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The equivalence of some variational problems for surfaces of prescribed mean curvature

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One method of finding non-parametric hypersurfaces of prescribed mean curvature which span a given curve in R^n is to find a function which minimizes a particular integral amongst all smooth functions satisfying certain boundary conditions. A new problem can be considered by changing the integral slightly and then minimizing over a larger class of functions. It is possible to show that a solution to this new problem exists under very general conditions and it is usually known as the generalized solution. In this paper we show that the two problems are equivalent in the sense that the least value for the original minimization problem and the generalized problem are the same even though no solution may exist. The case where the surfaces are constrained to lie above an obstacle is also considered.

Introduction

Let Ω be a bounded domain in R^n, $n \geq 2$, with locally Lipschitz boundary $\partial\Omega$. Let $H(x, t)$ be a real valued function on $\Omega \times R$ which is increasing in t and such that $H(x, t_0) \in L^n(\Omega)$ for each fixed t_0.

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Let φ be a given function in $L^1(\partial\Omega)$.

Then it is a problem of classical interest to find a function which minimizes the integral

$$M(v) = \int_\Omega \sqrt{1+|\nabla v|^2}\, dx + \int_\Omega \int_0^{v(x)} H(x, t)\,dt\,dx$$

in the set $C = \{v \in W^{1,1}(\Omega) : \text{trace } v = \varphi \text{ on } \partial\Omega\}$. If a solution u exists then it will define a surface with mean curvature $\frac{1}{n} H\big(x, u(x)\big)$ at each point x and with boundary determined by φ . However it is well known that, unless some further restrictions are placed on φ , $H(x, t)$, and $\partial\Omega$ (for example see Serrin [12]), it cannot be expected that a solution exists. Indeed examples have been constructed where no solution exists (see Santi [11], Giaquinta, Souček [5]). For this reason there has been great interest in the last few years in the notion of generalised solution (see Temam [14], Gerhardt [3], and the references found in this article). One of the main ideas in these generalisations consists in replacing $M(v)$ with

$$M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1} ,$$

where H_{n-1} is $(n-1)$-dimensional Hausdorff measure in R^n , and replacing the set C with the space $BV(\Omega)$. Here $BV(\Omega)$, the space of functions of bounded variation in Ω , denotes the set of functions in $L^1(\Omega)$ whose distributional derivatives are measures with finite total variation. Of course $M(v)$ must now be interpreted in a measure-theoretic sense and we must define trace v in a suitable way. (For a discussion of these points see Giusti [6, 7] or Miranda [9, 10].) In this new formulation it is possible to prove that a solution exists, provided some relatively mild restrictions are placed on $H(x, t)$ but none are needed on φ and $\partial\Omega$ (see Gerhardt [3]). We cannot, of course, expect that the solution so obtained will satisfy the boundary conditions, as in that case it would have to be a solution for our original problem. However, the generalised solution is still of great interest and numerous papers have been written in the last few years about its properties.

In this paper we show that

$$\text{(1)} \qquad \inf_{C} M(v) = \inf_{BV(\Omega)} \left\{ M(v) + \int_{\partial\Omega} |v-\varphi| dH_{n-1} \right\}$$

under the assumptions mentioned at the start. This equality holds true even if no solution exists to one or both of the problems. In fact we prove a slightly stronger version of this result by introducing an obstacle as well. Equation (1) is well known in the case where $\partial\Omega$ is C^2 and φ is Lipschitz continuous (see Gerhardt [2]) but relaxing the assumptions on Ω and φ makes the problem a good deal more complicated. The result is interesting in itself but has some even more important applications which use the fact that any minimizing sequence in C must also be a minimizing sequence in $BV(\Omega)$. However the functions in C are restricted on the boundary by the assumption $v = \varphi$ on $\partial\Omega$. Hence if we wish to examine the boundary behaviour of a generalised solution it is more productive to approach it by a sequence in C rather than one only in $BV(\Omega)$. This technique is used in the papers by Simon [13] and Williams [15].

This paper is split into two sections. The first section deals with some preliminary results required in the proof of equation (1) and the second section contains the main proof. There are two main results of Section 1 both of which generalize known results. The first result proves that φ may be extended in a suitably nice way by using the techniques of Gagliardo [1] and, in particular, it shows there is at least one function in C such that $M(v) < \infty$. The second theorem extends the work of Massari and Pepe [8] and is concerned with the approximation of Ω with smooth sets.

