Note on Dual Symmetric Functions

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§ 1. In an earlier paper, which this note is intended to supplement and in some respects improve, the writer gave a general theorem of duality relating to isobaric determinants with elements C_r and H_r, the elementary and the complete homogeneous symmetric functions of a set of variables. The result was shewn to include as special cases the dual forms of "bi-alternant" symmetric functions given by Jacobi[1] and Naegelsbach, as well as two equivalent forms of isobaric determinant used by MacMahon[2] as a generating function in an important problem of permutations.

The mode of proof employed in the earlier paper was gradational. It has since been pointed out to me, however, by Professor H. W. Turnbull that material for a simple direct proof is available in Jacobi's theorem on the minors of the adjugate determinant. Such a proof is given below, and certain corollaries deduced.

§ 2. We first define "bicomplementary sets" of indices. Consider for example the sets [1, 4] and [1, 2, 4] in relation to the complete set [0, 1, 2, 3, 4]. The indices not in [1, 4] are 0, 2, 3, and the defects of these from the highest index 4 reversed, give the set [1, 2, 4]. Such sets will be termed dual or bicomplementary. Clearly the relation is a reciprocal one.

Theorem. *An isobaric determinant with elements H_r is identically equal in value to another with elements C_r, provided that the suffixes of first rows, and also of last columns, in the respective determinants form bicomplementary sets.*

Proof. In virtue of the well-known joint recurrence-relations,

$$0 = C_0 H_r - C_1 H_{r-1} + C_2 H_{r-2} - \dots + (-)^r C_r H_0,$$

the matrices

$$H = \begin{bmatrix} H_0 & H_1 & H_2 & H_3 & \dots \\ . & H_0 & H_1 & H_2 & \dots \\ . & . & H_0 & H_1 & \dots \\ . & . & . & H_0 & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{bmatrix}, \quad C = \begin{bmatrix} C_0 & -C_1 & C_2 & -C_3 & \dots \\ . & C_0 & -C_1 & C_2 & \dots \\ . & . & C_0 & -C_1 & \dots \\ . & . & . & C_0 & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{bmatrix}$$

[1] See Muir's *History of Determinants*, 1, 341, 2, 146.

[2] *Combinatory Analysis* (Cambridge, 1915), 1, 205.

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are reciprocal matrices, and so, since $C_0 = H_0 = 1$, the determinants

$$|H|_t = \begin{vmatrix} H_0 & H_1 & H_2 & H_3 & \dots \\ . & H_0 & H_1 & H_2 & \dots \\ . & . & H_0 & H_1 & \dots \\ . & . & . & . & \dots \\ . & . & . & . & \dots \end{vmatrix}_t \quad \text{and } |C|_t = \begin{vmatrix} C_0 & . & . & \dots \\ C_1 & C_0 & . & \dots \\ C_2 & C_1 & C_0 & \dots \\ C_3 & C_2 & C_1 & \dots \\ . & . & . & \dots \\ . & . & . & \dots \end{vmatrix}_t$$

are mutually adjugate. (We here employ an adjugate with minors instead of co-factors for elements, as is permissible.) If a set of columns in $|H|$ is specified by the set of suffixes of H's in its *first* row, the complementary columns in $|C|$, if specified by the C's in its *last* row, give a set bicomplementary to the first. This is evident by inspection. Similarly for complementary *rows* in the determinants.

Now any isobaric determinant of H's with an element of highest weight, H_t, is a minor of $|H|_t$, and the elements of its first row and last column are in the first row and last column of $|H|_t$. By Jacobi's theorem on the adjugate such a minor is equal to the complementary minor in $|C|_t$, and this, by our observation above, is specified as to last row and first column by the rule of bicomplementary sets. A double reversal of order, in rows and columns, of the C-determinant now yields the desired result in the form enunciated. We may display it by the example

$$\begin{vmatrix} C_1 & C_2 & C_4 & C_5 \\ C_0 & C_1 & C_3 & C_4 \\ 0 & C_0 & C_2 & C_3 \\ 0 & 0 & C_0 & C_1 \end{vmatrix} = \begin{vmatrix} H_2 & H_5 \\ H_0 & H_3 \end{vmatrix},$$

where the sets [1, 2, 4, 5], [2, 5] and [1, 3, 4, 5], [3, 5] are bicomplementary pairs.

