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SYLOW NORMALIZERS WITH A NORMAL SYLOW 2-SUBGROUP

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Abstract If G is a finite solvable group and p is a prime, then the normalizer of a Sylow p-subgroup has a normal Sylow 2-subgroup if and only if all non-trivial irreducible real 2-Brauer characters of G have degree divisible by p.

Keywords: Brauer characters; Sylow normalizers; real characters; solvable groups

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1. Introduction

Let G be a finite group and let p be any prime. One of the main problems of the representation theory of finite groups is to relate the local properties of G (those of the so-called 'local subgroups' of G) with the properties of the group G. The most important local subgroup of G is the normalizer $\boldsymbol{N}_G(P)$ of a Sylow p-subgroup P of G. In particular, it is very relevant to study how the structure of $\boldsymbol{N}_G(P)$ determines (and is determined by) global properties of G.

In this note, we study when $\boldsymbol{N}_G(P)$ has a normal Sylow 2-subgroup and how this affects some of the representation theory of G.

Theorem A. *Let G be a finite solvable group, let p be any prime and let $P \in \mathrm{Syl}_p(G)$. Then $\boldsymbol{N}_G(P)$ has a normal Sylow 2-subgroup if and only if all the irreducible non-trivial 2-Brauer real characters of G have degree divisible by p.*

If $p = 2$, Theorem A is true for every finite group, and it is a restatement of a celebrated result of Fong on odd-degree real-valued 2-Brauer characters (see, for example, [**3**, Theorem 2.30]). However, for odd primes, it is unfortunate that Theorem A is not true for every finite group. For $p = 3$, all the irreducible non-trivial 2-Brauer real characters of M_{23} have degree divisible by 3, but on the other hand, its Sylow 3-normalizer does not have a normal Sylow 2-subgroup. (We would like to mention that this is the only counterexample that we know of for the 'if' direction in Theorem A.) For the 'only if' direction, $G = \mathrm{PSL}_2(7)$ with $p = 7$ has an odd 7-Sylow normalizer; but it does have a non-trivial 2-Brauer irreducible real character of degree 8.

In character theory, it is always pleasant to find new properties of the group that can be read off from its character table. For a solvable group G, Theorem A shows that the character table of G determines if its Sylow normalizers have normal Sylow 2-subgroups. We do not know whether this is true for every finite group.

Corollary B. *Let G be a finite solvable group and let p be any prime. Then the character table of G determines if the normalizer of a Sylow p-subgroup of G has a normal Sylow 2-subgroup.*

Proof. As an application of the Fong–Swan theorem (see Theorem 10.1 and Corollary 10.4 of [**3**]), we know that the character table of G determines the 2-Brauer modular table of G. Now, we apply Theorem A. $\square$

Is there a similar condition that determines if a Sylow normalizer of a finite group G has a normal Sylow q-subgroup for any odd prime q? This seems to be a natural question for which we do not know an answer at this time.

2. Proof of Theorem A

Our notation for characters is standard [**2**]. We begin with an extension lemma for ordinary characters.

Lemma 2.1. *Let p be a prime. Suppose that K and L are normal in G, where K/L is a p'-group, and G/K is a p-group. Let $L \subseteq H \subseteq G$ with $KH = G$ and $K \cap H = L$. Suppose that $\theta \in \mathrm{Irr}(K)$ is H-invariant, and that $\phi \in \mathrm{Irr}(L)$ is an irreducible H-invariant constituent of θ_L. Then θ extends to G if and only if ϕ extends to H.*

Proof. Suppose first that $\chi_K = \theta$ for some $\chi \in \mathrm{Irr}(G)$. Thus $[\chi_L, \phi] = [\theta_L, \phi]$ is not divisible by p by Corollary 11.29 of [**2**]. Now

$$[\chi_L, \phi] = \sum_{\xi \in \mathrm{Irr}(H|\phi)} [\chi_H, \xi][\xi_L, \phi],$$

and we deduce that there exists $\xi \in \mathrm{Irr}(H \mid \phi)$ such that $[\xi_L, \phi] = \xi(1)/\phi(1)$ is not divisible by p. By Corollary 11.29 of [**2**], we have that $\xi_L = \phi$.