1. Preliminary results

Throughout the paper we need many results and theorems about $BV(\Omega)$ functions. All these results may be found in the papers of Giusti [6, 7] and Miranda [9, 10].

DEFINITION. $BV(\Omega) = \left\{ v \in L^1(\Omega) : \int_\Omega |Dv| < \infty \right\}$ where

$$\int_\Omega |Dv| = \sup\left\{ \int_\Omega v \operatorname{div} g dx : g \in C_0^1(\Omega; \mathbb{R}^n) \text{ and } |g(x)| \le 1 \text{ for } x \in \Omega \right\} .$$

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With this definition it is possible to show that the distributional derivatives of v are measures and, if $\partial\Omega$ is locally Lipschitz, it is possible to define the trace of v on $\partial\Omega$.

The following three lemmas will be useful in what follows.

LEMMA 1. *Suppose* Ω *is a bounded open set in* R^n *with locally Lipschitz boundary* $\partial\Omega$ *and* $\{\Omega_k\}$ *is a sequence of bounded open sets with locally Lipschitz boundaries which are all contained in* Ω *and such that* $|\Omega-\Omega_k| \to 0$ *as* $k \to \infty$. *Let* f *be a given positive function in* $BV(\Omega)$. *Then*

$$\int_{\partial\Omega} f dH_{n-1} \le \liminf_{k\to\infty} \int_{\partial\Omega_k} f dH_{n-1} .$$

Proof. Define F and F_k in $BV(R^n)$ by

$$F(x) = \begin{cases} f(x) , & x \in \Omega , \\ 0 , & \text{otherwise,} \end{cases}$$

$$F_k(x) = \begin{cases} f(x) , & x \in \Omega_k , \\ 0 , & \text{otherwise.} \end{cases}$$

Then $F_k \to F$ in $L^1(R^n)$, and so

$$\int |DF| \le \liminf_{k\to\infty} \int |DF_k| .$$

Now

$$\int |DF| = \int_\Omega |Df| + \int_{\partial\Omega} f dH_{n-1} ,$$

$$\int |DF_k| = \int_{\Omega_k} |Df| + \int_{\partial\Omega_k} f dH_{n-1}$$

$$\le \int_\Omega |Df| + \int_{\partial\Omega_k} f dH_{n-1} ,$$

and the result follows. □

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LEMMA 2 (Co-area formula). *Let* $f \in L^1(R^n)$ *and* g *be Lipschitz continuous on* R^n. *Then the following formula holds:*

$$\int f|Dg| = \int_{-\infty}^{\infty} dt \int f|D\varphi_{G_t}| , \tag{3}$$

where φ_{G_t} *is the characteristic function of the set*

$$G_t = \{x \in R^n : g(x) < t\} . \qquad \square$$

LEMMA 3. *Let* Ω *be a bounded open set in* R^n *with locally Lipschitz boundary* $\partial\Omega$, *and suppose* g *is a given function in* $BV(R^n)$ *such that* $\int_{\partial\Omega} |Dg| = 0$ *and* $g \geq 0$. *Then*

$$\begin{aligned} \int_{\partial\Omega} g dH_{n-1} = \int g|D\varphi_\Omega| = \lim_{\varepsilon\to 0} \int g_\varepsilon |D\varphi_\Omega| = \lim_{\varepsilon\to 0} \int_{\partial\Omega} g_\varepsilon dH_{n-1} \\ \geq \lim_{\varepsilon\to 0} \int g|D\psi_\varepsilon|dx , \end{aligned} \tag{4}$$

where φ_Ω *is the characteristic function of* Ω, ψ_ε *is the mollification of* φ_Ω, *and* g_ε *is the mollification of* g.

Proof. The equations

$$\int_{\partial\Omega} g dH_{n-1} = \int g|D\varphi_\Omega| \quad \text{and} \quad \int_{\partial\Omega} g_\varepsilon dH_{n-1} = \int g_\varepsilon |D\varphi_\Omega|$$

are simple consequences of the fact that $\partial\Omega$ is locally Lipschitz (for example, see Giusti [7]).