This dual representation is of great value. In all theoretical work with symmetric functions, for example in the construction of generating functions in combinatory analysis, it is the H-functions which arise most naturally; in practical evaluation, however, the C-functions are usually more suitable, since they have far fewer terms and vanish for orders higher than the number of constituent variables. It is often expedient, therefore, to solve problems by H-determinants, transforming them finally into C-determinants.

§ 3. *Relation between Partitions, Compositions and Bicomplementary Sets.*

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The notion of bicomplementary sets includes in a very simple manner that of conjugate partitions[1] of integers and also MacMahon's conjugate "compositions." For by examining on the one hand diagrams, such as those appended, of conjugate partitions and

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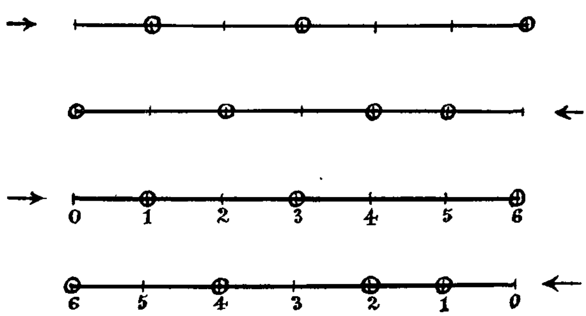
I. Conjugate Partitions (124), (1123) of 7.
II. Conjugate Compositions {123}, {1122} of 6.

conjugate compositions, and on the other hand the alternative representation of compositions and a similar one of bicomplementary sets, we remark by inspection:

If $(p, q, r, ..)$, $(p', q', r', ..)$ *are conjugate partitions, then* $\{p, q+1-p, r+1-q, ..\}$ *and* $\{p', q'+1-p', r'+1-q', ..\}$ *are conjugate compositions, and* $[p, q+1, r+2, ..]$ *and* $[p', q'+1, r'+2, ..]$ *are bicomplementary sets.*



III. Conjugate Compositions {123}, {1122}.
IV. Bicomplementary Sets [1, 3, 6], [1, 2, 4, 6].

This observation, combined with the theorem of duality, leads at once to the conclusion that in the Jacobi-Naegelsbach bi-alternant the bicomplementary property of first rows and last columns implies that the *diagonal suffixes of dual bi-alternants form conjugate partitions,* as was first observed[2] by Kostka; it shews equally that in the

[1] It has seemed more convenient to use *ascending* partitions, and to reverse the order in conjugate compositions from MacMahon's; the triple formulation is then simple and consistent.

[2] *J. für Math.*, 132 (1907), 159, 161.

MacMahon dual determinants (where the sub-diagonal elements are always H_0, or C_0) the same property implies that the *diagonal suffixes form conjugate compositions.* We may bring this under view by two examples.

(i) Jacobi-Naegelsbach.

$$\begin{vmatrix} H_1 & H_3 & H_6 \\ H_0 & H_2 & H_5 \\ 0 & H_1 & H_4 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_4 & C_6 \\ C_0 & C_1 & C_3 & C_5 \\ 0 & C_0 & C_2 & C_4 \\ 0 & 0 & C_1 & C_3 \end{vmatrix}.$$

(ii) MacMahon.

$$\begin{vmatrix} H_1 & H_3 & H_6 \\ H_0 & H_2 & H_5 \\ 0 & H_0 & H_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_4 & C_6 \\ C_0 & C_1 & C_3 & C_5 \\ 0 & C_0 & C_2 & C_4 \\ 0 & 0 & C_0 & C_2 \end{vmatrix}.$$

The application of the theorem of duality to the factorising of isobaric determinants was mentioned and used in the former paper, and a new diagrammatic representation of double bicomplementary sets introduced.

We may remark in conclusion that just as H and C in § 2 are reciprocal matrices, so all their compounds $H^{(m)}$ and $C^{(m)}$ must be reciprocal. There exist therefore compound Wronski recurrence-relations, and a theory of compound determinants of symmetric functions with an analogous principle of duality for each system. The nature of the results can be inferred, however, from reflections of a general kind, and there seems to be no necessity for detailed investigation.

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