Assume, conversely, that $\xi_L = \phi$ for some $\xi \in \mathrm{Irr}(H)$. Then

$$[\phi, \theta_L] = [\phi^K, \theta] = [(\xi_L)^K, \theta] = [(\xi^G)_K, \theta]$$

is not divisible by p, and we deduce that there exists some χ over θ such that $[\chi_K, \theta] = \chi(1)/\theta(1)$ is not divisible by p. By Corollary 11.29 of [**2**], we see that $\chi_K = \theta$. $\square$

For the rest of this paper, $\mathrm{IBr}(G)$ denotes a set of irreducible 2-Brauer characters of G. By the Fong–Swan theorem [**3**, Theorem 10.1], this set is canonically defined if G is solvable (see [**3**, Corollary 10.4]).

We shall need the following two elementary results, which we have recently proved, and which we restate for the reader's convenience.

Lemma 2.2. *Suppose that G/N has odd order. If $\theta \in \mathrm{IBr}(N)$ is real valued and G-invariant, then θ has a unique real-valued extension $\phi \in \mathrm{IBr}(G \mid \theta)$.*

Proof. This is Lemma 3.1 of [**5**]. $\square$

Lemma 2.3. *Let G be a finite group. Then G has no non-trivial real irreducible 2-Brauer characters if and only if G has a normal Sylow 2-subgroup.*

Proof. This is Lemma 2.1 of [**4**]. $\square$

From now on, we fix p to be an odd prime, and we write $\mathrm{IBr}_{\mathrm{rv},p'}(G)$ for the subset of irreducible real 2-Brauer characters of G of degree not divisible by p.

Lemma 2.4. *Suppose that G has a normal Sylow p-subgroup, where p is odd. If G is solvable, then $|\mathrm{IBr}_{\mathrm{rv},p'}(G)| = 1$ if and only if G has a normal Sylow 2-subgroup.*

Proof. First, if G has a normal Sylow 2-subgroup, then G does not have non-trivial real irreducible Brauer characters by Lemma 2.3, and the lemma follows. So assume that P is the normal Sylow p-subgroup of G and assume that $|\mathrm{IBr}_{\mathrm{rv},p'}(G)| = 1$. First notice that G/P has no non-trivial real-valued Brauer characters because every $\phi \in \mathrm{IBr}(G/P)$ has p'-degree (because, since G is solvable, every $\phi \in \mathrm{IBr}(G/P)$ has degree dividing $|G/P|$). By Lemma 2.3, we have that G/P has a normal Sylow 2-subgroup QP/P, where $Q \in \mathrm{Syl}_2(G)$. We claim that $[P, Q] = 1$. By coprime action, it suffices to show that $[P/\Phi(P), Q] = 1$. Otherwise, let $x \in Q$ such that $x\boldsymbol{C}_Q(P/\Phi(P))$ has order 2. Then, for every $\mu \in \mathrm{Irr}(P/\Phi(P))$, we have that x inverts $\mu^{-1}\mu^x$. Hence, x inverts some $1 \neq \lambda \in \mathrm{Irr}(P/\Phi(P))$. Now, if $\hat{\lambda}$ is the canonical extension of λ to its stabilizer T in G (by Corollary 8.16 of [**2**]), we have that $\hat{\lambda}$ is also inverted by x by the uniqueness of the canonical extension. Finally, since $\lambda \in \mathrm{IBr}(P)$, it follows that $\tau = \hat{\lambda}^0 \in \mathrm{IBr}(T)$ (because it extends λ). By the Clifford correspondence for Brauer characters, we have that $\phi = \tau^G \in \mathrm{IBr}(G)$. Now, ϕ has p'-degree, and

$$\bar{\phi} = (\bar{\tau})^G = (\tau^x)^G = \tau^G = \phi,$$

and we conclude that $1 \neq \phi$ is real valued of p'-degree. $\square$

In solvable groups, every irreducible Brauer character lifts to an ordinary character, by the Fong–Swan theorem. In fact, this lifting can be done in a canonical way. (For odd primes p, this is the well-known p-rational Isaacs lifting. The case $p = 2$, precisely the case that is needed here, is not that well known.) In [**1**], Isaacs proved that for every prime p and every p-solvable group G there is a canonical subset $B_{p'}(G)$ of $\mathrm{Irr}(G)$ such that restriction to p-regular elements defines a bijection $\chi \mapsto \chi^0$ from $B_{p'}(G) \to \mathrm{IBr}(G)$. (See [2] for all the details.) This canonical subset $B_{p'}(G)$ is closed under Galois action and under automorphisms of G. We shall also use the fact that normal irreducible constituents of $B_{p'}$-characters are $B_{p'}$-characters (again, see [**1**]). We use $B_{2'}$-characters in the following key lemma.

