamma<\alpha} C_\gamma$ is compact because X is $\aleph_1$-bounded. Cover it with compact neighbourhoods of cardinality less than or equal to $\aleph_1$ and choose a finite subcover. Let $C_{\alpha+1}$ be the union of these finitely many compact sets. Since $|\bar{D}| = \aleph_2$ the induction can continue for $\alpha < \omega_1$. Since each C_α is closed and $\bar{D}$ is first countable, $\bigcup_{\alpha<\omega_1} C_\alpha$ is closed in $\bar{D}$ hence compact because X is $\aleph_1$-bounded. The open cover

$$\{\mathrm{int}\, C_\alpha : \alpha < \omega_1\}$$

has a finite subcover, hence

$$\bigcup_{\alpha<\omega_1} C_\alpha = C_\gamma \quad \text{for some } \gamma < \omega_1,$$

but of course $x_{\gamma+1} \notin C_\gamma$. Hence $|X| \leqq \aleph_1$ and since X is $\aleph_1$-compact, X is compact.

The following corollary will be used repeatedly.

COROLLARY 4.4 [CH]. *If X is first countable non-compact then X contains either a countable closed discrete set or a set of cardinality $\aleph_1$ with no complete accumulation point.*

THEOREM 4.5 [CH]. *C[first countable] = almost compact.*

Proof. Let X be a first countable space that is not almost compact. First we show that X contains either

(i) two completely separated countable closed discrete sets or

(ii) two completely separated sets of cardinality $\aleph_1$ with no complete accumulation points.

Suppose that case (i) does not hold, i.e., there are no two completely separated discrete sets. Since X is not almost compact let Z_1 and Z_2 be disjoint non-compact zero sets of X. Since case (i) does not hold, we can assume Z_1 is countably compact. By Corollary 4.4 Z_1 contains a set H of cardinality $\aleph_1$ with no complete accumulation point. We now show that Z_2 also contains a set K with $|K| = \aleph_1$ and having no complete accumulation point. We are done if Z_2 is countably compact so let N be a countable closed discrete subset of Z_2. If Z_2 is not locally compact then there is point $x \in Z_2$ and a non-compact closed neighbourhood U of x such that $U \cap N$ is finite. U must be countably compact since U and $N \backslash U$ are completely separated but case (i) does not hold. Therefore by Corollary 4.5 U contains the required set K. Now suppose Z_2 is locally compact. This means each point has a neighbourhood of cardinality less than or equal to $\aleph_1$. If $|Z_2| \geqq \aleph_2$, we inductively choose open sets C_α: $\alpha < \omega_1$ of cardinality $\aleph_1$ and points $x_\alpha \in C_\alpha$ such that

$$\overline{\bigcup_{\gamma<\alpha} C_\gamma} \subset C_\alpha \quad \text{and} \quad x_\alpha \in C_\alpha \backslash \overline{\bigcup_{\gamma<\alpha} C_\gamma}.$$

This induction can be carried out because $|Z_2| > \aleph_1$ and $|\overline{\bigcup_{\gamma<\alpha} C_\gamma}| \leqq \aleph_1$

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and Z_2 is locally compact. By the countable tightness of X and the fact that $\overline{C_\alpha} \subset C_{\alpha+1}$ we see that $\cup (C_\alpha : \alpha < \omega_1)$ is closed in Z_2. The set $\{x_\alpha : \alpha < \omega_1\}$ has no complete accumulation point in Z_2 for if $x \in \cup C_\alpha$ then $x \in C_\alpha$ for some $\alpha < \omega_1$ which meets only countably many x_γ's.