By well known properties of mollifiers

$$g_\varepsilon \to g \quad \text{in} \quad L^1_{\text{loc}}(R^n) ,$$

$$\int |Dg_\varepsilon| \to \int |Dg| ,$$

and hence, as $\int_{\partial\Omega} |Dg| = 0$,

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$$\int_\Omega |Dg_\varepsilon| \to \int_\Omega |Dg| .$$

Thus, by Theorem 2.11 of [7],

$$\int_{\partial\Omega} g_\varepsilon dH_{n-1} \to \int_{\partial\Omega} g dH_{n-1} .$$

We only now need to show

$$\lim_{\varepsilon\to 0} \int g|D\psi_\varepsilon|dx \le \int g|D\varphi_\Omega| .$$

We assume τ is a positive symmetric mollifier and

$$\psi_\varepsilon(x) = \int_\Omega \tau_\varepsilon(x-y)dy = \varepsilon^{-n} \int_\Omega \tau\left(\frac{x-y}{\varepsilon}\right)dy .$$

Thus

$$\begin{aligned} |D\psi_\varepsilon(x)| &= \left|\int_\Omega \frac{\partial\tau_\varepsilon}{\partial x_i} (x-y)dy\right| \\ &= \left|\int_\Omega \frac{\partial\tau_\varepsilon}{\partial y_i} (x-y)dy\right| \\ &= \left|\int_{\partial\Omega} \nu_i(y)\tau_\varepsilon(x-y)dH_{n-1}(y)\right| , \end{aligned}$$

where $\nu(y)$ is the normal to $\partial\Omega$ at y. Hence

$$|D\psi_\varepsilon(x)| \le \int_{\partial\Omega} \tau_\varepsilon(x-y)dH_{n-1}(y) ,$$

and so

$$\begin{aligned} \int g|D\psi_\varepsilon|dx &\le \int g(x)\left\{\int_{\partial\Omega} \tau_\varepsilon(x-y)dH_{n-1}(y)\right\}dx \\ &= \int_{\partial\Omega} \left\{\int g(x)\tau_\varepsilon(x-y)dx\right\}dH_{n-1}(y) \\ &= \int_{\partial\Omega} g_\varepsilon dH_{n-1}(y) \\ &= \int g_\varepsilon|D\varphi_\Omega| . \qquad \square \end{aligned}$$

We now prove that the function φ may be extended to the whole of Ω in such a way that the extension has several nice properties. The proof of the theorem is basically due to Gagliardo [1] with a few minor changes to ensure that the extra conditions are fulfilled.

THEOREM 1. *Suppose*

(i) Ω *is a bounded open set in* $\mathbb{R}^n$ *with locally Lipschitz boundary* $\partial\Omega$,

(ii) $H(x, t)$ *is a given real valued function on* $\Omega \times \mathbb{R}$ *which is increasing in* t *and satisfies*

$$H(x, t_0) \in L^1(\Omega) \text{ for each fixed } t_0 \in \mathbb{R} ,$$

(iii) φ *is a given function in* $L^1(\partial\Omega)$.

Then φ *may be extended to a function* $\tilde{\varphi}$, *defined on* Ω , *such that*

(a) $\tilde{\varphi} \in W^{1,1}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$,

(b) $\int_\Omega \left|\int_0^{\tilde{\varphi}} H(x, t)dt\right|dx < \infty$,

(c) trace $\tilde{\varphi} = \varphi$ *on* $\partial\Omega$.

Proof. Since $\partial\Omega$ is locally Lipschitz, by taking the appropriate transformations and using a partition of unity, we may reduce the problem, as in Gagliardo, to the case where: $\varphi \in L^1(S)$ and has compact support in S with

$$S = \{(x_1, \ldots, x_n) : 0 < x_i < 1, i = 1, \ldots, n-1 \text{ and } x_n = 0\} ,$$

and we wish to extend φ to a function $\tilde{\varphi}$ having compact support and satisfying the conditions of the theorem with the set Ω replaced by

$$Q = \{(x_1, \ldots, x_n) : 0 < x_i < 1, i = 1, \ldots, n\} .$$

(Actually we would be dealing with transforms of φ and $H(x, t)$ but, as $\partial\Omega$ is locally Lipschitz, the assumption of the theorem would still hold for these transformations.) By mollifying φ in R^{n-1} we can find a sequence $\{\varphi_n\}$ such that $\varphi_0 \equiv 0$,