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Lemma 2.5. *Let G be a solvable group and let p be an odd prime. Let $K = \boldsymbol{O}^{p',p}(G)$ and let K/L be a chief factor of G. Let $H = L\boldsymbol{N}_G(P)$, where $P \in \mathrm{Syl}_p(G)$. Assume that G/K has a normal Sylow 2-subgroup. Then $|\mathrm{IBr}_{\mathrm{rv},p'}(G)| = 1$ if and only if $|\mathrm{IBr}_{\mathrm{rv},p'}(H)| = 1$.*

Proof. By elementary group theory, we have that $KH = G$ and $K \cap H = L$. Also, $KP \lhd G$ and $\boldsymbol{C}_{K/L}(P) = 1$. Suppose that $|\mathrm{IBr}_{\mathrm{rv},p'}(H)| = 1$ and let $\phi \in \mathrm{IBr}(G)$ be real valued of p'-degree. Let $\delta \in \mathrm{IBr}(KP)$ under ϕ so that $\bar{\delta} = \delta^x$ for some $x \in H$. Now, x normalizes the stabilizer of δ, x^2 stabilizes δ and therefore $x_{2'}$ stabilizes δ. So we may assume that x is a 2-element. Now δ has p'-degree, so $\delta_K = \theta \in \mathrm{IBr}(K)$ by Theorem 8.30 of [**3**] and Clifford's theorem, for instance. Now, let $\nu \in \mathrm{Irr}(KP)$ be the canonical $B_{2'}$-lift of δ. By uniqueness of the $B_{2'}$-lifting, we have that $\nu^x = \bar{\nu}$. (This is because $\bar{\nu}^x$ and ν are $B_{2'}$-characters lifting δ.) Also, ν has p'-degree and $\nu_K = \mu$ is the canonical lift of θ. Now, by Problem 13.4 of [**2**], let $\xi \in \mathrm{Irr}(L)$ be the unique P-invariant constituent of μ_L. Note that $\xi \in B_{2'}(L)$. We claim that $\bar{\xi} = \xi^x$. Notice that $\xi^{xy} = \xi^{yx} = \xi^x$ for every $y \in P$ because 2-elements and p-elements of H commute modulo L. (Recall that H/L has a normal Sylow 2-subgroup by hypothesis.) Hence $\bar{\xi}^x$ is a P-invariant constituent of μ_L (because $\bar{\mu}^x = \mu$). Now, by uniqueness, we conclude that $\bar{\xi} = \xi^x$. Since μ extends to KP, we have that ξ extends to LP, by Lemma 2.1. Also, it has an odd number of extensions by Gallagher's corollary [**2**, Corollary 6.17]. Now, let $\tau \in \mathrm{Irr}(KP)$ be any extension. Then $\bar{\tau}^x = \lambda\tau$ for some $\lambda \in \mathrm{Irr}(LP/L)$ by Gallagher. Notice that $\lambda^x = \lambda$, again because H/L has a normal Sylow 2-subgroup. It easily follows that $\tau^{x^2} = \tau$. This implies that the map $\psi \mapsto \bar{\psi}^x$ is a permutation of the set of the extensions of ξ to LP of order dividing 2. It follows that there is an extension $\tau \in \mathrm{Irr}(LP)$ of ξ, such that $\tau^x = \bar{\tau}$. Since LP/L is a $2'$-group and $\xi \in B_{2'}(L)$, it follows that $\tau \in B_{2'}(LP)$ by Theorem 7.1 of [**1**]. Thus, we have found $\rho = \tau^0 \in \mathrm{IBr}(LP)$ of p'-degree such that $\rho^x = \bar{\rho}$. Now, H/LP has a normal Sylow 2-subgroup Q/LP. Since $\mathrm{IBr}(Q \mid \rho) = \{\epsilon\}$ by Green's theorem [**3**, Theorem 8.11], it follows that $\epsilon^x = \bar{\epsilon}$. Since $x \in Q$, it follows that ϵ is real valued. Now, H/Q has odd order. By using Lemma 2.2 and the Clifford correspondence, we conclude that there is a real-valued Brauer character φ over ϵ that necessarily has p'-degree. By hypothesis, $\varphi = 1$. Hence, ϵ and ρ are also trivial. This implies that $\tau = 1$ by the uniqueness of the canonical lifting, and therefore $\xi = 1$. Now, by Problem 13.10 of [**2**], we conclude that $\mu = 1$ (again, by uniqueness). Thus $\phi \in \mathrm{IBr}(G/K)$. Since G/K has a normal Sylow 2-subgroup, it follows that $\phi = 1$, by Lemma 2.4.