Now we assume that $|Z_2| \leqq \aleph_1$. If $|\mathrm{cl}_{\beta X} Z_2 \setminus Z_2| > 1$ then we could find disjoint $Z_3, Z_4 \in Z(X)$ both contained in Z_2 which were not compact. Of these two non-compact zero sets at least one is countably compact because case (i) does not hold and this zero set contains the necessary set K. Therefore the remaining possibility is

$$|\mathrm{cl}_{\beta X} Z_2 \setminus Z_2| = 1.$$

In this case, let

$$Z_2 = \{x_\alpha : \alpha < \omega_1\}.$$

Let, for each $\alpha < \omega_1$, W_α be a non-compact zero set of X contained in Z_2 such that

$$x_\alpha \notin W_\alpha \quad \text{and} \quad W_\alpha \subseteq \bigcap_{\gamma<\alpha} W_\gamma.$$

Clearly $\bigcap_{\alpha<\omega_1} W_\alpha = \emptyset$. Suppose there is an $\alpha < \omega_1$ such that $\bigcap_{\gamma<\alpha} W_\gamma = \emptyset$. Let

$$\{p\} = \mathrm{cl}_{\beta X} Z_2 \setminus Z_2.$$

Since $\bigcap_{\gamma<\alpha} W_\gamma = \emptyset$ and $p \in \mathrm{cl}_{\beta X} W_\gamma$ for each γ, there exists an $f \in C^*(X)$ such that

$$\beta f(p) = 0 \text{ and } Z(f) \cap Z_2 = \emptyset.$$

We can choose $y_n \in Z_2$ such that

$$|f(y_{n+1})| < \tfrac{1}{2}|f(y_n)|.$$

Therefore $\{y_n : n \in \omega\}$ is C^*-embedded in βX but

$$\aleph_1 = |\mathrm{cl}_{\beta X} Z_2| \geqq |\mathrm{cl}_{\beta X}\{y_n : n \in \omega\}| = 2^c,$$

a contradiction. Hence, for each $\alpha < \omega_1$, $\bigcap_{\gamma<\alpha} W_\gamma \neq \emptyset$. Inductively choose, for $\alpha < \omega_1$, $\xi_\alpha < \omega_1$ such that

$$\bigcap_{\gamma<\alpha} W_{\xi_\gamma} \setminus \bigcap_{\gamma\leqq\alpha} W_{\xi_\gamma} \neq \emptyset.$$

Also choose

$$y_\alpha \in \bigcap_{\gamma<\alpha} W_{\xi_\gamma} \setminus \bigcap_{\gamma\leqq\alpha} W_{\xi_\gamma}.$$

The set $\{y_\alpha : \alpha < \omega_1\}$ does not have a complete accumulation point since each x_α has $X \setminus W_\alpha$ as a neighbourhood which contains only countably many y_α's. We have finished showing that either case (i) or case (ii) holds.

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If case (i) holds then the construction in Section 2 provides a first countable space in which X is embedded but not C^*-embedded. Now suppose case (ii) holds. Let H and K be completely separated closed subsets of X of cardinality $\aleph_1$ which have no complete accumulation points. Let $\{\lambda_\alpha: \alpha < \omega_1\}$ be an indexing of the limit ordinals less than ω_1. Also let

$$H = \{h_{\lambda_\alpha+2n}: \alpha < \omega_1 \text{ and } 1 \leqq n < \omega\} \text{ and}$$
$$K = \{k_{\lambda_\alpha+2n+1}: \alpha < \omega_1 \text{ and } n \in \omega\}.$$

Since H and K are locally countable we can assume they are indexed such that there are clopen subsets of H, $\{U_\alpha: \alpha < \omega_1\}$ and clopen subsets of K, $\{V_\alpha: \alpha < \omega_1\}$ such that

$$\{h_\beta: \lambda_\alpha < \beta < \lambda_{\alpha+1}\} \subseteq U_\alpha \cap H \subseteq \{h_\beta: \beta < \lambda_{\alpha+1}\}$$

and

$$\{k_\beta: \lambda_\alpha < \beta < \lambda_{\alpha+1}\} \subseteq V_\alpha \cap K \subseteq \{k_\beta: \beta < \lambda_{\alpha+1}\}.$$

We will topologize $Y = \omega \times \omega_1 \cup X$ so that Y is first countable, H is not completely separated from K in Y and X is a topological subspace of Y.

$\omega \times \omega_1$ will be an open subset of Y and is given the product topology. For each $\alpha < \omega_1$ we will define a clopen subset C_α of $\omega \times \omega_1$ such that

(i) $C_\alpha \subseteq \omega \times \lambda_{\alpha+1}$.