$$\varphi_n \in C_0^\infty(S) \cap L^1(S) ,$$

$$\varphi_n \to \varphi \quad \text{in } L^1(S) ,$$

and, choosing a subsequence if necessary,

$$\sum_{n=0}^{\infty} \int_S |\varphi_n - \varphi_{n+1}| dS < \infty .$$

Now let $\{t_n\}$ be any sequence in $(0, 1)$ which is decreasing and convergent to 0 . Define the function u on Q by

$$u(x_1, \ldots, x_n) = 0 , \quad t_0 \le x_n < 1 ,$$

$$u(x_1, \ldots, x_{n-1}, t_n) = \varphi_n(x_1, \ldots, x_{n-1}) , \quad n = 1, 2, \ldots ,$$

$$u(x_1, \ldots, x_{n-1}, \lambda t_{n+1} + (1-\lambda)t_n) = \lambda \varphi_{n+1}(x_1, \ldots, x_{n-1}) + (1-\lambda)\varphi_n(x_1, \ldots, x_{n-1}) \quad \text{for } 0 < \lambda < 1 \text{ and } n = 0, 1, 2, \ldots .$$

Gagliardo shows that trace $u = \varphi$ and $u \in L^1(Q)$ and, if the sequence $\{t_n\}$ is chosen so that the differences $t_n - t_{n+1}$ are sufficiently small, then $u \in W^{1,1}(Q)$. Obviously by the construction of u it has compact support in Q and is locally Lipschitz. Thus it remains only to show that condition *(b)* of the theorem may be fulfilled.

Let

$$M_n = \sup_S |\varphi_n(x_1, \ldots, x_{n-1})|$$

and

$$Q_n = \{(x_1, \ldots, x_n) : 0 < x_i < 1, i = 1, \ldots, n-1 \text{ and } t_{n+1} < x_n < t_n\} .$$

Then $|u(x)| \le M_{n+1}$ on Q_n and so

$$\int_{Q_n} \left|\int_0^u H(x, t)dt\right| dx \le M_{n+1} \int_{Q_n} \{|H(x, M_{n+1})| + |H(x, -M_{n+1})|\} dx .$$

However $H(x, M_{n+1})$ and $H(x, -M_{n+1})$ belong to $L^1(Q)$. Thus if when selecting the sequence $\{t_n\}$ we ensure that $t_n - t_{n+1}$, and hence measure (Q_n) , are sufficiently small, then we will have

$$\sum_{n=0}^{\infty} \int_{Q_n} \left| \int_0^u H(x,\ t)dt \right| dx < \infty ,$$

and the theorem is proved. □

REMARKS. (i) The theorem of course tells us nothing new if we know in advance that $\varphi \in L^{\infty}(\partial\Omega)$ or if we know that

$$\int_{\Omega} \int_0^u H(x,\ t)dtdx < \infty$$

for every function in $W^{1,1}(\Omega)$; (for example, $H(x,\ t)$ uniformly bounded or independent of t). Thus the only interesting cases are when $|H(x,\ t)| \to \infty$ as $|t| \to \infty$.

(ii) Choosing $H(x,\ t) = \frac{1}{p} \operatorname{sign} t|t|^{p-1}$ we have $\int_0^t H(x,\ t)dt = |t|^p$ if $p > 1$. Hence, if Ω has locally Lipschitz boundary $\partial\Omega$ and $\varphi \in L^1(\partial\Omega)$, then φ may be extended to a function $\tilde{\varphi}$ such that $\tilde{\varphi} \in W^{1,1}(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega) \cap L^p(\Omega)$ for any p , $1 \le p < \infty$. Indeed we could even ensure, by standard techniques, that $\tilde{\varphi} \in C^{\infty}(\Omega)$. □

The next theorem generalizes a result of Massari and Pepe [8].