Conversely, assume that $|\mathrm{IBr}_{\mathrm{rv},p'}(G)| = 1$. Since this direction is analogous to the other one, we just sketch the proof. Suppose that $\phi \in \mathrm{IBr}(H)$ is real valued of p'-degree. Let $\delta \in \mathrm{IBr}(LP)$ under ϕ, so that $\bar{\delta} = \delta^x$ for some 2-element in H. Let $\nu \in B_{2'}(LP)$ be the canonical lift of δ. Let $\mu = \nu_L$ and let $\xi \in \mathrm{Irr}(K)$ be the unique P-invariant over μ by Problem 13.10 of [**2**]. First of all, we claim that $\xi \in B_{2'}(K)$. We know that K/L is an elementary abelian q-group. If q is odd, then every irreducible character over $\mu \in B_{2'}(L)$ is in $B_{2'}(K)$. If $q = 2$, then there exists a unique $\psi \in B_{2'}(K)$ over μ (by Theorem 6.2.b of [**1**]). By uniqueness, ψ is P-invariant (because μ is P-invariant). Hence $\xi = \psi \in B_{2'}(K)$ as claimed. Also by uniqueness (and using the fact that G/K has a normal Sylow 2-subgroup), we have that $\bar{\xi} = \xi^x$, that ξ extends to KP by Lemma 2.1

and that some extension $\tau \in B_{2'}(KP)$ satisfies $\tau^x = \bar{\tau}$. Let $\rho = \tau^0 \in \mathrm{IBr}(KP)$. Now G/KP has a normal Sylow 2-subgroup V/KP and $\mathrm{IBr}(V \mid \rho) = \{\epsilon\}$. By uniqueness, $\bar{\epsilon} = \epsilon^x$. Since $x \in V$, we conclude that ϵ is real valued. So there is a real-valued irreducible Brauer character of G over ϵ with p'-degree. Hence $\epsilon = 1$ and $\mu = 1$. Now, $\phi \in \mathrm{IBr}(H/L)$ and we deduce that $\phi = 1$ by Lemma 2.4. □

We are finally ready to prove our main result.

Theorem 2.6. *Let G be a finite solvable group, let p be any prime and let $P \in \mathrm{Syl}_p(G)$. Then $\boldsymbol{N}_G(P)$ has a normal Sylow 2-subgroup if and only if all the irreducible non-trivial 2-Brauer real characters of G have degree divisible by p.*

Proof. By Fong's theorem [**3**, Theorem 2.30], we may assume that p is odd. We argue by induction on $|G|$.

Let $K = \boldsymbol{O}^{p',p}(G)$. If $K = 1$, then the theorem follows from Lemma 2.4. So we may assume that $K > 1$. Let K/L be a chief factor of G and let $H = L\boldsymbol{N}_G(P)$. As in Lemma 2.5, we know that $H < G$. In order to prove both directions, notice that we may always assume that G/K has a normal Sylow 2-subgroup.

If $\boldsymbol{N}_G(P)$ has a normal Sylow 2-subgroup, then we have that $|\mathrm{IBr}_{\mathrm{rv},p'}(H)| = 1$ by induction. Hence $|\mathrm{IBr}_{\mathrm{rv},p'}(G)| = 1$ by Lemma 2.5. Conversely, if $|\mathrm{IBr}_{\mathrm{rv},p'}(G)| = 1$, then $|\mathrm{IBr}_{\mathrm{rv},p'}(H)| = 1$ and the theorem follows by induction. □

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