(ii) $\beta < \alpha \rightarrow \exists\, n < \omega$ such that $C_\beta \cap [n, \omega) \times \lambda_{\beta+1} = C_\alpha \cap [n, \omega) \times \lambda_{\beta+1}$.

(iii) $\gamma < \lambda_{\alpha+1}$ and γ not a limit ordinal $\rightarrow \exists\, n < \omega$ such that $[n, \omega) \times \{\gamma\} \subseteq C_\alpha$.

Let $C_0 = \bigcup_{n<\omega} [n, \omega) \times \{n\}$. Let $\alpha < \omega_1$ and suppose for $\beta < \alpha$ we have defined C_β satisfying (i)–(iii).

If $\alpha = \beta + 1$ then let $C_\alpha = C_\beta \cup \bigcup_{0<n<\omega} [n, \omega) \times \{\lambda_\alpha + n\}$. Otherwise let $\{\alpha_n: n < \omega\}$ be an increasing sequence of ordinals converging to α. We let

$$C_\alpha = \bigcup_{n<\omega} [n, \omega) \times (\lambda_{\alpha_{n-1}+1}, \lambda_{\alpha_n+1}) \cap C_{\alpha_n} \cup \bigcup_{0<n<\omega} [n, \omega) \times \{\lambda_\alpha + n\}.$$

(i) and (iii) obviously hold. Suppose $\alpha_{i-1} < \beta \leqq \alpha_i$. By induction there is an $m < \omega$ such that

$$C_\beta \cap [m, \omega) \times \lambda_{\beta+1} = C_{\alpha_i} \cap [m, \omega) \times \lambda_{\beta+1}.$$

There is also a $p < \omega$ such that

$$C_{\alpha_j} \cap [p, \omega) \times \lambda_{\alpha_j+1} = C_\beta \cap [p, \omega) \times \lambda_{\alpha_j+1} \text{ for } j < i.$$

Therefore if $n = \max\{p, m, i\}$ we observe that

$$C_\beta \cap [n, \omega) \times \lambda_{\beta+1} = C_\alpha \cap [n, \omega) \times \lambda_{\beta+1}.$$

Hence (ii) holds. To see that C_α is clopen, note that for any $\beta \leqq \alpha + 1$ and any $n \in \omega$,

$$(n, \lambda_\beta) \notin \overline{C_\alpha}.$$

We can now continue to describe the topology of Y. $X \backslash (H \cup K)$ is open in Y and retains its topology. Let us define for $U \subseteq X$,

$$\hat{U} = \{\gamma < \omega_1: h_\gamma \in U\}.$$

Let $h_\gamma \in H$ and $\gamma < \lambda_\alpha + 1$. Let $\{U_n: n \in \omega\}$ be a neighbourhood base for h_γ in X such that $U_n \cap H \subseteq U_\alpha$ and is clopen in H. A neighbourhood base for h_γ in Y is

$$\{U_n \cup (C_\alpha \cap [n, \omega) \times \hat{U}_n): n < \omega\}.$$

Similarly define neighbourhoods for points in K. Condition (ii) in the definition of the C_α's implies that the above sets are in fact clopen subsets of Y.

Y is obviously first countable and completely regular. We will now show that H and K cannot be separated by disjoint closed neighbourhoods. Indeed, let U and V be neighbourhoods of H and K respectively. For each $h_\gamma \in H$ there is an $n_\gamma < \omega$ such that

$$[n_\gamma, \omega) \times \{\gamma\} \subseteq U$$

and similarly for each $k_\beta \in K$ there is an $m_\beta < \omega$ such that

$$[m_\beta, \omega) \times \{\beta\} \subseteq V.$$

Hence we can choose $n < \omega$ such that for uncountably many $h_\gamma \in H$,

$$[n, \omega) \times \{\gamma\} \subseteq U$$

and for uncountably many $k_\beta \in K$,

$$[n, \omega) \times \{\beta\} \subseteq V.$$

It is then easy to check that there is a λ_α such that $(n, \lambda_\alpha) \in \bar{U} \cap \bar{V}$.

This completes the proof because X is not C^*-embedded in Y since H and K are completely separated in X.

5. Closed sets have countable neighbourhood bases. Let Y be a Tychonoff space in which every closed subset has a countable neighbourhood base.