THEOREM 2. *Suppose* Ω *is a bounded open set in* R^n *with locally Lipschitz boundary. Suppose also that* f *is a given function in* $BV(\Omega)$. *Then there exists a sequence* $\{\Omega_k\}$ *of open sets satisfying:*

(i) $\partial\Omega_k$ *is an* $(n-1)$*-dimensional* C^{∞} *manifold;*

(ii) $\bar{\Omega}_k \subseteq \Omega_{k+1} \subseteq \Omega$ *for each* k *;*

(iii) $\lim_{k\to\infty} \varphi_{\Omega_k}(x) = \varphi_{\Omega}(x)$, $x \in R^n - \partial\Omega$, *where* φ_{Ω_k} *and* φ_{Ω} *are the characteristic functions of* Ω_k *and* Ω *respectively;*

(iv) $\lim_{k\to\infty} \int f|D\varphi_{\Omega_k}| = \int f|D\varphi_{\Omega}|$.

REMARKS. (i) Theorem 2 is proved in the special case where $f = 1$

by Massari and Pepe and in fact they also prove a similar theorem for approximation of Ω from the outside.

(ii) As pointed out in Lemma 3 we have

$$\int f|D\varphi_{\Omega_k}| = \int_{\partial\Omega_k} f dH_{n-1} ,$$

$$\int f|D\varphi_{\Omega}| = \int_{\partial\Omega} f dH_{n-1} .$$

Proof of Theorem 2. In [8] the following notation is introduced;

$\tau : R^n \to R$ is a positive symmetric mollifier,

$$\tau_k(x) = k^n \tau(kx) , \quad x \in R^n , \quad k = 1, 2, \ldots ,$$

$$\psi_k(x) = \int_\Omega \tau_k(x-y)dy , \quad x \in R^n ,$$

$$S_k(\lambda) = \left\{x \in R^n : \psi_k(x) > \lambda\right\} ,$$

and they prove that there exists a set $\Lambda \subseteq (0, 1)$ with $H^1(\Lambda) > 0$, and

(a) $\partial S_k(\lambda)$ is an $(n-1)$-dimensional C^∞ manifold for $k = 1, 2, \ldots$ and $\lambda \in \Lambda$,

(b) $\overline{S_k(\lambda)} \subseteq \Omega$,

(c) $\lim\limits_{k\to\infty} \varphi_{S_k(\lambda)}(x) = \varphi_\Omega(x) , \quad x \in R^n - \partial\Omega , \quad \lambda \in \Lambda ,$

(d) $\liminf\limits_{k\to\infty}\ \inf\limits_{\lambda\in\Lambda} |D\varphi_{S_k(\lambda)}|(R^n) = |D\varphi_\Omega|(R^n) .$

Then by taking an appropriate sequence $\{\lambda_k\}$ in Λ their result is proved. Clearly, if we can replace (d) by

(d') $\liminf\limits_{k\to\infty}\ \inf\limits_{\lambda\in\Lambda} \int f|D\varphi_{S_k(\lambda)}| = \int f|D\varphi_\Omega| ,$

then our result may also be proved. We show that (d') holds in much the same way that Massari and Pepe demonstrate (d), but now use Lemmas 1, 2, and 3 in place of the corresponding cases where $g \equiv 1$.

By using positive and negative parts we may assume that f is positive. Furthermore, by Theorem 1, we may extend f to a function, still denoted f, but now defined on the whole of R^n and satisfying

$$f \in BV(R^n) ,$$

on $\partial\Omega$, trace f/Ω = trace $f/R^n - \Omega$, and hence

$$\int_{\partial\Omega} |Df| = 0 .$$

Now suppose (d') is false. Then there exists $\varepsilon_0 > 0$ and a subsequence of $\{\psi_k\}$ which we still denote $\{\psi_k\}$ such that

$$\inf_{\lambda\in\Lambda} \int f|D\varphi_{S_k(\lambda)}| < \int f|D\varphi_\Omega| - \varepsilon_0 \tag{5}$$

or

$$\inf_{\lambda\in\Lambda} \int f|D\varphi_{S_k(\lambda)}| > \int f|D\varphi_\Omega| + \varepsilon_0 . \tag{6}$$

Suppose (5) holds. Then there exists a sequence $\{\lambda_k\} \subseteq \Lambda$ such that

$$\int f|D\varphi_{S_k(\lambda_k)}| < \int f|D\varphi_\Omega| - \varepsilon_0 + 1/k , \tag{7}$$

and hence

$$\limsup_{k\to\infty} \int f|D\varphi_{S_k(\lambda_k)}| \le \int f|D\varphi_\Omega| - \varepsilon_0 .$$

However it is known that for $\lambda_k \in \Lambda$,

$$\lim_{k\to\infty} \varphi_{S_k(\lambda_k)}(x) = \varphi_\Omega(x) , \quad x \in R^n - \partial\Omega .$$

Then noting that

$$\int f|D\varphi_\Omega| = \int_{\partial\Omega} f dH_{n-1} < \infty ,$$

we can use Lemma 1 to obtain

$$\int f|D\varphi_\Omega| \le \liminf_{k\to\infty} \int f|D\varphi_{S_k(\lambda_k)}| . \tag{8}$$

Clearly (7) and (8) give a contradiction, so that (5) cannot hold.

Now consider (6). Obviously we have

$$\int f|D\varphi_{S_k(\lambda)}| > \int f|D\varphi_\Omega| + \varepsilon_0 \; , \quad \lambda \in \Lambda \; .$$

Using Lemma 2 we obtain

$$\begin{aligned}\int f|D\psi_k(x)|dx &= \int_0^1 d\lambda \int f|D\varphi_{S_k(\lambda)}| \\ &= \int_\Lambda d\lambda \int f|D\varphi_{S_k(\lambda)}| + \int_{(0,1)-\Lambda} d\lambda \int f|D\varphi_{S_k(\lambda)}| \\ &> H^1(\Lambda)\left\{\int f|D\varphi_\Omega|+\varepsilon_0\right\} + \int_{(0,1)-\Lambda} d\lambda \int f|D\varphi_{S_k(\lambda)}| \; .\end{aligned}$$

From Lemmas 1 and 3 and result (c) above, on taking limit as $k \to \infty$, we see that

$$\int f|D\varphi_\Omega| \geq \lim_{k\to\infty} \int f|D\psi_k(x)|dx \geq \int f|D\varphi_\Omega| + \varepsilon_0 H^1(\Lambda)$$

giving the required contradiction. □

2. Main theorem

Using the approximation theorems of the previous section we can now prove our main result.

THEOREM 3. *Suppose*

(i) Ω *is a bounded open subset of* R^n *with locally Lipschitz boundary* $\partial\Omega$,

(ii) $H(x, t)$ *is a given real valued function on* $\Omega \times R$ *which is increasing in* t *and satisfies*

$$H(x, t_0) \in L^n(\Omega) \quad \textit{for each fixed} \quad t_0 \in R \; ,$$

(iii) $\varphi \in L^1(\partial\Omega)$ *is given,*

(iv) $\psi \in W^{1,\infty}(\Omega)$ *and* $\psi \leq \varphi$ *on* $\partial\Omega$.

Let

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$$M(v) = \int_\Omega \sqrt{1+|\nabla v|^2}\,dx + \int_\Omega \int_0^v H(x,\, t)\,dt\,dx \;,$$

$$K_1 = \{v \in W^{1,1}(\Omega) \;:\; v \geq \psi \text{ on } \Omega,\; v = \varphi \text{ on } \partial\Omega\} \;,$$

$$K_2 = \{v \in BV(\Omega) \;:\; v \geq \psi \text{ on } \Omega\} \;.$$

Then

$$\inf_{K_1} M(v) = \inf_{K_2} \left\{M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1}\right\} .$$

Proof. Noting that $K_1 \subseteq K_2$ and

$$M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1} = M(v) \quad \text{for} \quad v \in K_1 \;,$$

we have

$$\inf_{K_2} \left\{M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1}\right\} \leq \inf_{K_1} M(v) \;. \tag{9}$$

Let

$$K_3 = \{v \in C^\infty(R^n) \;:\; v \geq \psi \text{ on } \Omega\} \;.$$

We show

$$\inf_{K_3} \left\{M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1}\right\} = \inf_{K_2} \left\{M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1}\right\} . \tag{10}$$