LEMMA 5.1. *The set of non-isolated points of Y is countably compact.*

Proof. Suppose D is a countable closed discrete subset of the non-isolated points of Y. Let $D = \{d_n: n \in \omega\}$ and choose disjoint open sets $\{V_n: n \in \omega\}$ such that for each n, $d_n \in V_n$. Suppose that $\{U_n: n \in \omega\}$ is a neighbourhood base for D. Choose $x_n \in V_n \backslash \{d_n\} \cap U_n$. Since, for each $n \in \omega$, $V_n \cap \{x_i: i \in \omega\} = \{x_n\}$ we see that

$$\overline{\{x_i: i \in \omega\}} \cap D = \emptyset.$$

Therefore $X \backslash \overline{\{x_i: i \in \omega\}}$ is a neighbourhood of D but it does not contain any U_n. This contradicts the fact that D has a countable neighbourhood base.

Lemma 5.1 and the following lemma have been proven by C. E. Aull [**3**].

LEMMA 5.2. *Y is normal.*

Proof. Let H and K be disjoint closed subsets of Y. Let

$$H_1 = H \cap \text{isolated points of } Y \text{ and}$$
$$K_1 = K \cap \text{isolated points of } Y.$$

$H \backslash H_1$ is a closed subset of Y so let $\{U_n: n \in \omega\}$ be a shrinking neighbourhood base for $H \backslash H_1$. Suppose that

$$y \in (\cap \bar{U}_n) \backslash (H \backslash H_1)$$

and let $\{W_n: n \in \omega\}$ be a neighbourhood base at y. Choose

$$y_n \in W_n \cap U_n \backslash (H \backslash H_1).$$

y_n can be chosen so that it is not in $H \backslash H_1$ because $H \backslash H_1$ is closed and $y \notin H \backslash H_1$. $\{y\} \cup \{y_n: n \in \omega\}$ is closed and is disjoint from $H \backslash H_1$, but $Y \backslash \overline{\{y_n: n \in \omega\}}$ does not contain any U_n.

This contradiction means that $\bigcap_{n \in \omega} \bar{U}_n = H \backslash H_1$. Since $K \backslash K_1$ is disjoint from $H \backslash H_1$ we see that

$$K \backslash K_1 \cap \bigcap_{n \in \omega} \bar{U}_n = \emptyset.$$

$K \backslash K_1$ is countably compact by Lemma 5.1 so there is an $n \in \omega$ such that

$$\bar{U}_n \cap K \backslash K_1 = \emptyset.$$

Therefore $H_1 \cup \bar{U}_n \backslash K_1$ is a closed neighbourhood of H which does not intersect K. Hence Y is normal.

THEOREM 5.3. *C[closed sets have countable neighbourhood bases] = countably compact.*

Proof. First let X be a space in which closed sets have countable neighbourhood bases and X is countably compact. Let X be embedded in Y, where closed subsets of Y have countable neighbourhood bases; then Y

is normal by Lemma 5.2. X is closed in Y since X is countably compact and Y is first countable. Hence X is C^*-embedded [**4**, 3*D*].

Conversely, suppose X is not countably compact. Let D be a closed countable discrete subset of X. By Lemma 5.1 we can choose D to consist of isolated points. Therefore D is clopen in X. We can construct a space $Y = X \cup \{\mathscr{P}\}$ where neighbourhoods of $\mathscr{P}$ are $\{\mathscr{P}\}$ union cofinite subsets of D. It is easily seen that X is not C^*-embedded in Y and that closed subsets of Y have countable neighbourhood bases.

Remark. Countably compact spaces which have the property that closed sets have countable neighbourhood bases are also normal. W. Weiss [**7**] has shown that every countably compact perfectly normal space is compact if we assume Martin's Axiom plus the negation of the continuum hypothesis and thus is consistent with the usual axioms of set theory. On the other hand, Ostaszewski [**6**] constructs space θ, which is perfectly normal, countably compact and not compact. The construction of θ also requires special set theoretic assumptions. It is easily shown that the product of θ with the one-point compactification of the integers is a countably compact space which has the property that closed sets have countable neighbourhood bases and this space has infinite growth in its Stone–Čech compactification.

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