Suppose $v \in K_2$ and define

$$v_m = \begin{cases} m \;, & v > m \;, \\ v \;, & |v| < m \;, \\ -m \;, & v < -m \;, \end{cases}$$

where $m > \max\{0,\, \max_\Omega \psi(x)\}$. Now $v \in L^1(\Omega)$ and hence $v_m \to v$ in $L^1(\Omega)$. Furthermore, by Lemma A.4 of [4] and lower semicontinuity of the appropriate integral,

$$\int_\Omega \sqrt{1+|\nabla v_m|^2}\,dx \to \int_\Omega \sqrt{1+|\nabla v|^2}\,dx \;, \tag{11}$$

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$$v_m \to v \quad \text{in} \quad L^1(\partial\Omega) \ . \tag{12}$$

We note that

$$0 \le \int_0^{v_m} [H(x,\ t)-H(x,\ 0)]dt \le \int_0^{v} [H(x,\ t)-H(x,\ 0)]dt$$

and

$$\int_0^{v_m} [H(x,\ t)-H(x,\ 0)]dt \to \int_0^{v} [H(x,\ t)-H(x,\ 0)]dt$$

for almost all x in Ω. Assuming $M(v) < \infty$ and making use of the assumptions on $H(x,\ 0)$, we may apply Lebesgue's Theorem on dominated convergence to obtain

$$\int_\Omega \int_0^{v_m} H(x,\ t)dtdx \to \int_\Omega \int_0^{v} H(x,\ t)dtdx \ . \tag{13}$$

Combining (11), (12), and (13), we have

$$M(v_m) + \int_{\partial\Omega} |v_m-\varphi| dH_{n-1} \to M(v) + \int_{\partial\Omega} |v-\varphi| dH_{n-1} \ . \tag{14}$$

Now v_m is defined on Ω but may be extended to a function, still denoted v_m, but now defined on the whole of R^n and such that

$$v_m \in BV(R^n) \ ,$$

on $\partial\Omega$, trace v_m/Ω = trace $v_m/R^n-\Omega$, and so

$$\int_{\partial\Omega} |Dv_m| = 0 \ .$$

If $v_{m\varepsilon}$ are the mollifers of v_m on R^n then it is known (see [7]) that

$$v_{m\varepsilon} \to v_m \quad \text{in} \quad L^1(\partial\Omega) \quad \text{as} \quad \varepsilon \to 0 \ ,$$

$$\int_\Omega \sqrt{1+|\nabla v_{m\varepsilon}|^2}\, dx \to \int_\Omega \sqrt{1+|\nabla v_m|^2}\, dx \quad \text{as} \quad \varepsilon \to 0 \ ,$$

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$$|v_{m\varepsilon}| \leq m \quad \text{as} \quad \varepsilon \to 0 ,$$

$$v_{m\varepsilon} \geq \psi_\varepsilon \geq \psi - \varepsilon L ,$$

where L is the Lipschitz constant of ψ. Putting

$$v^1_{m\varepsilon} = v_{m\varepsilon} + \varepsilon L ,$$

we have that $v^1_{m\varepsilon} \in K_3$ and, if $0 < \varepsilon < 1$,

$$\left| \int_\Omega \int_{v_{m\varepsilon}}^{v_m} H(x, t) dt dx \right| \leq C \int_\Omega |v_m - v_{m\varepsilon}| dx$$

where $C = \|H(x, m+L)\|_n + \|H(x, -m)\|_n$. Thus we obtain

$$(15) \qquad M\left(v^1_{m\varepsilon}\right) + \int_{\partial\Omega} \left|v^1_m - \varphi\right| dH_{n-1} \to M\left(v_m\right) + \int_{\partial\Omega} |v_m - \varphi| dH_{n-1} .$$

Combining (14) and (15) with a diagonal argument shows that we can find a sequence $\{v_k\}$ in K_3 such that

$$M\left(v_k\right) + \int_{\partial\Omega} |v_k - \varphi| dH_{n-1} \to M(v) + \int_{\partial\Omega} |v - \varphi| dH_{n-1} .$$

Noting that $K_3 \subseteq K_2$, equation (10) is proved.

Now extend φ to a function $\tilde{\varphi}$ as in Theorem 1, and observing that since $\psi \leq \varphi$ on $\partial\Omega$ and ψ is Lipschitz continuous on $\bar{\Omega}$, we may assume, by replacing $\tilde{\varphi}$ with $\max(\tilde{\varphi}, \psi)$, that $\tilde{\varphi} \geq \psi$ and the conclusions of Theorem 1 still hold true. Thus $\tilde{\varphi} \in K$ and $M(\tilde{\varphi}) < \infty$.

Suppose $v \in K_3$; then by Theorem 2 there exists a sequence of open sets $\{\Omega_k\}$ satisfying

(i) $\partial\Omega_k$ is C^∞,

(ii) $\bar{\Omega}_k \subseteq \Omega_{k+1} \subseteq \Omega$ for all k,

(iii) $\lim_{k\to\infty} \varphi_{\Omega_k}(x) = \varphi_\Omega(x)$, $x \notin \partial\Omega$,

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(iv) $\int_{\partial\Omega_k} |v-\tilde{\varphi}|dH_{n-1} \to \int_{\partial\Omega} |v-\tilde{\varphi}|dH_{n-1}$.

On each Ω_k we have $\tilde{\varphi}, v \in C^{0,1}(\overline{\Omega}_k)$, and hence, by the methods of Section 4 of [2], for each k there exists a sequence $\{v^1_{kj}\}$ such that

(a) $v^1_{kj} \in C^{0,1}(\overline{\Omega}_k)$,

(b) $v^1_{kj} \geq \psi$ on $\overline{\Omega}_k$,

(c) $v^1_{kj} = \tilde{\varphi}$ on $\partial\Omega_k$,

(d) $$\lim_{j\to\infty} \int_{\Omega_k} \sqrt{1+|\nabla v^1_{kj}|^2}\, dx + \int_{\Omega_k} \int_0^{v^1_{kj}} H(x, t)dtdx$$
$$= \int_{\Omega_k} \sqrt{1+|\nabla v|^2}\, dx + \int_{\Omega_k} \int_0^{v} H(x, t)dtdx + \int_{\partial\Omega_k} |v-\tilde{\varphi}|dH_{n-1} .$$

Now putting

$$v_{kj} = \begin{cases} v^1_{kj} & \text{in } \Omega_k , \\ \tilde{\varphi} & \text{in } \Omega - \Omega_k , \end{cases}$$

we have $v_{kj} \in K_1$ and

$$\lim_{j\to\infty} M(v_{kj}) = M(v) + \int_{\Omega-\Omega_k} \{\sqrt{1+|\nabla\tilde{\varphi}|^2} - \sqrt{1+|\nabla v|^2}\}dx$$
$$+ \int_{\Omega-\Omega_k} \int_{\tilde{\varphi}}^{v} H(x, t)dtdx + \int_{\partial\Omega_k} |v-\tilde{\varphi}|dH_{n-1}$$
$$= M(v) + I_1 + I_2 + I_3 .$$

However $v, \tilde{\varphi} \in W^{1,1}(\Omega)$ so $I_1 \to 0$ as $k \to \infty$; also $v \in K_3$ and $\tilde{\varphi}$ satisfies *(b)* of Theorem 1 so that $I_2 \to 0$ as $k \to \infty$. Finally, by (iv) above, $I_3 \to \int_{\partial\Omega} |v-\varphi|dH_{n-1}$ as $k \to \infty$. Thus

$$\lim_{k\to\infty}\left(\lim_{j\to\infty} M(v_{kj})\right) = M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1} ,$$

and the theorem is proved. □

REMARK. In the proof we have actually shown that, if

$$K_3 = K_2 \cap C^\infty(\bar{\Omega}) ,$$

$$K_4 = K_1 \cap W^{1,\infty}_{\mathrm{loc}}(\Omega) ,$$

then

$$\inf_{K_1} M(v) = \inf_{K_4} M(v) = \inf_{K_2} \left\{M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1}\right\}$$

$$= \inf_{K_3} \left\{M(v) + \int_{\partial\Omega} |v-\varphi|\,dH_{n-1}\right\} .$$

If we assume greater regularity for φ *then we may place a greater* restriction on K_4 and still obtain the same infinum. For example if $\varphi \in L^\infty(\partial\Omega)$ then we may replace K_4 by $K_4 \cap L^\infty(\partial\Omega)$ and if φ is Lipschitz continuous on $\partial\Omega$ then we may replace K_4 by $K_4 \cap C^{0,1}(\bar{\Omega})$. In both these cases the proofs could be greatly simplified.

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